ON MEROMORPHIC SOLUTIONS OF A LINEAR DIFFERENTIAL-DIFFERENCE EQUATION WITH CONSTANT COEFFICIENTS

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INTRODUCTION

A. O. Gelfond considered in his paper [4], among other questions, the entire solutions of the double system of equations (S) $L[F(z)] = 0$, $M[F(z)] = 0$, where

$$L[F(Z)] = \sum_{k=1}^{m} c_k F(z+\gamma_k), \qquad M[F(z)] = \sum_{k=1}^{n} d_k F(z+\delta_k)$$

are linear difference operators with constant coefficients c_k, d_k and steps γ_k, δ_k. If $L[F(z)] = F(z+\alpha) - F(z)$, $M[F(z)] = F(z+\beta) - F(z)$, $\operatorname{Im}(\alpha/\beta) \neq 0$, then the meromorphic solutions of the system (S) are the much studied elliptic functions. Gelfond's result on the entire solutions of the general system (S) is, for this particular case, identical with the well known theorem which states that there are no entire elliptic functions other than $f(z) \equiv$ constant.

A. I. Markushevich suggested in connection with Gelfond's paper the following question: What can be said about the meromorphic solutions of the double system (S)?

The meromorphic solutions of the system (S), as well as of other systems (not necessarily double systems) of difference and differential-difference equations, were afterwards studied by the authors of this paper, by L. Navickaite, R. Sandler, T. S. Silver, V. Tevelis, and L. Trushina. (see [7], [8], [14]-[34], [37], [38]). Earlier material on meromorphic solutions of difference equations may be found in the monograph [1] of P. Appell and E. Lacour and in the monograph [35] of E. Picard. This subject has also been studied in the papers [2]-[6], [9]-[13], [39] of F. Erwe, G. Floquet, M. Ghermanesco, A. Hurwitz, H. Löwig, P. Montel, and J. M. Whittaker.

In this paper we consider the meromorphic solutions $f(z)$ of the differential-difference equation

$$(1) \qquad A[f(z)] \equiv A_0[f(z)] + A_1[f'(z)] + A_2[f''(z)] + \dots + A_n[f^{(n)}(z)] = 0,$$

where

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$$A_i[f(z)] = \sum_{j=1}^{p_i} a_{ij} f(z + \alpha_{ij}) \qquad (i = 0, 1, 2, \ldots, n) \tag{2}$$

is a linear difference operator with constant complex coefficients a_{ij} and complex steps α_{ij}. The operator A_i is said to be *associated* to A and the steps α_{ij} of A_i are said to have *index i.* Some of the operators A_i may vanish identically ($A_i \equiv 0$), but $A_n \not\equiv 0$. This operator A_n and its steps α_{ij} are the operator and steps of *highest index.* Similarly, if A_m is the first operator A_i which does not vanish identically, we call A_m and its steps α_{mj} the operator and steps of *lowest index.* The number $n - m$ will be called the *order* of both the equation (1) and operator A. We assume also that none of the coefficients a_{ij} of the operator A_i vanishes and that no two of its steps are equal ($\alpha_{is} \neq \alpha_{it}$, if $s \neq t$), if $A_i \not\equiv 0$.

If $f(z)$ is an entire function and $Df(z) = f'(z)$, then

$$f(z + \alpha) = f(z) + \frac{\alpha D}{1!} f(z) + \frac{\alpha^2 D^2}{2!} f(z) + \ldots = \exp(\alpha D) f(z)$$

for every z and α. Thus for entire $f(z)$ the equation (1) is equivalent to the differential equation

$$A(D) f(z) = 0, \tag{3}$$

where

$$A(D) = A_0(D) + DA_1(1D) + D^2 A_2(D) + \ldots + D^n A_n(D) \tag{4}$$

and

$$A_i(D) = \sum_{j=1}^{p_i} a_{ij} \exp(\alpha_{ij} D) \quad (i = 0, 1, 2, \ldots, n). \tag{5}$$

The differential equation (3) is generally of infinite order and its coefficients are constants. Its *characteristic function*

$$A(t) = A_0(t) + tA_1(t) + \ldots + t^n A_n(t), \quad A_i(t) = \sum_{j=1}^{p_i} a_{ij} \exp(\alpha_{ij} t), \tag{6}$$

(to get this function substitute t for D in $A(D)$) is an entire function of exponential type, i.e., there exist two positive constants B and C such that $|A(t)| \leq B \exp(C|t|)$ for every complex t.

In the following description of the entire solutions of the differential equation (3), the operator $A(D)$ is not necessarily of the form (4). It may be any linear differential operator with constant coefficients of finite or infinite order, whose characteristic function $A(t)$ is an entire function of exponential type.

Let λ be a zero of order k of the function $A(t)$ and $P(z)$ an arbitrary algebraic polynomial of degree not more than $k - 1$. Then the function $P(z)\exp(\lambda z)$ is an entire solution of the equation (3). Such a solution is called a principal solution. Obviously, every finite sum of principal solutions of the equation (3) is also a solution of this equation. Let $S_1(z)$, $S_2(z)$, $S_3(z)$, ... be a sequence of such solutions of (3) (every $S_k(z)$ is a finite sum of principal solutions) and let $S_n(z) \to f(z)$ uniformly on every bounded set of the complex plane. Then it is easy to see that $f(z)$ is also an entire solution of (3). That, conversely, every entire solution $f(z)$ of (3) can be obtained in the above described way has been proved by A. O. Gelfond [4] (see also [3], chapter 5).

Let us return to the differential-difference equation (1) and let us study its *properly meromorphic* solutions. We call a function $f(z)$ properly meromorphic, if it is meromorphic in the finite complex plane and has there at least one pole.

