SOME QUOTIENT SURFACES ARE SMOOTH

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Let $G\backslash \mathbf{V}$ be the quotient variety where G is a reductive group acting linearly on a vector variety $\mathbf{V}$. Then $G\backslash \mathbf{V}$ is an affine variety, whose regular functions may be identified with the regular functions on $\mathbf{V}$, which are invariant under the action of G.

It has become increasingly clear how very special the singularities of $G\backslash \mathbf{V}$ actually are [1,4,5 and 6]. In this note, we will show that, if G is a connected semi-simple group and $G\backslash \mathbf{V}$ is two dimensional, then $G\backslash \mathbf{V}$ must be non-singular. This result was conjectured by V. L. Popov, and it was brought to my attention by V. Kać.

A key step in my proof is the application of Mumford's smoothness criterion in terms of the local fundamental group. This idea was suggested by Mumford himself. The first step in the proof will be to check directly that the algebraic fundamental group of a large open subvariety of the quotient $G\backslash \mathbf{V}$ is trivial.

1. ETALE COVERINGS OF QUOTIENT VARIETIES

Let G be a reductive group over an algebraically closed field k of characteristic zero. We will denote the connected component of the identity of G by G_0. We will be working in the category of k-schemes of finite type.

Let X be an affine scheme with a given (morphic) action of the group G. Let $G\backslash X$ denote the quotient variety. Then we have the quotient morphism $\pi: X \to G\backslash X$. Let $f: S \to G\backslash X$ be a morphism. Form the Cartesian square,

$$\begin{array}{ccc} X_S & \to & X \\ \pi_S \downarrow & & \downarrow \pi \\ S & \xrightarrow{f} & G\backslash X. \end{array}$$

Then, G acts naturally on X_S through its action on X so that S may be regarded as the quotient $G\backslash X_S$ [10].

Next we will study the connected components of X_S.

LEMMA 1.1. *Assume that S is connected. Then,*

Received September 30, 1978.
This research was partly supported by NSF Grant #MCS75-05578.

Michigan Math. J. 27 (1980).

a) Each connected component of X_S is invariant under G_0.

b) These components are permuted transitively by $\pi_0 G = G/G_0$.

c) Each component maps surjectively onto S.

Proof. As G_0 is connected, its orbits in X_S are connected. Hence, G_0 perserves connected components. Thus, *a*) is true. For any connected component Y of X_S, the set $G \cdot Y$ is a finite union of connected components of X_S and $G \cdot Y$ is then an open and closed G-invariant subset of X_S. Clearly, X_S is a finite disjoint union $G \cdot Y_1 \amalg \dots \amalg G \cdot Y_n$ of such subsets. By invariant theory, S is the finite disjoint union $G \backslash G \cdot Y_1 \amalg \dots \amalg G \backslash G \cdot Y_n$ of closed subsets $G \backslash G \cdot Y_i =$ image of Y_i in S. As S is connected and not empty, we must have $S =$ image of Y and $X_S = G \cdot Y$ for one particular component Y of X_S. Thus, we have proven *c*); and *b*) follows because $X_S = G \cdot Y = \pi_0(G) \cdot Y$ using *a*).

Remark. The same result is true in characteristic *p*. Its proof requires slight technical changes.

We may now apply this lemma to study etale coverings. Recall that a scheme Z is simply connected if it is connected and any finite etale morphism $g: W \to Z$ from a connected variety W is an isomorphism. If V is an open subscheme of a simply connected smooth variety Z such that the complement $Z - V$ has codimension at least two in Z, then V is simply connected by the theorem of purity of Zariski-Nagata [3,$X - 3.4.i$].

We may now proceed to the proof of

LEMMA 1.2. *Let $\pi: X \to G \backslash X$ be the quotient morphism where G is connected and X is a smooth simply connected affine variety. Let U be an open subvariety of $G \backslash X$ such that $X - \pi^{-1} U$ has codimension at least two in X. Then, U is simply connected.*

Proof. Let $r: S \to U$ be a finite etale covering of U, where S is connected. Then, the pullback morphism $r': X_S \to \pi^{-1} U$ is again a finite etale morphism, where X_S is formed using the composition $f: S \to U \to G/X$ as before. By the above remark, our codimension assumption and the simple connectivity of X imply that $\pi^{-1} U$ is simply connected.

As $G = G_0$, the Lemma 1.1 implies that X_S is connected. Therefore,

$$r': X_S \to \pi^{-1} U$$

must be an isomorphism. Hence, $r: S = G \backslash X_S \to U = G \backslash \pi^{-1} U$ is also an isomorphism. Thus, U is simply connected.

We will use the last lemma by means of

PROPOSITION 1.3. *Let X be a smooth simply connected affine variety. Assume that*

a) $\Gamma(X, \mathcal{O}_X)$ is a unique factorization domain with only units being constants, and

b) a connected semi-simple group G acts on X.

Let U be an open subvariety of the quotient $G\backslash X$ such that is complement $G\backslash X - U$ has codimension at least two in $G\backslash X$. Then U is simply connected.

Proof. As G has no nontrivial characters, our assumption on $\Gamma(X, \mathcal{O}_X)$ implies that, if Z is a closed G-invariant subset of X, which has codimension $d = 1$ or 0, then its image $\pi(Z)$ in $G\backslash X$ has codimension d (See [5] for instance). Therefore, if W is a closed subset of $G\backslash X$ of codimension greater than or equal to 2, then $\pi^{-1} W$ is a closed subset of X of codimension greater than or equal to 2.

In our case, this means that the codimension of $X - \pi^{-1} U$ in X must be at least two. Thus, Lemma 1.2 shows that U is simply connected.

