SPLITTING THE PL INVOLUTIONS OF NONPRIME 3-MANIFOLDS

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INTRODUCTION

We describe four basic operations (I-operations) involving the connected sum construction with which PL involutions of 3-manifolds can be built up from involutions of simpler 3-manifolds. The main result (Theorem 1) is that every PL involution of a compact 3-manifold arises from involutions on its prime summands by repeated application of these four I-operations. It is well-known that every compact 3-manifold can be uniquely expressed (up to order) as the connected sum of prime 3-manifolds in normal form. Thus the study of PL involutions of compact 3-manifold is now reduced to problems involving PL involutions of prime 3-manifolds.

Section 1 is devoted to the descriptions of the I-operations and stating the main results. An application of Theorem 1 to double-coverings of S^3 branched over a link is also given here. Theorem 1 has also been applied to $P^3 \# P^3$ to show that there exist exactly seven distinct nonconjugate involutions on $P^3 \# P^3$ (see [5]). Section 2 contains the proof of Theorem 1. Finally, in Section 3, we prove a basic lemma for splitting 3-manifolds with involution along disks and suggest a further reduction for PL involutions of compact irreducible 3-manifolds with boundary with respect to the multi-disk sum operation.

1. STATEMENT OF RESULTS

We work exclusively in the PL category throughout this paper. All orientable 3-manifolds are assumed to be oriented. We let M^- denote the 3-manifold obtained from an oriented 3-manifold $M = M^+$ by reversing its orientation. Recall that the *connected sum* $M_1 \# M_2$ of two connected 3-manifolds M_1 and M_2 is obtained by removing the interior of a closed 3-cell from the interior of each and identifying the resulting 2-sphere boundaries by a homeomorphism (orientation reversing if both M_1 and M_2 are oriented). A compact 3-manifold M is said to be *prime* if it cannot be written as a connected sum of two 3-manifolds, each distinct from S^3. Recall that S^3 is the identity element for this operation. According to the unique decomposition theorem of Kneser [7] and Milnor [11] (see Hempel [4]), every compact 3-manifold can be written uniquely (up to order) as a connected sum of prime 3-manifolds in normal form (in the normal form $S^1 \times S^2$ is allowed to appear as a summand only when M is orientable). It follows that a compact 3-manifold can be built up in an essentially unique way from prime 3-manifolds.

Received February 7, 1975. Revision received October 29, 1979.

The work of the second author research was supported in part by the National Science Foundation.

Michigan Math. J. 27 (1980).

Our goal (Theorem 1) is to show that every involution of a compact 3-manifold can be built up in a canonical fashion from involutions on prime 3-manifolds. There are four basic operations used to build involutions by forming an equivariant connected sum of two 3-manifolds with involution. For the description of these I-operations we introduce the notation $M_1 \bar{\#} M_2$ to denote the 3-manifold obtained from two connected 3-manifolds M_1 and M_2 by removing the interiors of two disjoint 3-cells from each and identifying the resulting 2-sphere boundaries to obtain a connected 3-manifold homeomorphic to either $M_1 \# M_2 \# (S^1 \times S^2)$ or $M_1 \# M_2 \# N$ (here N is used to denote the non-orientable 2-sphere bundle over S^1).

Consider given involutions h_1 and h_2 on the 3-manifolds M_1 and M_2, respectively. In the first and third I-operations we construct involutions $\tilde{h}_1$ on $M_2 \# M_1 \# M_2^{\pm}$ and $h_1 \bar{\#} h_2$ on $M_1 \bar{\#} M_2$ using 3-cells for the connected sum operation which are disjoint from their image under the corresponding involution. For the other two I-operations we choose components of the fixed point sets of h_1 and h_2 and use invariant 3-cells meeting these preferred components for the connected sum construction. We obtain involutions $h_1 \# h_2$ on $M_1 \# M_2$ and h_1^* on $M_1 \bar{\#} S^3$.

Operation I-1. Let M_1 be a prime 3-manifold and let h_2 be the identity map. Choose a pair of disjoint closed 3-cells $(C_1, h_1(C_1))$ in $\mathring{M}_1$ and one 3-cell C_2 in $\mathring{M}_2$. Now take two copies of M_2 and form $M_2 \# M_1 \# M_2^{\epsilon}$ by sewing $M_2 - \mathring{C}_2$ to $M_1 - \mathring{C}_1$ with a homeomorphism $f : \partial C_1 \to \partial C_2$ and then sewing the second $M_2^{\epsilon} - \mathring{C}_2$ to $(M_2 \# M_1) - h_1(\mathring{C}_1)$ with $fh_1 : \partial h_1(C_1) \to \partial C_2 (\epsilon = +$ or $-$ as h_1 preserves or reverses the orientation in the oriented case). We let $\tilde{h}_1$ denote the involution agreeing with h_1 on $M_1 - (C_1 \cup h_1(C_1))$ and interchanging the two copies of M_2 via the identity map.

Operation I-2. Let F_1 and F_2 be given components of $\text{Fix}(h_1)$ and $\text{Fix}(h_2)$, respectively, having the same dimension. Choose an invariant closed 3-cell $C_i \subset \mathring{M}_i$ meeting F_i (for $i = 1$ and 2). Let $f: \partial C_1 \to \partial C_2$ be a homeomorphism such that $fh_1 | \partial C_1 = h_2 f$. Let $h_1 \# h_2$ denote the involution induced by h_1 and h_2 on $M_1 \# M_2$, where the connected sum is formed by identifying $M_1 - \mathring{C}1$ to $M_2 - \mathring{C}_2 2$ via f.

Operation I-3. Let $(C_i, h_i(C_i))$ be a pair of disjoint 3-cells in $\mathring{M}_i$, for $i = 1, 2$. Let $f : \partial C_1 \to \partial C_2$ be a homeomorphism with which we can form $M_1 \bar{\#} M_2$ by removing the interiors of the four given 3-cells, sewing ∂C_1 to ∂C_2 with f and sewing $\partial h_1(C_1)$ to $\partial h_2(C_2)$ with $h_2 f h_1$. Then h_1 and h_2 define an involution on $M_1 \bar{\#} M_2$ which we denote by $h_1 \bar{\#} h_2$.

Operation I-4. Let F_1 and F_1' be given components of $\text{Fix}(h_1)$ having the same dimension (perhaps $F_1 = F_1'$). Let C_1, C_1' be disjoint invariant 3-cells in M_1 meeting F_1, F_1', respectively. Let h_2 be a standard involution of $M_2 = S^3$ such that $\text{Fix}(h_2)$ has the same dimension as F_1 and F_1'. Choose a pair of disjoint invariant 3-cells C_2 and C_2' in $\mathring{M}_2$. Let $f : \partial C_1 \to \partial C_2$ and $f' : \partial C_1' \to \partial C_2'$ be homeomorphisms such that $fh_1 | \partial C_1 = h_2 f$ and $f' h_1 | \partial C_1' = h_2 f'$. Form $M_1 \bar{\#} S^3$ by sewing the two pairs of invariant 2-sphere boundaries together using the maps f and f' and let h_1^* denote the involution defined on $M_1 \bar{\#} S^3$ by h_1 and h_2.