Consider first the pure differential equation

$$a_0 f(z) + a_1 f'(z) + \dots + a_n f^{(n)}(z) = 0 \tag{7}$$

with constant coefficients a_k and the pure difference equation

$$B[f(z)] \equiv \sum_{i=1}^{p} b_i f(z + \beta_i) = 0, \qquad p \geq 2, \tag{8}$$

with constant coefficients b_i and steps β_i such that $b_i \neq 0$ and $\beta_i \neq \beta_j$ if $i \neq j$ (these restrictions guarantee that $B[f(z)]$ contains more than one term). Both these equations are particular cases of (1).

Every solution of (7) is a finite sum of principal solutions of this equation and is therefore an entire function. Thus the equation (7) has no properly meromorphic solutions. On the other hand, there exist, and with a large degree of arbitrariness, properly meromorphic solutions of the equation (8) (see Theorem 1.2).

This observation shows that so far as the properly meromorphic solutions are concerned there are two types of differential-difference equations (1): there are *D-equations* which, like the differential equation (7), have no properly meromorphic solutions, and there are *Δ-equations,* which like the difference equation (8), have properly meromorphic solutions.

One may ask how to determine the type of the equation (1). Let us agree to say that the equation (1) (the operator A) is a *P-equation* (*P-operator*), if and only if all its steps are real and it has an extremal step of highest index, i.e., either the maximal or the minimal of all steps α_{ij} is a step of highest index. Note that the extremal step, being necessarily a step of highest index n, may also be simultaneously a step of index less than n.

The main result of this paper may be expressed approximately as follows:

The P-equation $A[f(z)] = 0$ is a D-equation, except in case the differential-difference operator A contains a difference operator as a factor, i.e., when

$$A[f(z)] = A^*[B[f(z)]],$$

where $B[f(z)]$ *is a difference operator containing more than one term* (see (8)) *and* A^* *is a differential-difference operator.*

In the exceptional case every solution, and in particular every properly meromorphic solution, of the equation $B[f(z)] = 0$ is also a solution of the equation (1). Thus (1) is obviously in this case a Δ-equation. A less obvious class K of Δ-equations has been studied by A. Naftalevich in [21]. The full description of this class will be presented in the next section. Here we confine ourselves to describing the equations with real steps of the class K. We call these equations *Q-equations* and define them as follows:

The equation (1) (operator A) is a Q-equation (Q-operator), if it contains more than one step and both its minimal and maximal steps are steps of lowest index, but none of them is a step of any higher index. Thus, if (1) is a Q-equation, and α_{mj} are its steps of lowest index, and $\beta = \min_j \alpha_{mj}$, $\gamma = \max_j \alpha_{mj}$, then $\beta < \gamma$ and either $m = n$ or $\beta < \alpha_{ij} < \gamma$ for every $m < i \leq n$ and every j. Note that every equation (1) with real steps and of order one is either a P-equation or a Q-equation. Every equation with real steps and of order zero is simultaneously a P and Q-equation, if it contains more than one step.

Suppose A is a Q-operator and A^* an arbitrary differential-difference operator with real steps. Then, obviously, the equation $AA^*[f(z)] = 0$ is a Δ-equation. But it remains an open question whether there are any Δ-equations not of this type.

The type of equation (1) has been discussed above only in the case when all steps of the operator A are real. In what follows we will treat also the case in which the steps are arbitrary complex numbers.

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1. THE SPACES $\mathscr{S}(A)$ AND $\mathscr{S}(E;A)$

Denote by $P_1, P_2, \ldots, P_s$ arbitrary linear difference or differential-difference operators and let $\mathscr{S}(P_1, P_2, \ldots, P_s)$ be the set of all meromorphic functions (here and hereafter we mean meromorphic in the entire complex plane) $f(z)$ such that $P_k[f(z)]$ is an entire function for every $k = 1, 2, \ldots, s$. If $f(z)$ is a meromorphic solution of the equation (1), then obviously $f(z) \in \mathscr{S}(A)$. Conversely if $\varphi(z) \in \mathscr{S}(A)$, then there exists an entire function $g(z)$ such that $\varphi(z) + g(z)$ is a solution of (1). Indeed, put $A[\varphi(z)] = h(z)$. Since $h(z)$ is an entire function there exists an entire solution $g(z)$ of the equation $A[g(z)] = -h(z)$ [5]. The function $\varphi(z) + g(z)$ is therefore a meromorphic solution of (1). Thus (1) is a D-equation, if and only if $\mathscr{S}(A)$ contains no properly meromorphic function ($\mathscr{S}(A)$ contains obviously every entire function), and is a Δ-equation, if and only if $\mathscr{S}(A)$ contains properly meromorphic functions. *The question of the type of equation* (1) *is thus reduced to this one: does* $\mathscr{S}(A)$ *contain properly meromorphic functions?*

It will be convenient to use for the operators A and A_i the expressions (4) and (5) also when operating on not necessarily entire functions. Note that the product P_1P_2 of two differential-difference operators P_1 and P_2 reduces then to the formal algebraic product $P_1(D) \cdot P_2(D)$. There is also an obvious one-to-one correspondence between the operators $A = A(D)$ and their characteristic functions $A(t)$.

Besides the characteristic function $A(t)$ of the operator A we introduce also the *characteristic system* $A_0(t), A_1(t), A_2(t), \ldots, A_n(t)$ of this operator. Note that each $A_i(t)$ is an exponential polynomial and is the characteristic function of the difference operator A_i.

The study of the space $\mathscr{S}(A)$ which follows depends on the existence and uniqueness of the GCD (*greatest common divisor*) of a given system of exponential polynomials. We call $Q(t)$ a *divisor* of the exponential polynomial

$$P(t) = \sum_{k=1}^{m} p_k \exp(\pi_k t)$$

(p_k and π_k are complex numbers), if $P(t) = Q(t)R(t)$, where $Q(t)$ and $R(t)$ are both exponential polynomials. An arbitrary monomial $C\exp(\sigma t)$ is obviously a divisor of every exponential polynomial. We call such a monomial a *trivial divisor*. Thus a *nontrivial divisor* contains at least two terms. Note also that if $Q(t)$ is a divisor of $P(t)$, then $C\exp(\gamma t)Q(t)$ is also a divisor of $P(t)$ for arbitrary complex numbers C and γ. The exponential polynomial $Q(t)$ is said to be the GCD of the system $A_1(t), A_2(t), \ldots, A_n(t)$, if $Q(t)$ is a divisor of each of the exponential polynomials $A_i(t)$ and the system $A_1/Q, A_2/Q, \ldots, A_n/Q$ has no common nontrivial divisor.