Remark. This proposition applies to the case $X = \mathbf{A}^n$ as $\mathbf{A}^n$ is simply connected (in characteristic zero) and verifies the assumption *a*).

2. CONICAL AND QUOTIENT VARIETIES

Let $A = k \oplus A_1 \oplus A_2 \ldots$ be a finitely generated graded k-algebra. Then the affine scheme $X = \operatorname{Spec} A$ will be called conical. The zeroes of the ideal $A_1 \oplus A_2 \oplus \ldots$ consists of a single point called the vertex of X.

Geometrically, a conical scheme X is an affine scheme with a morphic $\mathbf{G}_m$-action such that $\lim_{t\to 0} t \cdot x$ exists in X for all x in X and these limits are equal (to the vertex).

In coordinates $(x_1, \ldots, x_n)$, a conical scheme X may be written as the zeroes of a system $\{f_j(x_1, \ldots, x_i)\}$ of weighted-homogeneous polynomials. Here the weight p_i of the variable x_i is a positive integer and we are requiring that each polynomial f_j satisfies a functional equation $f_j(t^{p_1}x_1, \ldots, t^{p_x}x_n) = t^p f_j(x_1, \ldots, x_m)$ for some positive integer p depending on f_j. The vertex of X is the origin 0.

Assume that our ground field k is the complex numbers $\mathbf{C}$. Consider the function $g(x) = \max |x_i|^{1/p_i}$ on X. Then f satisfies the equation $g(t^{p_1}x_1, \ldots, t^{p_n}x_n) = |t| g(x_1, \ldots, x_n)$. Let U_i be the subset of X where $g(x) < i$ for some $0 < i \leq \infty$. Thus the U_i's form an increasing family of open neighborhoods of the vertex 0 in the associated complex analytic variety X_{an}. Better yet, when $i \leq j$, the inclusion $U_i - \{0\} \subseteq U_j - \{0\}$ is easily seen to be a homotopic equivalence by using the functional equations.

The consequence, that we need, is

LEMMA 2.1. *The local fundamental group of a conical $\mathbf{C}$-variety X_{an} at its vertex v is isomorphic to the fundamental group of $X_{an} - v$.*

Proof. By definition, the local fundamental group of X_{an} at $v = 0$ is $\varinjlim \pi_1(U - \{0\})$ where U runs through the system of open neighborhood U of 0. This system $\{U\}$ is partial ordered by inclusion and, as s approaches zero, $\{U_s\}$ is a cofinal subsystem of it. By the homotopy equivalence of the U_s's, the local fundamental group $\approx \pi_1(U_s - \{0\})$ for any s. The lemma is the particular case of this statement when $s = \infty$.

Next we will see how much of the local fundamental group may be computed algebraically.

LEMMA 2.2. *The pro-finite completion of local fundamental group of a conical* **C***-variety* X_{an} *at its vertex* v *is isomorphic to the (algebraic) fundamental group of* $X - v$.

Proof. As Grothendieck has remarked for any **C**-algebraic variety Y, the (algebraic) fundamental group of Y is isomorphic to the pro-finite completion of the fundamental group of Y_{an}. This follows from a result of Grauert-Remmert. Thus, our Lemma 2.1 implies the present one.

To achieve our objective, we will need to apply the following modification of Mumford's smoothness criterion.

THEOREM 2.3. *Let* S *be a conical surface over* **C** *with vertex* s. *If* $S - s$ *is simply-connected and* S *has rational singularities, then* S *is smooth.*

Proof. By definition, the local homology group of S_{an} at s is

$$H \equiv \underset{\rightarrow}{\text{limit}}\, H_1(U - s, \mathbf{Z})$$

where U runs through the neighborhoods of s as before. If $S - s$ is simply-connected, then by Lemma 2.2 the local fundamental group of S_{an} at s has no nontrivial finite quotient. As H is a finite generated abelian quotient of the local fundamental group, H is trivial. By a variant of Korollar 1.6 of [2], the local ring of S_{an} at s is factorial. By Satz 3.3 of [2], either S_{an} is smooth at s or it is locally isomorphic to the conical analytic subvariety $X^2 + Y^3 + Z^5 = 0$ of $\mathbf{C}^3$. As this famous icosahedral singularity of Klein has a finite (non-abelian) fundamental group, we now know that the local fundamental group of S_{an} at s is finite and hence, trivial. Therefore, by Mumford's smoothness criterion [9], S_{an} is smooth at s. As s is the only possible singularity on the normal conical variety S, we have shown that S is in fact smooth.

Remark. One would like to have a more algebraic proof of Theorem 2.3 along the lines of [8].

Now it only remains to combine the results of the two sections to prove

THEOREM 2.4. *Given a linear representation of a connected semi-simple group* G *on a vector variety* **V**, *if the quotient* $G\backslash\mathbf{V}$ *is a surface, then* $G\backslash\mathbf{V}$ *is smooth or, equivalently,* $\Gamma\mathbf{V}, \mathcal{O}_{\mathbf{V}})^G$ *is a free graded* k*-algebra with two generators.*

Proof. The scalar multiplication on **V** commutes with the action G. Thus $G\backslash\mathbf{V}$ inherits an action of $\mathbf{G}_m$ such that it is conical. By the Lefschetz principle, we may assume that the ground field k is **C**. By Proposition 1.3, we know that $G\backslash\mathbf{V}$ − {vertex} is simply-connected. By the Theorem in [5 or 1], we know that $G\backslash\mathbf{V}$ has rational singularities. Thus, as the assumptions of Theorem 2.3 are verified, the result follows.

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