A GEOMETRIC CONDITION WHICH IMPLIES BMOA

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1. INTRODUCTION

The space BMOA is the collection of analytic functions on the unit disc $D$ which are in the Hardy space $H^1$ and whose boundary values belong to the space BMO of John and Nirenberg [6].

Recently, Hayman and Pommerenke [5] discovered a geometric characterization of all regions $\Omega$ with the property that an analytic function with values in $\Omega$ will belong to BMOA. Their characterization uses logarithmic capacities.

At about the same time I independently discovered the sufficiency result along with several applications and generalizations to known results. These applications are given below along with the best norm result which involves a property of logarithmic capacity which may be of independent interest.

2. STATEMENT OF THE RESULTS

The geometric characterization given in [5] is that there exist an $r > 0$ and $\delta > 0$ such that $\text{Cap}(D(w, r) \setminus \Omega) \geq \delta$ for all $w$ in $\Omega$. Here $D(w, r)$ is the closed disc of radius $r$ centered at $w$ and "Cap" denotes the logarithmic capacity of a set.

For a region $\Omega$ ($\Omega$ is an open, connected subset of $\mathbb{C}$) let

$$\phi(r) = \inf_{w \in \Omega} \frac{\text{Cap}(D(w, r) \setminus \Omega)}{\text{Cap}(D(w, r))}$$

so that $0 \leq \phi \leq 1$. We could replace the denominator with $r$ since the capacity of a disc is its radius. If in the definition of $\phi$ we replace Cap with a measure, then the condition $\phi(r_0) = \delta > 0$ would not imply a stronger result for large $r$, i.e., the ratio could remain constant. Surprisingly, the situation with capacities is quite different.

THEOREM 1. For a region $\Omega$, $\lim_{r \to \infty} \phi(r) = 1$ provided that $\phi(r) \neq 0$ for some $r > 0$. In addition, there exists an $r > 0$ with $2^{-5} \leq \phi(r) \leq 2^{-1/5}$.

The next is a refinement of that given in [5].

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THEOREM 2. There are positive functions $c_1(t)$ and $c_2(t)$ defined on $[0, 1)$ satisfying the following conditions:

Let $\Omega$ be a region with $0 < a \leq \phi(r) \leq b < 1$ for some $r > 0$. Then

$$c_1(b)r \leq \sup ||f||_* \leq c_2(a)r$$

where the supremum is taken over all analytic functions on $D$ with values in $\Omega$ and $||\cdot||_\ast$ denotes the BMO norm.

Moreover, $c_2(a)$ is dominated by a constant multiple of $\log 2/a$ and $c_1(b)$ is bounded from below by a positive constant on the interval $(0, 1/2]$ but tends to zero as $b$ tends to one.

We remark that the above result restricted to the case where $b$ is small can be obtained by the methods in [5]. However, for small $b$ the constants $c_1, c_2$ are not comparable and Theorem 1 is needed to guarantee that $\phi(r)$ can be chosen sufficiently large so that these constants are comparable.

COROLLARY 1. Let $\Omega$ be a region with $\phi(r_0) > 0$ for some $r_0$. Then there exist an $r > 0$ with $2^{-5} \leq \phi(r) \leq 2^{-1/6}$ and the supremum of $||f||_\ast$ for $f$ with values in $\Omega$ will be comparable to $r$.

We now give some geometric conditions of a more elementary nature which imply BMOA. Let $m_w(t)$ denote the Lebesgue measure of the set of numbers $r$, $0 \leq r \leq t$, for which the circle $|z - w| = r$ is contained in $\Omega$.

COROLLARY 2. The condition $\sup_{w \in \Omega} \frac{m_w(r)}{r} = d < 1$ implies that $\phi(r) \geq 1/4 (1 - d)$.

Proof. The circular projection mapping $z$ into $|z|$ decreases distances and hence decreases capacity, see Pommerenke’s book [7, Theorem 11.3, p. 337]. Taking $w$ to be the origin we see that the circular projection of $D(w, r) \setminus \Omega$ is a set whose complement in the interval $[0, r]$ has measure $m_w(r)$. Since the capacity of linear set is at least one quarter of its length the result follows.