J. F. Ritt's [36] factorization theory of exponential polynomials easily yields the following:

For any finite system of exponential polynomials $A_1(t), A_2(t), \ldots, A_n(t)$ *there exists a* GCD. *The* GCD *is determined uniquely up to a factor* $C\exp(\gamma t)$. *If the exponents of each* $A_i(t)$ *(the numbers* α_{ij} *in* (6)) *are all real, then the* GCD *of this system may be chosen with all its exponents real.*

We are now able to formulate the main theorem of this paper.

THEOREM 1.1. *Suppose A is the operator* (4), $A_i(t)$ $(i = 0,1,2,\ldots,n)$ *its characteristic system,* $B(t)$ *the* GCD *of this system, and* $B = B(D)$ *the difference operator corresponding to* $B(t)$. *If A is a P-operator, then*

$$\mathscr{S}(A) = \mathscr{S}(B) = \mathscr{S}(A_0, A_1, \ldots, A_n).$$

This theorem is proved in section 3. Here we offer some comment on it.

If the characteristic system $\{A_i(t)\}$ has only trivial common divisors (which is usually the case), we may assume $B(t) \equiv 1$, and B is then the identity operator: $B[f(z)] \equiv f(z)$. Thus in this case $\mathscr{S}(A)$ contains (in accordance with the result on P-equations announced in the introduction) no properly meromorphic function.

Consider now the case when the GCD, $B(t)$, contains more than one term. We may assume that all exponents of $B(t)$ are real. Thus the steps of the corresponding operator B are all real. Nevertheless, we will describe the space

$\mathscr{S}(B)$ in the more general case when B is the difference operator (8) with arbitrary complex steps β_i, $(i = 1,2,\ldots,p)$.

We will find it convenient to speak about a *strip parallel to a line,* about *parallel strips* and about the *angle between a strip and a line* or *between two strips.* For example, a strip S will be said to be parallel to the line L, if the boundary lines of S are parallel to L.

Fix now in the complex plane an arbitrary line L and an arbitrary point z_0 and consider the set $V = \{z_0 + \beta_i, i = 1,2,\ldots,p, p \geq 2\}$. Let $\bar{\pi}(z_0,L)$ be the smallest closed strip which contains the whole set V and is parallel to the line L. Each of the boundary lines L_1 and L_2 of $\bar{\pi}(z_0,L)$ contains at least one point of the set V. Suppose that L_1 as well as L_2 contains only one point of V and remove from $\bar{\pi}(z_0,L)$ one of its boundary lines (either L_1 or L_2). We denote the remaining half closed strip $\Pi = \Pi\,(z_0,L)$ and call it a *fundamental strip* of the operator B.

It is easy to see that there exist two fundamental strips $\Pi\,(z_0,L)$ for every line L which is not parallel to any of the lines joining two points of the set V. Thus there is only a finite number of exceptional lines L passing through a fixed point for which no fundamental strip $\Pi\,(z_0,L)$ exists, and for every line L, including the exceptional ones, there exists a fundamental strip of B which makes an arbitrarily small angle with L.

Note that in the case when all steps of the operator B are real and arranged such that $\beta_1 < \beta_2 < \ldots < \beta_p$ and L is the imaginary axis, then we may take for $\Pi\,(z_0,L)$ the strip

$$\Pi\,(z_0,L) = \{z\colon \operatorname{Re}(z_0 + \beta_1) \leq \operatorname{Re} z < \operatorname{Re}(z_0 + \beta_p)\}.$$

It will be convenient to have a fundamental strip for B also in the case when B contains only one term. In this case we identify every fundamental strip of B with the empty set.

THEOREM 1.2 [17]. *Let B be the difference operator* (8), Π *a fundamental strip of B and*

$$G(z,\lambda_i) = \sum_{k=1}^{m_i} (C_{ik}/(z-\lambda_i)^k), \quad \lambda_i \in \Pi,$$

an arbitrary sequence of rational functions whose poles are restricted only by the condition $\lambda_i \to \infty$ as $i \to \infty$. Then there exists a function $f(z) \in \mathscr{S}(B)$, whose principal part at each λ_i $(i = 1,2,3,\ldots)$ is $G(z,\lambda_i)$ and which has no other poles in Π. If $f_1(z) \in \mathscr{S}(B)$ and $f_1(z) - f(z)$ is regular in Π, then $f_1(z) - f(z)$ is an entire function.

Theorem 1.2 may be extended to some differential-difference operators.

Let $A = A(D)$ be the operator (4) and A_m its associated difference operator of lowest index. *If A_m contains only one term* (if $p_m = 1$), *then $\mathscr{S}(A)$ contains no properly meromorphic function* [21]. Let A_m contain at least two terms and let Π be a fundamental strip of A_m. Π is also said to be then a fundamental strip of A. The operator A is called an *R-operator,* if it has a fundamental strip Π such that each step α_{ij} of A with index $i > m$ is an inner point of Π. Such a fundamental strip Π we will call an *R-fundamental strip.* Note that in the case of real steps an R-operator is identical to a Q-operator.

THEOREM 1.3 [21]. *Theorem* 1.2 *remains valid if B is an arbitrary differential-difference R-operator and Π is an R-fundamental strip of B.*

We continue to study the operator A but do not restrict it to be an R-operator.