There are several choices involved in each of these I-operations and their effect on the resulting involution will be examined in the Appendix. We are now in a position to state our main theorem.

THEOREM 1. *Every PL involution of a compact 3-manifold M is equivalent to an involution built up from a collection of involutions on prime summands of M by performing a sequence of I-operations.*

In case no summand of the 3-manifold is a 2-sphere bundle over S^1, this splitting theorem can be expressed very nicely.

COROLLARY 1. *Let M be a nonprime, compact 3-manifold with no 2-sphere bundle summands. If h is an involution of M then M and h can be viewed as $M = M_1 \# \ldots \# M_n$ and $h = h_1 \# \ldots \# h_n$, where each h_i is an involution on $M_i = A_i \# Q_i \# A_i^{\epsilon}$ arising from operation I-1 (thus Q_i is a prime 3-manifold).*

COROLLARY 2. *Let h be an involution on a nonprime, compact 3-manifold M that has no 2-sphere bundle summands and for which no two prime summands are homeomorphic. Then there exists a splitting $M = M_1 \# \ldots \# M_n$ and $h = h_1 \# \ldots \# h_n$ where each h_i is an involution on the irreducible 3-manifold M_i.*

COROLLARY 3. *Let h be an orientation-reversing involution on a connected sum M of lens spaces. Then $h = h_1 \# \ldots \# h_n$ and $M = M_1 \# \ldots \# M_n$ where each h_i is an involution on $M_i = A_i \# Q_i \# A_i^{\epsilon}$ arising from operation I-1, each Q_i is either p^3 or S^3, and the A_i are connected sums of lens spaces.*

Proof. Since a lens space $L(p,q), p > 2$, does not admit an orientation-reversing involution [8], this follows directly from Theorem 1.

Given a link L in S^3, we can apply Theorem 1 to compare the primeness of L with that of the unique 2-sheeted covering space of S^3 branched over L, denoted by $M(L)$. Recall that a link L is *splittable* if there exists a 3-cell D in S^3 such that $L \cap \partial D = \emptyset$, $D \cap L \neq \emptyset$ and $(S^3 - D) \cap L \neq \emptyset$. The link is said to be *prime* if for each 3-cell D in S^3 such that ∂D meets L transversally and in two point, either $D \cap L$ or $(S^3 - \mathring{D}) \cap L$ is an unknotted arc in D or $(S^3 - \mathring{D})$, respectively. It is well-known for a nonsplittable link L that if $M(L)$ is prime then L must also be prime [15]. We show that the converse to this also holds.

COROLLARY 4. *Suppose that L is a nonsplittable link in S^3. Then the 3-manifold $M(L)$ is prime if and only if L is a prime link.*

Proof. Suppose that $M(L)$ is not prime. The nontrivial covering transformation h of the branched covering space $p : M(L) \to S^3$ is an involution of $M(L)$ with $p(\text{Fix}(h)) = L$. It follows from Theorem 1 that there exists a 2-sphere S not bounding a 3-cell in $M(L)$ such that either $h(S) = S$ and S meets $\text{Fix}(h)$ transversally or else $h(S) \cap S = \emptyset$. Since L is nonspittable, we cannot have $h(S) \cap S = \emptyset$. Thus $h(S) = S$ and S must meet $\text{Fix}(h)$ in exactly two points. Therefore it follows that $p(S)$ is a 2-sphere splitting L into two nontrivial links.

2. CONNECTED SUM SPLITTING

The object of this section is the proof of Theorem 1. Our first goal is to prove the existence of suitable 2-spheres along which we can equivariantly split compact, nonprime 3-manifolds with involutions.

Suppose that h is a simplicial involution of a triangulated 3-manifold M. Consider a surface F properly embedded in M as a subcomplex. We move F into what we call *h-general position* by the following isotopy. First move F into general position with respect to Fix(h). Then, by using only isotopies that keep Fix(h) constant, we move $F - (F \cap \text{Fix}(h))$ into general position with respect to $h(F) - (h(F) \cap \text{Fix}(h))$. This can be done using the usual methods of shifting subcomplexes into general position. Finally we subdivide M such that $F \cup h(F)$ and h are simplicial. Observe that there are no isolated points in $F \cap h(F)$ since F is in general position with respect to Fix(h).

Given such a surface F in h-general position, we define the complexity $c(F)$ as the sum of the number of components in $(F \cap h(F)) - \text{Fix}(h)$ together with the number of components in $F \cap \text{Fix}(h)$. If F is invariant and in general position with respect to Fix(h) we set $c(F) = 0$. A closed disk E contained in $h(F)$ is said to be *innermost* if $E \cap F \subset \partial E$ and $\partial E - (E \cap F) \subset \partial M$. Two surfaces F and G in M are said to be *parallel* if there exists an embedding of $F \times [-1,1]$ in M such that $F = F \times \{1\}$ and $G = F \times \{-1\}$.

LEMMA 1. *Let h be an involution of a compact 3-manifold M that is not irreducible. Then there exists a 2-sphere S in $\mathring{M}$ not bounding a 3-cell such that either $h(S) \cap S = \emptyset$ or $h(S) = S$ and S is in general position with respect to* Fix(h).

Proof. If ∂M contains a 2-sphere component then we may take S to be a boundary component in $\mathring{M}$ of a regular neighborhood of such a 2-sphere component of ∂M. Thus let us assume that M has no 2-sphere boundary components. Let Σ denote the collection of all 2-spheres S in $\mathring{M}$ such that (i) S does not bound a 3-cell and (ii) either S is in h-general position or S is invariant and in general position with respect to Fix(h). Clearly $\Sigma \neq \emptyset$. Choose a 2-sphere S from Σ having minimal complexity among all members of Σ. If $c(S) = 0$ then we are done. Thus we will be finished if we show that $c(S) > 0$ implies the existence of another 2-sphere $S'' \in \Sigma$ with $c(S'') < c(S)$.

Suppose $c(S) > 0$ and let E denote an innermost disk in $h(S)$. Then $J = \partial E$ is a simple closed curve separating S into two open disks E_1 and E_2. Since S does not bound a 3-cell, it follows that at least one of the 2-spheres $E_1 \cup E$ and $E_2 \cup E$ does not bound a 3-cell. Let $S' = E_i \cup E$ denote one of the 2-spheres which does not bound a 3-cell. If $h(S') = S'$ then $S' \in \Sigma$ and $c(S') = 0$ (it is elementary to check that S' is in general position with respect to Fix(h)). If $h(S') \neq S'$ then we want to move S' by a small isotopy so as to shift it into h-general position and obtain $c(S') < c(S)$.