COROLLARY 3. If the image $f(D)$ of an analytic function $f$ does not contain circles centered in $f(D)$ of radius larger than $r$ then $f$ is in BMOA and $||f||_\ast \leq cr$ for some constant $c$ independent of $f$ and $r$.

COROLLARY 4. If the vertical cross-sectional measures of $f(D)$ are bounded by $d$ then $f$ is in BMOA and $||f||_\ast \leq cd$.

Proof. Take $\Omega = f(D)$ and $r = d$ then $D(w, r) \setminus \Omega$ contains a linear set of measure at least equal to $d$.

We remark that Corollary 3 is a generalization of Pommerenke’s result [8] that a univalent function $f$ is in BMOA if and only if $f(D)$ contains no discs of arbitrary large radii. Obviously, if a circle is contained in a simply connected region then the entire disc is also. Also, Corollary 4 is a generalization of Baernstein’s result [2] that a nonvanishing univalent function $f$ satisfies $\log f \in$ BMOA. In this case $\log f(D)$ has vertical cross-sectional measures bounded by $2\pi$. See also [9].
Finally, let $\Omega_w(r)$ be the component of $D(w, r) \cap \Omega$ containing $w$.

**COROLLARY 5.** If $\Omega$ is a region and $\sup_{w \in \Omega} \frac{\text{area}(\Omega_w(r))}{\pi r^2} = d^2 < 1$ for some $r > 0$ then $f(D) \subset \Omega$ implies that $f$ is in BMOA and $\|f\|_* \leq c r \log \frac{2}{1 - d}$.

**Proof.** A calculation shows that $\pi m_w(r)^2 \leq \text{area}(\Omega_w(r))$ and hence Corollary 2 and Theorem 2 imply the norm estimate.

A special case of Corollary 5 is the case that area $(f(D))$ is finite and the resulting norm inequality is that $\|f\|_* \leq c (\text{area } f(D))^{1/2}$. This problem was first considered in [1, Theorem 1] where it is shown that finite area implies $H^2$. Later, this result was improved in [3] to all $H^p$ for $p < \infty$. Since BMOA is contained in $H^p$ for all $p < \infty$, see [6], the above corollary generalizes these results.

3. PROOFS OF THE THEOREMS

The proofs of Theorem 1 and Theorem 2 are based on the following lemmas.

**LEMMA 1.** There is a positive constant $t_0$, satisfying

\[ (*) \quad \phi(2tr) \geq \exp \left( -\frac{1}{t - 2} \left[ \log \frac{t}{\phi(r)} + 4 \right] \right) \]

whenever $\phi(r) > 0$ and $t \geq t_0$.

**Proof.** Assume that $\phi(r/2) = \delta > 0$ then $\text{Cap}(D(w, r/2) \setminus \Omega) \geq \delta r/2$ for all $w$ in $\Omega$. It follows that $\text{Cap}(D(w, r) \setminus \Omega) \geq \delta r/2$ for all $w$ in $\mathbb{C}$. Fix an odd integer $n = 2k + 1$ and $R \geq (1 + n/3) r$. Put

\[ A_m = \left\{ z : R - 2r \leq |z| \leq R; \frac{2\pi}{n} \left( m - \frac{1}{2} \right) \leq \text{arg } z \leq \frac{2\pi}{n} \left( m + \frac{1}{2} \right) \right\} \]

for $m = 0, 1, \ldots, n - 1$. If $n$ is sufficiently large a computation shows that $A_m$ contains a disc of radius $r$ and hence there is a subset $E_m$ of $A_m \setminus \Omega$ with $\text{Cap}(E_m) \geq \delta r/2$. Then there exist a positive measure $\mu_m$ on $E_m$ with unit mass and such that the potential

\[ U^\mu_m(z) = \int \log \frac{1}{|z - \zeta|} d\mu_m(\zeta) \]

is bounded by $\log [2/\delta r]$ [4, p. 235].