Let $\Pi = \Pi(z_0,L)$ be a fundamental strip of A, $W = W(0)$ the set of all steps α_{ij} of A, and $W(z_0) = \{z: z = z_0 + \alpha_{ij},\ \alpha_{ij} \in W\}$. Denote $e = e(z_0,L)$ the smallest closed strip which contains both Π and $W(z_0)$. Each of the boundary lines of e contains at least one point of $W(z_0)$. We call the operator A a *P^*-operator* (with respect to Π), if at least one of the boundary lines of e, say the line 1, contains only one point of the set $W(z_0)$ and this single point is $z_0 + \alpha_{nj}$, where α_{nj} is a step of highest index (α_{nj} may simultaneously be a step of some index lower than n). In this special case we remove from the set e the line 1, denote the remaining half-closed strip $E = E(z_0,L)$, and call it an *extended fundamental strip* of A.

Note that a P^*-operator with real steps is identical to a P-operator. It is thus natural to ask whether theorem 1.1 holds in case A is a P^*-operator. The following example offers some answer to this question.

Suppose α and β are two complex numbers such that $\mathrm{Im}(\alpha/\beta) \neq 0$ and $A = A_0 + DA_1$, where $A_0[f(z)] = f(z+\alpha) - f(z)$ and $A_1[f(z)] = f(z+\beta) - f(z)$. It is easy to see that: (1) Every elliptic function with periods α and β is a properly meromorphic function belonging to $\mathscr{S}(A)$; (2) A is a P^*-operator (with respect to every fundamental strip of A which is not parallel to the line $z = \beta t$, $-\infty < t < \infty$); and (3) the GCD of the characteristic system $\{A_0(t) = \exp(\alpha t) - 1,\ A_1(t) = \exp(\beta t) - 1\}$ is $B(t) \equiv 1$ (see [36]). Thus B is the identity operator and $\mathscr{S}(A) \neq \mathscr{S}(B)$. Furthermore, there is also $\mathscr{S}(A) \neq \mathscr{S}(A_0,A_1)$. To see this consider any pair of meromorphic functions $f_1(z)$ and $f_2(z)$ which have the following poles and principal parts:

(1) The poles of $f_1(z)$ are at the points $m\alpha + n\beta$ ($m = 0, \pm 1, \pm 2, \ldots$; $n = \pm 1, \pm 2, \pm 3, \ldots$; note that $n \neq 0$) and the corresponding principal parts are $n/(z - m\alpha - n\beta)$.

(2) The poles of $f_2(z)$ are at the points $m\alpha + n\beta$ ($m = \pm 1, \pm 2, \pm 3, \ldots$; $n = 0, \pm 1, \pm 2, \ldots$; note that $m \neq 0$) and the corresponding principal parts are

$$m/(z - m\alpha - n\beta)^2.$$

A direct verification shows that: (1) $f_1(z) \in \mathscr{S}(A_0)$; (2) $f_2(z) \in \mathscr{S}(A_1)$; and (3) $A_1[f_1(z)](A_0[f_2(z)])$ is a meromorphic function which has the principal part $1/(z - m\alpha - n\beta)(1/(z - m\alpha - n\beta)^2)$ at each point $m\alpha + n\beta$ ($m,n = 0, \pm 1, \pm 2, \ldots$) and has no other poles (in other words

$$A_1[f_1(z)] = \zeta(z) + g_1(z), \quad A_0[f_2(z)] = \rho(z) + g_2(z),$$

where $g_1(z)$ and $g_2(z)$ are entire functions and $\zeta(z)$ and $\rho(z)$ are the elliptic Weierstrass functions). It follows now easily that $f(z) = f_1(z) + f_2(z) \in \mathscr{S}(A)$, but $f(z) \notin \mathscr{S}(A_0,A_1)$.

This example shows that none of the statements of Theorem 1.1 holds for some P^*-operators. Nevertheless, Theorem 1.1 will be easily deduced (in section 3) from the theorem on P^*-operators stated a few lines below.

Let Ω be an arbitrary set in the complex plane and $P_1, P_2, \ldots, P_n$ arbitrary linear difference or differential-difference operators. Denote the set $\mathscr{S}(P_1, P_2, \ldots, P_n) \cap \mathscr{G}(\Omega)$, where $\mathscr{G}(\Omega)$ is the space of all meromorphic functions which have in Ω at most a finite number of poles, by $\mathscr{S}(\Omega; P_1, P_2, \ldots, P_n)$.

THEOREM 1.4. *Suppose A is the operator* (4), $\{A_i(t), (i = 0, 1, 2, \ldots, n)\}$ *its characteristic system, $B(t)$ the* GCD *of this system, and $B = B(D)$ the difference operator corresponding to $B(t)$. If A is a P^*-operator and E is an extended fundamental strip of A, then*

$$\mathscr{S}(E; A) = \mathscr{S}(E; B) = \mathscr{S}(E; A_0, A_1, A_2, \ldots, A_n).$$

The proof of this theorem will be given in Section 2.

We will call the operator A a *reduced* operator if $A_0 \not\equiv 0$. If $A_0 \equiv A_1 \equiv \ldots \equiv A_{m-1} \equiv 0$, but $A_m \not\equiv 0$, then $A = D^m A^*$, where A^* is a reduced differential-difference operator. It is easy to see that $\mathscr{S}(A) = \mathscr{S}(A^*)$. Without loss of generality we will assume in the rest of this paper that A is a reduced operator.

Consider now the rational function

$$G(z, \lambda) = \sum_{k=1}^{n} \frac{a_k}{(z - \lambda)^k} \tag{1.1}$$

and an arbitrary meromorphic function $f(z)$, and introduce their i-components (or components of order i) $G_i(z, \lambda)$ and $f_i(z)$. The i-component $G_i(z, \lambda)$ is the i-th term $a_i/(z - \lambda)^i$ of the sum (1.1) ($G_i(z, \lambda) = a_i/(z - \lambda)^i$), if $i \le n$, and $G_i(z, \lambda) = 0$, if $i > n$. Note that $G_i(z, \lambda)$ may vanish also for $i \le n$. To define the i-component $f_i(z)$ let $\{\lambda_k\}$ be the set of poles of $f(z)$, $G(z, \lambda_k)$ the principal part of $f(z)$ at the pole λ_k, and $G_i(z, \lambda_k)$ the i-component of $G(z, \lambda_k)$. For every natural number i construct the set Ω_i of all poles λ_k for which $G_i(z, \lambda_k) \not\equiv 0$. The i-component $f_i(z)$ is defined only up to an arbitrary additive entire function and may be taken to be any meromorphic function whose principal part at each $\lambda_k \in \Omega_i$ is $G_i(z, \lambda_k)$ and which has no poles outside Ω_i. Note that the set Ω_i may be empty and $f_i(z)$ is then an arbitrary entire function.