Let U be a small regular neighborhood of E in $\mathring{M}$ such that $U \cap S$ is a regular neighborhood of J in S. Choose a disk E' close to E in U such that

(i) $E' \cap S = \partial E'$;

(ii) $E' \cap h(S) = J \cap \text{Fix}(h)$;

(iii) in E_i, $\partial E' \cup J$ bounds an annulus A pinched along $J \cap \text{Fix}(h)$ (that is, A is homeomorphic to the quotient space of $J \times I$ obtained by identifying $\{y\} \times I$ to a point for each $y \in J \cap \text{Fix}(h)$);

(iv) the interior of the 3-cell in U bounded by $E \cup A \cup E'$ is disjoint from $S \cup h(S)$. Define S'' to be the 2-sphere $E' \cup (S' - (A \cup E))$, which is isotopic to S'. The 2-sphere S'' may fail to intersect Fix(h) transversally along some points of $J \cap \text{Fix}(h)$. If this occurs then we merely equivariantly push S'' away from

$h(S'')$ near those points where S'' is tangent to $\mathrm{Fix}(h)$ (the construction of this isotopy moving S' to S'' is referred to as an α-operation in [6]).

Observe that we have constructed a new 2-sphere S'' in h-general position such that S'' does not bound a 3-cell. Furthermore, since the intersection of S and $h(S)$ has been simplified along J and has not been added to elsewhere, we have $c(S'') < c(S)$. In view of our choice of S with minimal complexity, this completes the proof of the lemma.

Let h be an involution of a connected 3-manifold M. Suppose there exists a 2-sphere in $\mathring{M}$ such that $h(S) \cap S = \emptyset$ and $M - S$ is connected.

LEMMA 2. (a) *If S is parallel to $h(S)$ then either M is a 2-sphere bundle over S^1 or there exists an invariant 2-sphere in $\mathring{M}$ parallel to S.*

(b) *Suppose that S is not parallel to $h(S)$ and $M - (S \cup h(S))$ is not connected. If h interchanges the two components of $M - (S \cup h(S))$ then there exists a 2-sphere S' in $\mathring{M}$ defining a splitting $M = A \# B \# A'$ in which h interchanges A and A' and B is an invariant 2-sphere bundle over S^1 containing S.*

(c) *Suppose that S is not parallel to $h(S)$ and $M - (S \cup h(S))$ is connected. Then there exists either a nonseparating, invariant 2-sphere in $\mathring{M}$ which is in general position with respect to $\mathrm{Fix}(h)$ or a separating 2-sphere S' in $\mathring{M}$ such that $h(S') \cap S' = \emptyset$ and the 2-spheres S and $h(S)$ lie in different components of $M - S'$.*

Proof. (a) Since S is parallel to $h(S)$ we have M separated into two components by $S \cup h(S)$, say M_1 and M_2. Assume M_1 is a component homeomorphic to $S^2 \times I$. If h interchanges M_1 and M_2 then M is obviously a 2-sphere bundle over S^1. On the other hand, if $h(M_1) = M_1$ then it follows from [9], [10], [12] and [15] that there exists an invariant 2-sphere S' in $\mathring{M}_1$ parallel to a boundary component of M_1.

(b) Let M' be a component of $M - (S \cup h(S))$ and choose an arc γ in $cl(M')$ joining S to $h(S)$ such that γ meets $S \cup h(S)$ only at its endpoints. Consider the 2-sphere boundary S' of a regular neighborhood of $S \cup h(S) \cup \gamma$ in $cl(M')$. Then $S' \cup h(S')$ splits M into three components $\tilde{A}$, $\tilde{B}$, and $\tilde{A}'$ such that h interchanges $\tilde{A}$ and $\tilde{A}'$, leaving $\tilde{B}$ invariant. We finish by capping the 2-sphere boundaries corresponding to S' and $h(S')$ to obtain the 3-manifolds A, B, and A', respectively, such that $M = A \# B \# A'$, where B is a 2-sphere bundle over S^1 containing S.

(c) First suppose that $\mathrm{Fix}(h) \neq \emptyset$. Choose an arc γ in $\mathring{M} - h(S)$ joining S and $\mathrm{Fix}(h)$ such that $\gamma \cap (S \cup \mathrm{Fix}(h) \cup h(\gamma)) = \partial\gamma$. Let N be an invariant regular neighborhood of $S \cup h(S) \cup \gamma \cup h(\gamma)$ in M such that ∂N meets $\mathrm{Fix}(h)$ in general position. Then ∂N consists of three components, two of which are interchanged by h. The remaining component S' of ∂N is the desired nonseparating invariant 2-sphere.

Now suppose that $\mathrm{Fix}(h) = \emptyset$. There exists a noncontractible loop γ in $\mathring{M} - h(S)$ such that $\gamma \cap S$ consists of a single point and $h(\gamma) \cap \gamma = \emptyset$. Let S' denote the boundary of a regular neighborhood of $S \cup \gamma$ such that $h(S') \cap S = \emptyset$. It follows that S' separates M in such a way that S and $h(S)$ fall into opposite components. This completes the proof of the lemma.

Proof of Theorem 1. Let h be an involution of a compact nonprime 3-manifold M. We show that h can be decomposed into involutions of prime 3-manifolds from which we can recover h by repeated application of the four operations described in Section 1. Recall that the number of pairwise disjoint and nonparallel 2-spheres not bounding 3-cells in M is bounded (see [4], Lemma 3.14). Thus, the maximal collections of certain types of such 2-spheres which we consider below are always finite sets.

Step 1. Let Ω be a maximal collection $\{S_i\}$ of invariant 2-spheres in $\mathring{M}$ such that $\cup\ S_i$ does not separate M and each S_i is in general position with respect to $\mathrm{Fix}(h)$. Define M' to be the compact 3-manifold obtained by splitting M along the 2-spheres $\cup\ S_i$ and capping the resulting 2-sphere boundary components with 3-cells. Let h' denote the involution defined on M' by extending the involution induced by h on the split M over the added 3-cells by coning. Observe that we can recover M and h from M' and h' by reattaching the handles we cut along $\cup\ S_i$ by means of operation *I*-4. Now M' and h' have the property that every invariant 2-sphere in M' which is in general position with respect to $\mathrm{Fix}(h')$ separates M'.

Step 2. Consider a maximal collection $\Phi = \left\{C_i\right\}_{i=1}^{n-1}$ of pairwise disjoint 2-spheres in $\mathring{M}'$ such that

(i) $h'(C_i) = C_i$

(ii) no two C_i's are parallel,

(iii) each C_i does not bound a 3-cell,

(iv) C_i is disjoint from Ω, and

(v) each C_i is in general position with respect to $\mathrm{Fix}(h')$.