If $\mu = \frac{1}{n} \sum_m \mu_m$ then the inequality $\sup_{z \in E} U^\mu(z) \leq c$ where $E = \bigcup_m E_m$ implies the lower bound $\text{Cap } E \geq e^{-c}$. Since $E \subset D(0, R) \setminus \Omega$ this will result in a lower bound estimate for $\phi(R)$.
Let \( z \in E_{m'} \), then for \( U_m = U^m \) we have

\[
U^m(z) = \frac{1}{n} \left[ U_{m'-1}(z) + U_{m'}(z) + U_{m'+1}(z) + \sum_{\text{rest}} U_m(z) \right]
\]

\[
\leq \frac{1}{n} \left[ 3 \log \frac{2}{\delta r} + \sum_{\xi \in F_m} \log \frac{1}{\|z - \xi\|} \right]
\]

\[
\leq \frac{1}{n} \left[ 3 \log \frac{2}{\delta r} + 2 \sum_{m=2}^{k} \frac{\log \left( \frac{1}{(R - 2r)e^{2\pi i(m-1)/n} - 1} \right)}{(R - 2r)e^{2\pi i(m-1)/n} - 1} \right]
\]

\[
\leq \frac{1}{n} \left[ 3 \log \frac{2}{\delta r} + 2n \int_0^{(k-1)/n} \log \frac{1}{(R - 2r)e^{2\pi it} - 1} \, dt \right]
\]

Since \( \frac{k-1}{n} = \frac{1}{2} - \frac{3}{2n} \) and \( \int_0^{1/2} \log |e^{2\pi it} - 1| \, dt = 0 \) we get after some simplification that

\[
\sup_{z \in E} U^m(z) \leq \frac{3}{n} \left[ \log \frac{2}{\delta} + \log \left( \frac{R}{r} - 2 \right) + \log 2 \right]
\]

\[
+ \log \frac{R}{R - 2r} - \log R.
\]

We now set \( R = tr \) and assume that \( (1 + n/3)r \leq R \leq (1 + n/3)r \). If \( t \geq t_0 \)

where \( t_0 \) is sufficiently large then the value of \( n \) will be large enough to apply the above argument. Since \( n/3 \geq t - 2 \) we deduce that

\[
\log \phi(tr) \geq -\frac{3}{n} \left[ \log \frac{t}{\phi(r/2)} + \log \left( 1 - \frac{2}{t} \right) + \log 4 \right] + \log \left( 1 - \frac{2}{t} \right)
\]

\[
> \frac{1}{t - 2} \left[ \log \frac{t}{\phi(r/2)} + \log 4 \right] - \frac{2}{t - 2}.
\]

Replacing \( r \) with \( 2r \) in the above yields (*)

**Lemma 2.** If \( \phi(r) \neq 0 \) for some \( r > 0 \) then there exists an \( R > 0 \) with \( 2^{-5} \leq \phi(R) \leq 2^{-1/5} \).

**Proof.** Since \( r\phi(r) \) is a nondecreasing function, \( \phi(r) \) has left and right limits everywhere. In fact, by the outer regularity of capacity we see that \( \phi \) is continuous from the right. By (*), the set \( \{ r : \phi(r) \geq 2^{-6} \} \) is nonempty. Let \( R \) be the infimum of this set so that \( R > 0 \), \( \phi(R - 0) \leq 2^{-5} \), and \( \phi(R) = \phi(R + 0) \geq 2^{-5} \).

Let \( \varepsilon > 0 \). Since the capacity of a semicircle of radius \( R \) is \( R/\sqrt{2} \), an open neighborhood will have capacity bounded by \( (1 + \varepsilon)R/\sqrt{2} \). Let \( w \in \Omega \) and \( R' < R \). If \( R' \) is sufficiently close to \( R \) then there exists \( w \in \Omega \) with
A GEOMETRIC CONDITION WHICH IMPLIES BMOA

\[ \text{Cap}(D(w,R) \setminus D(w', R')) < (1 + \varepsilon) R / \sqrt{2}. \]