Consider now the i-components $F_i(z)$ of the function $F(z) = A[f(z)]$. It is easy to see that

$$\begin{aligned} F_1(z) &= A_0[f_1(z)], F_2(z) = A_0[f_2(z)] + DA_1[f_1(z)], \\ F_3(z) &= A_0[f_3(z)] + DA_1[f_2(z)] + D^2 A_2[f_1(z)], \ldots. \end{aligned} \tag{1.2}$$

LEMMA 1.1. *If $f(z) \in \mathscr{S}(A)$, then the i-component $f_i(z)$ of $f(z)$ belongs to $\mathscr{S}(A_0^i)$ for every $i = 1, 2, 3, \ldots$.*

Proof. The condition $f(z) \in \mathscr{S}(A)$ means that $F(z) = A[f(z)]$ is an entire function. Each component $F_i(z)$ of $F(z)$ is also an entire function and the first equation of (1.2) shows that $f_1(z) \in \mathscr{S}(A_0)$. To get $f_2(z) \in \mathscr{S}(A_0^2)$ operate with A_0 on both sides of the second equation in (1.2) and note that $A_0[F_2]$ and $DA_1[A_0[f_1(z)]]$ are both entire functions. Operating with A_0^{n-1} on the n-th equation of (1.2) and using induction we easily get $f_n(z) \in \mathscr{S}(A_0^n)$.

Suppose that $\varphi(z)$ is a meromorphic function and B is the operator (8). Denote by $\mathscr{S}(B;\varphi(z))$ the space of all meromorphic functions $f(z)$ such that $B[f(z)] - \varphi(z)$ is an entire function. Note that $\mathscr{S}(B;\varphi(z)) \equiv \mathscr{S}(B)$ if $\varphi(z) \equiv 0$. In [17] it has been proved that theorem 1.2 remains true if the space $\mathscr{S}(B)$ is replaced by $\mathscr{S}(B;\varphi(z))$.

Consider an arbitrary meromorphic function $F(z)$ and associate to it two functions $p(z;F)$ and $P(\Omega;F)$, where

$$p(z;F) = \begin{cases} 0 \text{ if } F(\zeta) \text{ is regular at the point } z, \\ k \text{ if } F(\zeta) \text{ has a pole of order } k \text{ at } z, \end{cases}$$

and $P(\Omega;F)$ is defined for every set Ω of the complex plane by the formula

$$P(\Omega;F) = \sup p(z;F),\ z \in \Omega.$$

LEMMA 1.2. *Let* $\mathbf{C}$ *denote the complex plane,* B *a difference operator,* Π *a fundamental strip of* B*, and* $\varphi(z)$ *a meromorphic function. If* $f(z) \in \mathscr{S}(B;\varphi(z))$*, then*

$$P(\mathbf{C};f) = \max\,[P(\Pi;f), P(\mathbf{C};\varphi)]. \tag{1.3}$$

COROLLARY 1.1. *If* $f(z) \in \mathscr{S}(B)$*, then* $P(\mathbf{C};f) = P(\Pi{:}F)$*. In particular,* $f(z)$ *is an entire function if* $f(z) \in \mathscr{S}(B)$ *and* f *is regular in* Π.

Proof. If B has only one step $(B = b\exp(\beta D))$ and $f(z) \in \mathscr{S}(B;\varphi(z))$, then $f(z) = (1/b)\varphi(z-\beta) + g(z)$, where $g(z)$ is an arbitrary entire function. In this case Π is the empty set and obviously $P(\mathbf{C};f) = P(\mathbf{C};\varphi)$. Thus (1.3) holds.

Suppose now that B is the operator (8) and $f(z) \in \mathscr{S}(B;\varphi(z))$. Then $B[f(z)] = \varphi(z) + g(z)$, where $g(z)$ is an entire function. Substitute in this equation $z - \beta_p$ for z and get

$$f(z) = \frac{1}{b_p}[\varphi(z-\beta_p) + g(z-\beta_p)] - \frac{1}{b_p}\sum_{i=1}^{p-1} b_i f(z+\beta_i-\beta_p). \tag{1.4}$$

Without loss of generality we may assume that the fundamental strip Π is parallel to the imaginary axis Y and that the steps β_i of B are arranged such that $\operatorname{Re}\beta_1 < \operatorname{Re}\beta_2 \le \operatorname{Re}\beta_3 \le \dots \le \operatorname{Re}\beta_{p-1} < \operatorname{Re}\beta_p$. Let

$$\Pi = \Pi(z_0, Y) = \{z: \operatorname{Re}(z_0+\beta_1) \le \operatorname{Re} z < \operatorname{Re}(z_0+\beta_p)\}$$

and $\delta = \operatorname{Re}(\beta_p - \beta_{p-1}) > 0$. Apply (1.4) to a point z of the strip

$$Q_1 = \{z: \operatorname{Re}(z_0+\beta_p) \le \operatorname{Re} z < \operatorname{Re}(z_0+\beta_p+\delta)\}.$$

For such a z all points $z+\beta_i-\beta_p$ $(i=1,2,\dots,p-1)$ belong to Π and therefore the right side of (1.4) is either regular at this point z or has a pole whose order is at most $K=\max(P(\mathbf{C};\varphi),P(\Pi,f))$. Thus we proved that $P(\Pi_1;f)\le K$, where $\Pi_1=\Pi\cup Q_1=\{z:\operatorname{Re}(z_0+\beta_1)\le\operatorname{Re}z<\operatorname{Re}(z_0+\beta_p+\delta)\}$. We prove by induction that $P(\Pi_m;f)\le K$, where $\Pi_m=\{z:\operatorname{Re}(z_0+\beta_1)\le\operatorname{Re}z<\operatorname{Re}(z+\beta_p+m\delta)\}$. Consequently $P(H_1;f)\le K$, where $H_1=\{z:\operatorname{Re}(z_0+\beta_1)\le\operatorname{Re}z<\infty\}$: Similarly we prove $P(H_2;f)\le K$, where $H_2=\{z:-\infty<\operatorname{Re}z<\operatorname{Re}(z_0+\beta_1)\}$, and (1.3) follows immediately.