It follows from the maximality of Ω that each C_i separates M'. Split M' along $\cup\ C_i$ to obtain a collection $\left\{R_i'\right\}_{i=1}^{n}$ of compact 3-manifolds. If we cap the 2-sphere boundary components arising from the cuts along $\cup\ C_i$ with 3-cells, we obtain 3-manifolds $\left\{R_i\right\}_{i=1}^{n}$ such that $M' = R_1 \# \ldots \# R_n$, where the sum is formed by identifying the manifolds R_i' back along the C_i's.

Case 1. h' interchanges the sides of some C_i. Choose an invariant regular neighborhood $\tilde{U}$ of C_i in $\mathring{M}'$ such that $\tilde{U}$ is disjoint from Ω. Observe that $\tilde{U}$ is homeomorphic to $S^2 \times I$. Let $\tilde{A}$ and $\tilde{A}'$ denote the two components of $cl(M' - \tilde{U})$ which are interchanged by h'. Cap the 2-sphere boundary components of $\tilde{A}$, $\tilde{A}'$ and $\tilde{U}$ corresponding to $\partial\tilde{U}$ with 3-cells to obtain A, A' and U, respectively, where U is homeomorphic to S^3. Then $M' = A \# U \# A'$, where the sum is formed by reattaching $\tilde{A}$, $\tilde{A}'$ and $\tilde{U}$ back together as they were. Thus, in this case, h' is an involution of M' arising from an involution of S^3 by operation *I*-1.

Case 2. h' does not interchange the sides of any C_i. Then each R_i' is invariant under h' and $h' \mid R_i'$ can be extended to an involution g_i of R_i by coning over the 3-cells attached along the 2-spheres $\{C_i\}$ (note that this coning over a 3-cell attached to an invariant 2-sphere introduces at least one fixed point for g_i). Since we now have the splitting $M' = R_1 \# \ldots \# R_n$ and $h' = g_1 \# \ldots \# g_n$, it follows that h' can be obtained from $\left\{R_i, g_i\right\}_{i=1}^{n}$ by $n - 1$ applications of operation *I*-2.

Step 3. We now examine the involution $g = g_k$ of $R = R_k$, to look for a further splitting ($1 \leq k \leq n$). We note that every invariant 2-sphere in R in general position with respect to $\mathrm{Fix}(g)$ bounds a 3-cell. From the previous steps we have some invariant 2-spheres in R from Ω and Φ which now bound 3-cells. We shall avoid intersecting these with any new 2-spheres used in our further splitting of R.

Consider a maximal collection $\Gamma = \left\{(D_i, g(D_i)\right\}_{i=1}^{m-1}$ of pairs of 2-spheres in $\mathring{R}$ such that

(i) $g(D_i) \cap D_i = \emptyset$,

(ii) no two D_i's are parallel,

(iii) D_i is not parallel to $g(D_j)$ if $i \neq j$,

(iv) D_i does not separate R but $D_i \cup g(D_i)$ does separate R,

(v) $\bigcup_{i=1}^{m-1} (D_i \cup g(D_i))$ separates R into exactly m components, and

(vi) $\bigcup_{i=1}^{m-1} (D_i \cup g(D_i))$ is disjoint from Ω and Φ.

We emphasize that in this step we allow D_i and $h(D_i)$ to be parallel when these 2-spheres cobound a manifold $N(\approx S^2 \times I)$ and $N \cap \mathrm{Fix}(h) \neq \emptyset$ (in this case, although there exists an invariant 2-sphere S in N (see Lemma 2(a)), this S cannot be contained in Ω since S is not in general position with respect to $\mathrm{Fix}(h)$).

Case 1. g interchanges the sides of $R - (D_i \cup g(D_i))$ for some i. It follows from Lemma 3 that either $M' = B$ or we can write $M' = A \# B \# A'$ where B is an invariant 2-sphere bundle over S^1 and g interchanges A and A'. Hence the involution g arises by applying operation *I*-1 to the involution induced on B by g. Moreover, since g is obviously fixed point free, it follows that the collection Φ from Step 2 must be vacuous and hence $M' = R$.

Case 2. g does not interchange the sides of $R - (D_i \cup g(D_i))$ for any i. Split R along $\bigcup_{i=1}^{m-1} (D_i \cup g(D_i))$ to obtain the collection $\left\{M_i'\right\}_{i=1}^{m}$ of 3-manifolds. Cap the 2-sphere boundaries of each M_i' coming from Γ with 3-cells to obtain the 3-manifolds $\{M_i\}$ such that $R = M_1 \mathbin{\bar{\#}} \ldots \mathbin{\bar{\#}} M_m$, where M_i is joined to M_{i+1} along the 2-spheres $D_i \cup g(D_i)$. Since each M_i' is invariant under g, we have involutions h_i defined on M_i ($i = 1, \ldots, m$) by g such that $g = h_1 \mathbin{\bar{\#}} \ldots \mathbin{\bar{\#}} h_m$.

Consequently, we can obtain the involution g of R by applying operation I-3 (m-1 times) to $\left\{M_i, h_i\right\}_{i=1}^{m}$.

It remains to show that each involution h_i of M_i is one of the desired type arising from operation I-1. If M_i is prime there is nothing to show, so suppose that M_i is not prime. It follows from Lemma 1 that there exists a 2-sphere S in $\mathring{M}_i$ such that S does not bound a 3-cell and either $h_i(S) \cap S = \emptyset$ or S is invariant and in general position with respect to $\mathrm{Fix}(h_i)$. We may also assume that S is disjoint from $\Omega \cup \Phi \cup \Gamma$ since all the 2-spheres in these sets now bound 3-cells. It follows from the maximality of Ω and Φ that S cannot be invariant. Thus $h_i(S) \cap S = \emptyset$. Let us first suppose that S does not separate M_i. It follows from the maximality of Γ that $S \cup h_i(S)$ does not separate M_i. It then follows from Lemma 3 (c) and the maximality of Ω and Φ that there exists a separating 2-sphere S' in $\mathring{M}_i$ such that $S' \cap h_i(S') = \emptyset$ and S and $h(S)$ fall into opposite components of $M_i - S'$ (we may assume that S' is disjoint from $\Omega \cup \Phi \cup \Gamma$). Hence we may as well assume that S itself originally separated $\mathring{M}_i$.

At this point we find ourselves with a 2-sphere S in $\mathring{M}_i$ such that S separates M_i, $h_i(S) \cap S = \emptyset$, and S does not bound a 3-cell. If we split M_i along $S \cup h(S)$ and cap the resulting 2-sphere boundaries, we obtain $M_i = A \# Q \# B$ where h_i interchanges A and B and leave Q invariant. We will be finished if we show that the 2-sphere S can be chosen such that Q is prime, since it then follows that h_i arises from operation I-1.