Put \( E = D(w,R) \setminus \Omega \) and \( E' = D(w', R') \setminus \Omega \). Then \( E \) can be split into two sets \( E_1, E_2 \) where \( E_1 \subset E' \) and \( \text{Cap}(E_2) \leq (1 + \varepsilon) R / \sqrt{2} \). Now the subadditivity of capacity gives

\[
1 / \log \frac{2R}{\text{Cap} E} \leq 1 / \log \frac{2R}{\text{Cap} E_1} + 1 / \log \frac{2R}{\text{Cap} E_2} \\
\leq 1 / \log \frac{2R}{\text{Cap} E'} + 1 / \log \frac{2\sqrt{2}}{1 + \varepsilon}.
\]

By letting \( R' \) tend to \( R \) so that \( \text{Cap} E' \) tends to \( R\phi(R - 0) \) and by letting \( \varepsilon \) tend to zero we deduce that

\[
1 / \log \frac{2}{\phi(R)} \leq 1 / \log \frac{2}{\phi(R - 0)} + 1 / \log 2 \sqrt{2}
\]

Since \( \phi(R - 0) \leq 2^{-5} \) this implies that \( \phi(R) \leq 2^{-1/5} \). Thus, \( 2^{-5} \leq \phi(R) \leq 2^{-1/5} \). See [7, Chapter 11.1] for the regularity and subadditivity results used in the above.

The author is indebted to the referee for suggesting the above lemma.

**Proof of Theorem 1.** Clearly (*) implies \( \lim_{r \to \infty} \phi(r) = 1 \) and the remaining statement is Lemma 2.

**Proof of Theorem 2.** In [5] it is shown that:

1. \( \phi(r) \geq \delta > 0 \) implies \( ||f||_* \leq c(\delta) r \) whenever \( f \) takes values in \( \Omega \).

2. There exist \( 0 < \delta_0 < 1 \) such that \( \phi(r) \leq \delta_0 \) implies there exists a function \( f \) with values in \( \Omega \) and \( ||f||_* \geq c r \).

Actually, the upper estimate given in [5] is of the form \( c(\delta, r) \) but an easy dilation argument places it in the above form.

Since \( \lim_{r \to \infty} \phi(r) = 1 \) we can define \( r_0 = \inf \{ r : \phi(r) \geq \delta_0 \} \). Thus, there exists \( r_0 \leq r_1 \leq 2r_0 \) with \( \phi(r_1) \geq \delta_0 \) and hence (1) implies \( \sup ||f||_* \leq c(\delta_0) 2r_0 \). In addition, \( \phi(r_0 / 2) < \delta_0 \) so (2) implies \( \sup ||f||_* \geq c r_0 / 2 \). Thus, the best norm estimate is given by \( r_0 \).

By (1) \( c_2(t) \) can be taken to be constant on \( [\delta_0, 1) \). By (2) \( c_1(t) \) can be constant on \( (0, \delta_0) \).

Let \( \delta < \delta_0 \) and \( \phi(r) \geq \delta \). Assuming as we may that \( \delta_0 \) is small we use \( t = \log 2 / \delta \geq t_0 \in (\ast) \) to get \( \phi(2(\log 2 / \delta) r) \geq \delta_1 \) where \( \delta_1 \) is independent of \( \delta < \delta_0 \). Hence by (1) \( ||f||_* \leq c(\delta_1) 2(\log 2 / \delta) r \leq c(\log 2 / \delta) r \) whenever \( f \) takes values in \( \Omega \). It follows that \( c_2(\delta) \) can be chosen to be a constant multiple of \( \log 2 / \delta \) on \( (0, \delta_0) \) and hence also on \( (0, 1) \).
Finally, we must determine $c_1(\delta)$ for $\delta_0 < \delta < 1$. Let $\delta_0 < \delta < 1$ and $\phi(r) \leq \delta$. Now $r = 2tr_1$ for some $t$. From (*) and the fact that $\phi(r_1) \geq \delta_0$ we obtain an upper bound for $t$ in terms of $\delta$, say $8t \leq \psi(\delta)$. Since $\|f\|_* \geq cr_0/2$ for some $f$ with values in $\Omega$ and $2r_0 = r_1$ we obtain $\|f\|_* \geq c\psi(\delta)^{-1}r$. Thus, taking $c_1(\delta) = c\psi(\delta)^{-1}$ for $\delta_0 < \delta < 1$ we are done.

REFERENCES


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