Fix a real number α and two arbitrary points z_1,z_2 on the line

$$L=\{z: z=z_0+t\exp(i\alpha),\ -\infty<t<+\infty\}.$$

Take an arbitrary positive number γ, $0<\gamma<(\pi/2)$ and denote by $U=U(L,z_1,z_2,\gamma)$ the union of the two angular sets

$$U_1=U_1(L,z_1,\gamma)=\{z: z=z_1+t\exp(i\varphi),\ 0<t<\infty,\ \alpha-\gamma<\varphi<\alpha+\gamma\}$$

and

$$U_2=U_2(L,z_2,\gamma)=\{z: z=z_2-t\exp(i\varphi),\ 0<t<\infty,\ \alpha-\gamma<\varphi<\alpha+\gamma\}.$$

We agree to say that U is an *asymptotic neighborhood of* L and that the meromorphic function $\varphi(z)$ is *L-regular* if φ is regular in some asymptotic neighborhood of L. Note that there may be points of L not covered by U but the set of such points is bounded. If $\varphi(z)$ is L-regular, then it is also L_1-regular for every line L_1 which is either parallel to L or makes a small enough angle with L.

LEMMA 1.3. *Suppose L is a line of the complex plane, Π is a fundamental strip of the difference operator B parallel to L, and $\varphi(z)$ is an L-regular function. If $f(z)\in\mathscr{S}(B;\varphi(z))\cap\mathscr{G}(\Pi)$, then $f(z)$ is also L-regular.*

Proof. The lemma is evident if B contains only one step.

If B contains more than one step, we let its steps β_k, the strips Π, Q_1, Π_k and the halfplanes H_1,H_2 be exactly as in the proof of the previous lemma. The line L we assume to be the axis of symmetry of Π and take on it the vertices z_1,z_2 of the angular sets $U_1=U_1(L,z_1,\gamma)$ and $U_2=U_2(L,z_2,\gamma)$ which are parts of the asymptotic neighborhood $U=U_1\cup U_2$ of the line L. The points z_1,z_2, and the number γ are required to meet the following conditions: (1) $\varphi(z)$ is regular in U; (2) $f(z)$ is regular in $U\cap\Pi$; and (3) the asymptotic neighborhood $U(Y,0,0,\gamma)$ of the imaginary axis Y contains neither the points $\beta_p-\beta_i$ $(i=1,2,\dots,p-1)$ nor $\beta_1-\beta_j$ $(j=2,3,\dots p)$.

In the angle $U_1(U_2)$ fix on L a point $z_1^1(z_2^1)$ such that the distance of $z_1^1(z_2^1)$ to the boundary of $U_1(U_2)$ is more than $d=\max(|\beta_p|,|\beta_1|)$, and apply (1.4) to a point z of the set $Q_1\cap U^1$, where $U^1=U(L,z_1^1,z_2^1,\gamma)$. Each point $z+\beta_i-\beta_p$ $(i=1,2,\dots,p-1)$ belongs then to $U^1\cap\Pi$ and $z-\beta_p$ to U. Thus the right side, and therefore also the left side, of (1.4) is regular in $U^1\cap\Pi_1$. By induction we prove that $f(z)$ is regular in $U^1\cap\Pi_k$ $(k=1,2,3,\dots)$ and we conclude that $f(z)$ is regular in $U^1\cap H_1$. In the same way we find that $f(z)$ is regular also in $U^1\cap H_2$. Thus $f(z)$ is L-regular.

LEMMA 1.4. *Let* $\mathbf{C}$ *be the complex plane,* Π *a fundamental strip of* A, *and* $f(z) \in \mathscr{S}(\Pi; A)$. *If* L *is a line parallel to* Π *and* $P(\mathbf{C}; f) < \infty$, *then* $f(z)$ *is* L*-regular.*

Proof. Let $P(\mathbf{C}; f) = k < \infty$. Then

$$f(z) = f_1(z) + f_2(z) + \dots + f_k(z) + g(z), \tag{1.5}$$

where $f_i(z)$ $(i = 1, 2, \dots k)$ are the i-components of $f(z)$ and $g(z)$ is an entire function. Since $f(z) \subset \mathscr{S}(\Pi; A) \subset \mathscr{S}(A)$, the functions $F_i(z)$ of (1.2) are entire functions, and $f_1(z)$ as well as every other i-component of $f(z)$ has only a finite number of poles in Π. Thus $f_1(z) \in \mathscr{S}(A_0) \cap \mathscr{G}(\Pi)$ and Lemma 1.3 shows that $f_1(z)$ is L-regular. From (1.2) it follows that $f_2(z) \in \mathscr{S}(A_0; \varphi(z))$, where

$$\varphi(z) = -DA_1[f_1(z)],$$

which is L-regular (since $f_1(z)$ is L-regular). Thus $f_2(z)$ is also L-regular by Lemma 1.3. We prove by induction that every i-component $f_i(z)$ of $f(z)$ is L-regular and conclude from (1.5) that $f(z)$ is also L-regular.

2. OPERATORS WITH COMPLEX STEPS

LEMMA 2.1. *Suppose* Π *is a fundamental strip of* A, H *a half plane parallel to* Π, *and* $\mathbf{C}$ *the complex plane. If* $f(z) \in \mathscr{S}(A)$, *then* $P(\mathbf{C}; f) = P(H; f)$.