If $\mathrm{Fix}(h_i) \neq \emptyset$ then choose an arc γ in $\mathring{M}_i - h_i$ joining S and $\mathrm{Fix}(h_i)$ such that $\gamma \cap (S \cup \mathrm{Fix}(h_i) \cup h_i(\gamma)) = \partial\gamma$. Let N be an invariant regular neighborhood of $S \cup h_i(S) \cup \gamma \cup h_i(\gamma)$ in $\mathring{M}_i$ such that ∂N meets $\mathrm{Fix}(h_i)$ in general position (we may also assume N is disjoint from $\Omega \cup \Phi \cup \Gamma$). Then ∂N consists of three components, two of which are interchanged by h_i. The remaining component S' is an invariant 2-sphere which separates M_i into two components, one of which contains both A and B. By the maximality of Φ, the closure of other component of $M_i - S'$ is a 3-cell and hence Q is homeomorphic to S^3.

Suppose then that $\mathrm{Fix}(h_i) = \emptyset$ and assume that S has been chosen such that Q has the fewest number of nontrivial summands among all such Q arising in this way. If Q is not prime then, using the argument employed to select S in M_i, we can find a separating 2-sphere Σ in $\mathring{Q}$ which does not bound a 3-cell, such that $h_i(\Sigma) \cap \Sigma = \emptyset$; we may further assume that Σ is disjoint from and not parallel to any element of $\Omega \cup \Phi \cup \Gamma \cup \{S, h_i(S)\}$. If Σ separates S from $h_i(S)$ then $\Sigma \cup h_i(\Sigma)$ clearly defines a splitting $M_i = A' \# Q' \# B'$ in which h_i interchanges A' and B', leaving Q' invariant, and where Q' is a proper summand of Q. Since this contradicts our choice of S we must have S and $h_i(S)$ in the same component of $M_i - \Sigma$. We can choose an arc J in $\mathring{Q} - h_i(\Sigma)$ joining S and Σ such that $J \cap (S \cup h_i(J) \cup h_i(\Sigma)) = \partial J$. Let K denote a regular neighborhood of $S \cup J \cup \Sigma$ in $\mathring{Q}$ such that $h_i(K) \cap = \emptyset$. We may assume as usual that K is disjoint from $\Omega \cup \Phi \cup \Gamma$. Let Σ' denote the boundary component of K contained in the component of $Q - (S \cup \Sigma)$ which contains J. This 2-sphere Σ' defines a splitting $M_i = A' \# Q' \# B'$ in which h_i interchanges the summands A' and B', leaving Q' invariant. Again, Q' is a proper summand of Q, contradicting our choice of S. Hence Q must be a prime 3-manifold. This completes the proof since all involutions in this last step arises from an application of operation I-1. We have shown that the involution h arises by repeated applications of operation I-1, followed by applications of operation I-3, then operation I-2, and finally operation I-4. Moreover, the collection of 2-spheres $\Omega \cup \Phi \cup \Gamma \cup \{\Sigma, h(\Sigma)\}$ used in the construction of h was chosen with the 2-spheres pairwise disjoint.

3. FURTHER SPLITTING ALONG DISKS

We have seen in Section 2 that an involution of a compact 3-manifold can be decomposed into involutions of 2-sphere bundles over S^1 and of compact irreducible 3-manifolds. If these latter irreducible 3-manifolds have boundary, one may wish to further decompose the involution by splitting along disks. The purpose of this section is to suggest that a further splitting along disks is possible and to provide the necessary tool (Lemma 3) for carrying out such a program.

Given two disjoint, compact 3-manifolds M and M' (with nonempty boundaries) we can form a *multi-disk sum* of M and M' by identifying disks on ∂M with corresponding disks on $\partial M'$, allowing at most one disk in each component of ∂M and $\partial M'$. An irreducible 3-manifold is *m-prime* if it is not a 3-cell and every multi-disk sum decomposition of M contains a 3-cell summand. In [2], J. Gross proves that every compact, oriented, irreducible 3-manifold with nonvacuous boundary has an essentially unique multi-disk sum decomposition into m-prime 3-manifolds. It is easy to see that there exist infinitely many m-prime 3-manifold which contain nonseparating, properly embedded disks. This is in contrast to the connected sum operation in which the only prime 3-manifolds containing nonseparating 2-spheres are 2-sphere bundles over S^1. Thus by splitting along disks, one could decompose an involution of a compact, irreducible 3-manifold with boundary even further than into the m-prime summands, although the uniqueness of the decomposition would be lost. The following lemma provides the necessary disks for a splitting. The final reduction could be phrased in terms of either m-prime 3-manifolds or 3-manifolds in which every properly embedded disk splits off a 3-cell.

LEMMA 3. *Let h be an involution on a compact 3-manifold M. Suppose that there exists a properly embedded disk D in M such that ∂D lies in a given component B of ∂M and ∂D does not bound a disk in B. Then there exists a disk S properly embedded in M with the properties*

(i) $\partial S \subset B$,

(ii) *∂S does not bound a disk in B, and*

(iii) *either $h(S) \cap S = \emptyset$ or $h(S) = S$ and S is in general position with respect to* Fix(h).

Proof. Let Σ denote the collection of all disks D properly embedded in M with the following properties:

(1) $\partial D \subset B$,

(2) ∂D does not bound a disk in B, and

(3) D is either invariant and in general position with respect to $\mathrm{Fix}(h)$ or D is in h-general position.

Clearly $\Sigma \neq \emptyset$. In what follows we show that whenever $D \in \Sigma$ and $c(D) > 0$, there exists another disk D' in Σ such that $c(D') < c(D)$. Thus if we choose a disk D in Σ having the smallest possible complexity, it follows that $c(D) = 0$; that is, D satisfies the conclusion of the lemma.

Suppose that $D \in \Sigma$ and $c(D) > 0$. Let E be an innermost disk in $h(D)$. Such an E always exists and we have two cases to consider.

Case 1. $E \subset \mathrm{Int}(h(D))$. In this case we can use an α-operation (as in the proof of Lemma 1) to obtain the desired disk D'.

Case 2. E is not contained in $\mathrm{Int}(h(D))$. We may assume that $E \cap D$ is an arc contained in ∂E. We use a β-operation from [6] to obtain the desired disk D' in this case and thus complete the proof.

APPENDIX

We have shown that every *PL* involution of a compact 3-manifold arises from involutions on its prime summands by repeated application of the four operations defined in Section 1. There are several choices involved in each of these *I*-operations and we examine their effect on the resulting involution. This is done in Theorem 2 and in the examples followed by its proof.

In order to state Theorem 2 we need some additional notation. Let h_1 and h_2 be given involutions on the 3-manifolds M_1 and M_2, respectively. Let $C_1 \subset \mathring{M}_1$, $C_2 \subset \mathring{M}_2$ denote closed 3-cells as used in the *I*-operations. Another 3-cell $C_i' \subset \mathring{M}_i$ with $\partial C_i' \cap \partial C_i = \emptyset$ is said to be of the same type as C_i $(i = 1, 2)$ if either C_i and C_i' are both disjoint from $h_i(C_i \cup C_i')$ or both 3-cells are invariant under h_i and meet the same component of $\mathrm{Fix}(h_i)$. We let $\sum(h_1, h_2)$ denote the set of all homeomorphisms $f' : \partial C_1' \to \partial C_2'$, where the C_i' range over all 3-cells in $\mathring{M}_i$ having the same type as C_i and $f' h_1 | \partial C_1 = h_2 f'$ whenever C_1 and C_2 are invariant under h_1 and h_2, respectively. For $f, f' \in \sum(h_1, h_2)$ we say that $f \sim f'$ if

(i) there exists a 3-cell $D_i \subset \mathring{M}_i$ of the same type as C_i such that $C_i \cup C_i' \subset \mathring{D}_i$,

(ii) there exists a homeomorphism $k_i : D_i \to D_i$ that commutes with h_i whenever D_i is invariant such that $k_i | \partial D_i$ is the identity and $k_i(C_i) = C_i'$ $(i = 1, 2)$, and

(iii) f' and $k_2 f k_1^{-1} | \partial C_1'$ are isotopic as maps of pairs

$$(\partial C_1', \mathrm{Fix}(h_1) \cap \partial C_1') \to (\partial C_2', \mathrm{Fix}(h_2) \cap \partial C_2').$$

Let $[f]$ denote the equivalence class of f with respect to the equivalence relation on $\sum(h_1, h_2)$ generated by the relation $\sim$. Since one can always find a disk D_i

satisfying condition (i) the set $\sum(h_1,h_2)$ is decomposed into at most four equivalence classes. To write them, we let $\rho:\partial C_1 \to \partial C_1$ be a homeomorphism isotopic to the identity which reverses the orientation on $\partial C_1 \cap \mathrm{Fix}(h_1)$ (when this set is nonempty) and we let $r:\partial C_1 \to \partial C_1$ be an orientation-reversing homeomorphism constant on $\partial C_1 \cap \mathrm{Fix}(h_1)$. We also assume that ρ and r each commute with $h_1|\partial C_1$. All equivalence classes are given by $[f]$, $[fr]$, $[f\rho]$ and $[fr\rho]$, where often some of these coincide.

The set we are really interested in is the set of all triples $(h_1,h_2;[f])$, where $f \in \sum(h_1,h_2)$ and h_i is an involution of M_i. Consider the equivalence relation on this set generated by $(h_1,h_2;[f]) \sim (g_1h_1g_1^{-1},g_2h_2g_2^{-1};[g_2f(g_1|\partial C_1)^{-1}])$, where $g_i:M_i \to M_i$ is a homeomorphism $(i=1,2)$. Let $\{h_1,h_2;[f]\}$ denote the equivalence class containing $(h_1,h_2;[f])$. Corresponding to a given pair of involutions h_1, h_2 there clearly exist at most four such equivalence classes and at most two whenever $\partial C_1 \cap \mathrm{Fix}(h_1)=\emptyset$.

For pairs of sewing maps used in operation *I*-4, we say that $(f,f') \sim (k,k')$ if one of the following holds:

(i) $[f]=[k]$ and $[f']=[k']$, (ii) $[fr]=[k]$ and $[f'r']=[k']$, (iii) $[f\rho]=[k]$ and $[f'\rho']=[k']$ (r' and ρ' are homeomorphisms of $\partial C_1'$ serving the same purpose as r and ρ, respectively). The relation generates an equivalence relation on the set of all eligible pairs of sewing maps and we write $[f,f']$ for the equivalence class containing (f,f'). Analagous to the notation established for use with the first three *I*-operations, we consider the equivalence relation generated by

$$(h_1,h_2;[f,f']) \sim (g_1h_1g_1^{-1},g_2h_2g_2^{-1};[g_2f(g_1|\partial C_1)^{-1},g_2f'(g_1|\partial C_1')^{-1}]),$$

where $g_i:M_i \to M_i$ is a homeomorphism for $i=1,2$. Under this equivalence relation the set of all triples $(h_1,h_2;[f,f'])$, where h_i is an involution of M_i and f, $f' \in \sum(h_1,h_2)$ is a suitable pair of the *I*-4 operations decomposed into at most two equivalence classes. We write $\{h_1,h_2;[f,f']\}$ for the equivalence class containing $(h_1,h_2;[f,f'])$.

We can now indicate the dependence of the *I*-operations on the various choices we made during their description. Any variable which is not specifically mentioned in the next theorem does not influence the equivalence class of the constructed involution.

THEOREM 2. (i) *The equivalence class of the involution $\tilde{h}_1$ constructed in operation I-1 depends only on the (oriented) homeomorphism type of M_2 and the equivalence class $\{h_1,1;[f]\}$.*

(ii) *The equivalence class of $h_1 \# h_2$ constructed in I-2 depends only on the components F_1, F_2 and on the class $\{h_1,h_2;[f]\}$.*

(iii) *The equivalence class of $h_1 \bar{\#} h_2$ built in I-3 depends only on the class $\{h_1,h_2;[f]\}$.*

(iv) *The equivalence class of* h_1^* *built in I-4 depends only on the components* F_1, F_1' *and the class* $\{h_1, h_2;\ [f, f']\}$.

The equivalence class of an involution which results from a sequence of I-operations is independent of the choice of the 3-cells used for the connected sum operations. Since we can choose these 3-cells to be pairwise disjoint, we have the following useful result (also see the proof of Theorem 1).

COROLLARY 5. *The equivalence class of an involution built up by a sequence of I-operations is independent of the order of the sequence as long as all the I-1 operations are performed first.*

Thus the I-operations provide a well-defined procedure for building involutions. From the splitting theorem (Theorem 1) we see that every involution of a compact 3-manifold can be constructed in this way by a sequence of I-operations. In the following we give five examples which illustrate the nontrivial effects of changing the variables involved in the I-operations. The first example concerns the choice of the preferred components of the fixed point-sets.