Proof. Let A be the operator (4) and A_0 the difference operator of smallest index associated to A.

If A_0 contains only one step, then $\mathscr{S}(A)$ contains only entire functions and $P(\mathbf{C}; f) = P(H; f) = 0$ for every $f(z) \in \mathscr{S}(A)$.

Consider the case when A_0 contains more than one step. If $P(H; f) = \infty$, then obviously $P(\mathbf{C}; f) = \infty$. Suppose $f(z) \in \mathscr{S}(A)$ and $P(H; f) = k < \infty$. Fix $i > k$ and consider the i-component $f_i(z)$ of $f(z)$. It is regular on H and $f_i \in \mathscr{S}(A_0^i)$ as Lemma 1.1 shows. Note also that the operator A_0^i has a fundamental strip parallel to the fundamental strip Π_0 of A_0. (To prove the last statement assume that Π is parallel to the imaginary axis and write A_0 in the form (5)). It follows now from Corollary 1.1 that f_i is an entire function for $i > k$. Hence $P(\mathbf{C}; f) = P(H; f) = k$.

LEMMA 2.2. *Suppose* A *is a* P^**-operator,* E *an extended fundamental strip of* A, *and* $\mathbf{C}$ *the complex plane. If* $f(z) \in \mathscr{S}(A)$, *then* $P(\mathbf{C}; f) = P(E; f)$.

Proof. If the difference operator A_0 associated to A contains only one step, then $P(\mathbf{C}; f) = P(E; f) = 0$ for every $f(z) \in \mathscr{S}(A)$ [21].

Consider the case when A_0 contains more than one step and assume that E is parallel to the imaginary axis Y. Let $E = E(z_0, Y)$ be bounded to the left by the line L_1 and to the right by the line L_2. One of these lines belongs to the half-closed strip E. To be concrete assume $L_1 \subset E$. Then there is, on L_2, one and only one point ζ such that $\gamma = \zeta - z_0$ is a step of A and this $\gamma = \alpha_{nj}$ is a step of highest index. Note that γ may be simultaneously a step of several indexes (not only of the highest one) and write $A(D) = \exp(\gamma D) P(D) + \tilde{A}(D)$, where $P(D)$

is an algebraic polynomial of degree n in D and the differential-difference operator $\tilde{A}$ does not contain the step γ.

Suppose now that $f(z) \in \mathscr{S}(A)$. Then

$$A[f(z)] \equiv \exp(\gamma D) P(D) f(z) + \tilde{A}(D) f(z) = g(z),$$

where $g(z)$ is an entire function. The last equation implies

$$P(D) f(z) = g(z-\gamma) - \exp(-\gamma D)\tilde{A}(D) f(z). \tag{2.1}$$

Denote by $\chi_k (k = 1,2,\ldots,s)$ the steps of the operator $\exp(-\gamma D)\tilde{A}(D)$. Each χ_k is equal to some $\alpha_{ij} - \gamma$, where α_{ij} is a step of A different from γ. Therefore $\operatorname{Re}\chi_k \leq -d$, where $d = \min[\operatorname{Re}(\gamma - \alpha_{ij})]$ and the minimum is taken over all steps α_{ij} of A different from γ.

Note that $d > 0$ and apply (2.1) for $z \in Q_1$, where

$$Q_1 = \{z: \operatorname{Re}(z_0 + \gamma) \leq \operatorname{Re} z < \operatorname{Re}(z_0 + \gamma + d)\}.$$

The terms of the right side of (2.1) are (not counting the entire function $g(z-\gamma)$) $C_{ik} f^{(i)}(z + \chi_k)$, where $i \leq n$ and C_{ik} are constants. The relation $z \in Q_1$ implies $z + \chi_k \in E$. Thus the right side of (2.1) is at such a point z either regular or has a pole of order $m \leq P(E;f) + n$. Since the degree of $P(D)$ is n, the function $f(z)$ is either regular at the point z or has there a pole of order at most $P(E;f)$. Thus we proved that $P(E_1;f) = P(E;f)$, where $E_1 = E \cup Q_1$. Proceeding in the same way we easily get $P(H_2;f) = P(E;f)$, where

$$H_2 = \{z: \operatorname{Re}(z_0 + \gamma) \leq \operatorname{Re} z < \infty\}.$$

It remains now to use Lemma 2.1 to get $P(\mathbf{C};f) = P(E;f)$.

Remark. Let A be a P^*-operator and E an extended fundamental strip of A. Lemmas 1.4 and 2.2 show that the space $\mathscr{S}(E;A)$ previously defined as $\mathscr{S}(A) \cap \mathscr{G}(E)$ may be defined as follows: $f(z) \in \mathscr{S}(E;A)$ if and only if $P(E;f) < \infty$ and $f(z) \in \mathscr{S}(\Pi;A)$, where Π is an arbitrary fundamental strip of A parallel to E.

Suppose $f(z) \in \mathscr{S}(\Pi;B)$, where B is the difference operator (8) and Π a fundamental strip of B. Corollary 1.1 and Theorem 1.2 show that $P(\mathbf{C};f) < \infty$ and that for every natural k exists a function $f(z) \in \mathscr{S}(\Pi;B)$ such that $P(\mathbf{C};f) = k$. This property is characteristic for the difference operator. The next lemma will show that if $f(z) \in \mathscr{S}(\Pi;A)$, where A is a differential-difference operator (of order more than zero) and Π a fundamental strip of A, then generally either $P(\mathbf{C};f) = 0$ or $P(\mathbf{C};f) = \infty$. It will be proved that the relation $0 < P(C;f) < \infty$ may hold for $f(z) \in \mathscr{S}(\Pi;A)$ only in the exceptional case when $A = BA^*$, where B is a difference operator containing at least two terms and A^* is a differential-difference operator.