Example 1. Let h denote the unique orientation-reversing involution of real projective 3-space p^3 (see [8]). The fixed point set of h has two components, an isolated point and a projective plane. We can build an involution $h \# h$ of $p^3 \# p^3$, using operation I-2, by removing the interior of an invariant 3-cell neighborhood of the isolated fixed point from each copy of p^3 and then identifying along the resulting invariant 2-spheres. The resulting involution has a fixed point set homeomorphic to the disjoint union of two projective planes. On the other hand, we can build an involution $h \# h$ of $p^3 \# p^3$ by removing the interior of an invariant 3-cell meeting the p^2 component of $\mathrm{Fix}(h)$ in a disk from each copy of p^3 and then identifying along the resulting 2-sphere boundaries. This second construction gives an involution with two isolated points and a Klein bottle for a fixed point set. A similar example using operation I-4 is easily obtained.

In the next example we see the effect of changing the isotopy class of the sewing map f between the 2-sphere boundaries ∂C_1 and ∂C_2.

Example 2. Consider oriented lens spaces $M_1 = L(2,1) = p^3$ and $M_2 = L(4,1)$. Recall that $L(4,1)$ does not admit an orientation-reversing homeomorphism (e.g. see [4]). Let h_1 denote the free involution of M_1 that has $L(4,1)$ as an orbit space. Choose 3-cells $C_1 \subset \mathring{M}_1$, $C_2 \subset \mathring{M}_2$ and a homeomorphism $f : \partial C_1 \to \partial C_2$ which is orientation-reversing with respect to the induced orientations. Also, let $r : \partial C_1 \to \partial C_1$ be any orientation-reversing homeomorphism of the 2-sphere ∂C_1. Apply operation I-1, using the above maps and 3-cells, to build an involution $\tilde{h}$ on $M = M_2 \# M_1 \# M_2$ corresponding to $\{h_1, 1;\ [f]\}$. The orbit space of $\tilde{h}$ is homeomorphic to $M_2 \# M_2$. Now let $\tilde{h}^-$ denote the involution on $M' = M_2 \# M_1^- \# M_2$ corresponding to $\{h_1, 1;\ [fr]\}$ which results if we substitute fr for f in the above construction. The orbit space of $\tilde{h}^-$ is homeomorphic to $M_2 \# M_2^-$, which is not homeomorphic to $M_2 \# M_2$. However, since M_1 admits an orientation-reversing homeomorphism the 3-manifolds M and M' are homeomorphic but the involutions $\tilde{h}$ and $\tilde{h}^-$ are clearly not equivalent.

If we use operation I-3 instead of I-1, then we can build involutions $h_1 \bar{\#} h_1$ and $h_1 \bar{\#} h_1^-$ on $M_1 \bar{\#} M_1$ and $M_1 \bar{\#} M_1^-$ corresponding to $\{h_1, h_1; [1]\}$ and

$\{h_1, h_1; [r]\}$, respectively. These two involutions also have distinct orbit spaces $M_2 \# M_2$ and $M_2 \# M_2^-$.

In the third example we examine the effect of changing the involution h_1 by conjugating it with a homeomorphism g of M_1. This example is essentially the previous example viewed from a different perspective and one should notice here the role of the attaching map f when the involutions are replaced by their conjugates.

Example 3. As in the beginning of Example 2, let $M_1 = P^3$, $M_2 = L(4,1)$ and h_1 be a free involution of M_1 with orbit space $L(4,1)$. Let g be the orientation-reversing involution of M_1 as described in Example 1. After perhaps replacing g by some conjugate of g, we can find a closed 3-cell $C_1 \subset \mathring{M}_1$ such that C_1 and $h_1(C_1)$ are disjoint and both are invariant under g. Clearly $g|\partial C_1$ reverses the orientations of the 2-spheres ∂C_1. Apply operation *I*-1 to build the involutions $\tilde{h}$ on $M_2 \# M_1 \# M_2$ (using $f : \partial C_1 \to \partial C_2$), $\tilde{h}^-$ on $M_2 \# M_1^- \# M_2$ (using $fg: \partial C_1 \to \partial C_2$) and $g^{-1}hg$ on $M_2 \# M_1 \# M_2$ (using $f: \partial C_1 \to \partial C_2$). An equivalence between $\tilde{h}^-$ and $g^{-1}hg$ is easily defined by piecing together a homeomorphism $M_2 \# M_1 \# M_2 \to M_2 \# M_1^- \# M_2$, using g on the M_1 summand and the identity on the M_2 summands. By example 2, the involution $\tilde{h}$ is not in the same equivalence class as $\tilde{h}^-$ and $g^{-1}hg$. Thus the two classes $\{h, 1; [f]\}$ and $\{h, 1; [gf]\} = \{g\,hg, 1; [f]\}$, with all other variables the same, yield nonequivalent involutions under operation *I*-1. This type of example can easily be modified to apply to operation *I*-3.

The next example concerns operation *I*-4, where a pair of independent sewing maps are used, and the effect of reversing the isotopy class of only one sewing map.

Example 4. Let $h_1 = h_2$ be the involution of $M_1 = M_2 = S^3$ with a 2-sphere for the fixed point set. Choose invariant closed 3-cells C_1, $C_1' \subset \mathring{M}_1$ and C_2, $C_2' \subset \mathring{M}_2$ that are pairwise disjoint and each meets the corresponding fixed point set in a disk. Choose a pair of orientation-reversing homeomorphisms $f: \partial C_1 \to \partial C_2$, $f' : \partial C_1' \to \partial C_2'$ and apply operation *I*-4 (using h_1, h_2 and (f, f')) to build the involution h_1^* on $M_1 \bar{\#} M_2 \approx S^1 \times S^2$ having a fixed point set homeomorphic to $S^1 \times S^1$. On the other hand, let $\rho : \partial C_1 \to \partial C_1$ be a homeomorphism isotopic to the identity, such that ρ commutes with $h_1|\partial C_1$ and $\rho|(\partial C_1 \cap \mathrm{Fix}(h_1))$ is orientation-reversing. Apply *I*-4 again (this time using h_1, h_2 and $(f\rho, f')$) to build an involution h_1^{**} on $M_1 \bar{\#} M_2 \approx S^1 \times S^2$ where h_1^{**} has a Klein bottle for a fixed point set.

Example 5. Let L be a link in S^3 with the property that we can form the composition $L \cdot L$ [3] in two distinct ways to obtain links L_1 and L_2 with the property that no homeomorphism of S^3 carries L_1 onto L_2. Let h denote the nontrivial covering transformation of $p : M(L) \to S^3$, the two-sheeted covering of S^3 branched along L. We can apply operation *I*-2 to build an involution $h \# h$ of $M(L) \# M(L)$ in two ways; one with $p(\mathrm{Fix}(h \# h)) = L_1$ and the other with $p(\mathrm{Fix}(h \# h)) = L_2$. Obviously, those two involutions are not equivalent. This type of example also exists for operation *I*-4 and here we can additionally alter the number of components of $\mathrm{Fix}(h^*)$ by the choice of the preferred components of $\mathrm{Fix}(h)$.

Proof of Theorem 2. We continue the notation used in defining the *I*-operations.

Thus we have an involution h_i of the 3-manifold M_i, the special 3-cells $C_i \subset \mathring{M}_i$ and a homeomorphism $f : \partial C_1 \to \partial C_2$ (orientation-reversing in the oriented case).

(i) Let $\tilde{h}$ denote the involution of $M = M_2 \# M_1 \# M_2^\epsilon$ obtained by operation I-1 using h_1 and f. Consider a second homeomorphism $f' : \partial C_1 \to \partial C_2$ isotopic to f and let $\tilde{h}'$ denote the involution of $M' \approx M$ obtained by operation I-1 when f' is substituted for f. We want to define an equivalence between these two involutions. Choose a product neighborhood $U = \partial C_1 \times [0,1]$ for $\partial C_1 = \partial C_1 \times \{0\}$ in $(M_1 - \mathring{C}_1)$ which is disjoint from $h_1(U)$. Let $H_t : \partial C_1 \to \partial C_1$ be an isotopy from $H_0 = f^{-1}f'$ to $H_1 =$ identity. Define the homeomorphism $g : M' \to M$ by letting $g|(M' - (U \cup h_1(U))) =$ identity, $g|U(x,t) = (H_t(x),t) \in U$ and $g|h_1(U) = h_1 g h_1^{-1}|h_1(U)$ and observe that $\tilde{h} = g\tilde{h}'g^{-1}$. Hence isotopic sewing maps f and f' in operastion I-1 produce equivalent involutions.

Now consider homeomorphisms $k_i : M_i \to M_i$, for $i = 1, 2$, and compare the above involution $\tilde{h}$ of M with the involution $\widetilde{k_1 h_1 k_1^{-1}}$ of $M'' = M_2 \# M_1 \# M_2^\epsilon$ obtained by I-1 using $k_1 h_1 k_1^{-1}, k_2 f k_1^{-1}, k_1(C_1), k_2(C_2)$ in place of h_1, f, C_1, C_2, respectively. Define the homeomorphism $g : M'' \to M$ by letting $g(x) = k_1(x)$ for $x \in M_1$ and $g(x) = k_2(x)$ for $x \in M_2$. Then $g\tilde{h}g^{-1} = \widetilde{k_1 h_1 k_1^{-1}}$ and thus the equivalent triples $(h_1, 1; [f])$ and $(k_1 h_1 k_1^{-1}, 1; [k_2 f k_1^{-1}])$ determine equivalent involutions. Furthermore, we can also conclude that h is independent of the choice of the 3-cells C_1 and C_2. This is because a change of 3-cells from C_i to C_i' is accomplished by means of a homeomorphism $k_i : M_i \to M_i$ that commutes with h_i, carries C_i to C_i' and which defines a new sewing map $k_2 f k_1^{-1}$ (up to isotopy). This last observation completes the proof that $\tilde{h}$ depends only on the homeomorphism class of M_2 and the equivalence class $\{h_1, 1; [f]\}$.

(ii) In this case the 3-cells C_1 and C_2 are invariant under the given involutions and we have $fh_1 = h_2 f$. Let $h_1 \# h_2$ denote the involution obtained by operation I-2 using h_1, h_2, f, C_1 and C_2. Consider a second homeomorphism $f' : \partial C_1 \to \partial C_2$ such that $f'h_1 = h_2 f'$ and f' is isotopic to f as maps of pairs

$$(\partial C_1, \partial C_1 \cap \text{Fix}(h_1)) \to (\partial C_2, \partial C_2 \cap \text{Fix}(h_2)).$$

Denote by $(h_1 \# h_2)'$ the involution on $M' = M_1 \# M_2$ obtained by applying I-2 as before but with f' substituted for f. Find an invariant product neighborhood $U = \partial C_1 \times [0,1]$ of $\partial C_1 = \partial C_1 \times \{0\}$ in $(M_1 - \mathring{C}_1)$ such that the product structure has the property that $h_1|U(x,t) = (\alpha(x),t)$ for $\alpha = h_1|\partial C_1$ (see [6]). Since $f^{-1}f'$ is a homeomorphism of the 2-sphere ∂C_1 that commutes with α and is isotopic to the identity, it is easy to find an isotopy $H_t : \partial C_1 \to \partial C_2$ from $H_0 = f^{-1}f'$ to $H_1 =$ identity such that $\alpha H_t = H_t \alpha$ for each t (also see [1]). Define a homeomorphism $g : M' \to M$ by letting $g(x) = x$ for $x \notin U$ and $g(x,t) = (H_t(x),t)$ for $x \in U$. Observe that $(h_1 \# h_2)' = g^{-1}(h_1 \# h_2)g$. The remainder of this case is very similar to Case (i) and we leave the details to the reader.

(iii) This case is also similar to Case (i) and will be omitted.

(iv) Again, this case is very close to Case (ii) except for showing that the equivalent triples $(h_1, h_2; [f, f'])$, $(h_1, h_2; [rf, r'f']$ and $(h_1, h_2; [\rho f, \rho' f])$ give rise to equivalent involutions. Here $r : \partial C_2 \to \partial C_2$, $r' : \partial C'_2 \to \partial C'_2$ are orientation-reversing homeomorphisms constant on the fixed-point sets and $\rho : \partial C_2 \to \partial C_2$, $\rho' : \partial C'_2 \to \partial C'_2$ are isotopic to the identity but reverse the orientation on the fixed-point sets. We will treat only the case where the fixed-point set of h_2 is a circle as the other two cases are similar. We view S^3 as $\{(z_1,z_2) : |z_1|^2 + |z_2|^2 = 1\} \subset C^2$ and let $h_2(z_1,z_2) = (-z_1,z_2)$. We also use the involutions $g(z_1,z_2) = (\bar{z}_1,z_2)$ and $g'(z_1,z_2) = (\bar{z}_1,\bar{z}_2)$. Choose the 3-cells C_2 and C'_2 in S^3 to be regular neighborhoods of (0,1) and (0,−1), respectively, such that each is invariant under the involutions h_2, g and g'. Observe that $r = g|\partial C_2$ and $r' = g|\partial C'_2$ are orientation-reversing and constant on the fixed points of h_2. Also note that $\rho = g'|\partial C_2$ and $\rho' = g'|\partial C'_2$ are orientation-preserving and interchange each pair of fixed points. As in Case (ii), we can construct equivalences between the three involutions which arise by operation *I*-4 when we use the triples $(h_1,h_2;\ [f,f'])$, $(h_1,gh_2g^{-1};\ [gf,gf'])$ and $(h_1,g'h_2g'^{-1};\ [g'f,g'f'])$. However, since g and g' each commute with h_2 it follows that up to equivalence, a unique involution corresponds to the equivalence class $\{h_1,h'_2;\ [f,f']\}$. This completes the proof of Theorem 2.

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