LEMMA 2.3. *Suppose A is the operator* (4), Π *a fundamental strip of A and $\{A_i(t)\}$, $(i = 0,1,2,\ldots,n)$ its characteristic system, $B(t)$ the* GCD *of the characteristic system, and $B = B(D)$ the difference operator corresponding to $B(t)$. If $f(z) \in \mathscr{S}(\Pi;A)$ and $P(\mathbf{C};f) < \infty$* ($\mathbf{C}$ is the complex plane), *then $f(z) \in \mathscr{S}(B)$.*

Note that if $P(\mathbf{C};f) > 0$ and $f(z) \in \mathscr{S}(B)$, then B contains more than one term.

The proof of this lemma is based on the following theorem proved in our paper [8].

THEOREM 2.1. *Let $P_1, P_2, \ldots, P_n$ be difference operators, $P_i(t)$ $(i = 1,2,\ldots,n)$ their characteristic functions, $B(t)$ the* GCD *of the system $\{P_i(t)\}$, and $B = B(D)$ the difference operator corresponding to $B(t)$. If Π_i is a fundamental strip of P_i and $f(z) \in \mathscr{S}(\Pi_i; P_1, P_2, \ldots, P_n)$ for at least one i $(i = 1,2,\ldots,n)$, then $f(z) \in \mathscr{S}(B)$.*

Proof of Lemma 2.3. Fix a nonnegative integer k and denote by $\mathscr{S}^k(\Pi;A)$ the set of all functions $f(z)$ such that

$$f(z) \in \mathscr{S}(\Pi;A) \quad \text{and} \quad P(\mathbf{C};f) \le k. \tag{2.2}$$

We need to prove that $\mathscr{S}^k(\Pi;A) \subset \mathscr{S}(B)$. We prove it by induction on $p = k + n$, where n is the order of the operator A. The number p will be called the order of the pair $(A, \mathscr{L}^k(\Pi;A))$.

If $p = 0$, then $k = 0$, and $\mathscr{L}^0(\Pi;A)$ contains only entire functions. Hence $\mathscr{L}^0(\Pi;A) \subset \mathscr{L}(B)$. Let t be a nonnegative integer. Suppose that $\mathscr{L}^k(\Pi;A) \subset \mathscr{S}(B)$ for $p = k + n \le t$ and consider the case $p = k + n = t + 1$. Since $B(t)$ is the GCD of the system $\{A_i(t)\ (i = 0,1,2,\ldots,n)\}$ there exists a system of exponential polynomials $\{A_i^*(t)\}$ such that $A_i(t) = A_i^*(t)B(t)$. The GCD of the last system $\{A_i^*(t)\}$ is 1.

Consider the differential-difference operator

$$\tilde{A} = A_0^* \cdot A = A_0 \sum_{j=0}^{n} D^j A_j^*, \tag{2.3}$$

the function $f(z) \in \mathscr{S}^k(\Pi;A)$, where $k + n = t + 1$, and its first component $f_1(z)$.

The relation (2.2) and

$$f_1(z) \in \mathscr{S}(A_0) \tag{2.4}$$

(see Lemma 1.1) combined with (2.3) imply that both $f(z)$ and $f_1(z)$ belong to $\mathscr{S}(\tilde{A})$. Thus $F^*(z) = f(z) - f_1(z) \in \mathscr{S}(\tilde{A})$ and $F(z) = \int F^*(z)\,dz \in \mathscr{S}(\tilde{A})$. Note also that $P(\mathbf{C};F) \le k - 1$ (see (2.2)) and apply Lemma 1.4 to the function $f(z) \in \mathscr{S}(\Pi;A)$. We conclude then that $f(z)$ is L-regular, where L is a line parallel to Π. It follows easily that $F(z)$ is also L-regular and it has therefore only a finite number of poles in Π and in every strip which makes a small enough angle with L. Thus there exists a fundamental strip $\tilde{\Pi}$ of $\tilde{A}$ such that $F(z) \in \mathscr{S}^{k-1}(\tilde{\Pi};\tilde{A})$. The order of the pair $(\tilde{A}, \mathscr{S}^{k-1}(\tilde{\Pi};\tilde{A}))$ is t and the GCD of the characteristic system of $\tilde{A}$ is $A_0(t)$. Our assumption of induction shows that $F(z) \in \mathscr{S}(A_0)$ which implies $F'(z) = f(z) - f_1(z) \in \mathscr{S}(A_0)$. The last relation, together with (2.4) and (2.2), gives

$$f(z) \in \mathscr{S}(\Pi; A_0). \tag{2.5}$$

Use again (2.2) and the L-regularity of $f(z)$ to get $f(z) \in \mathscr{S}^k(\tilde{\Pi};\hat{A})$, where $\hat{A} = A - A_0$ and $\tilde{\Pi}$ is a fundamental strip of $\hat{A}$ which makes a small enough angle with L. The order of the operator $\hat{A}$ is at most $n - 1$, thus the order of the pair $(\hat{A}, \mathscr{S}^k(\tilde{\Pi};\hat{A}))$ is at most t. We can therefore use again the assumption of induction to get $f(z) \in \mathscr{S}(B_1)$, where B_1 is the difference operator corresponding to the GCD of the characteristic system $\{A_i(t)(i = 1,2,...,n)\}$ of $\hat{A}$. Obviously,

$$\mathscr{S}(B_1) \subset \mathscr{S}(A_1, A_2, ..., A_n),$$

thus $f(z) \in \mathscr{S}(A_1, A_2, ..., A_n)$. It follows now from (2.5) and Theorem 2.1 that $f(z) \in \mathscr{S}(\Pi; A_0, A_1, ..., A_n) = \mathscr{S}(B)$ and the proof is complete.

Proof of Theorem 1.4. Suppose A is a P^*-operator and $f(z) \in \mathscr{S}(E;A)$, where E is an extended fundamental strip of A. The relation $f(z) \in \mathscr{S}(E;A)$ shows that $f(z)$ has only a finite number of poles in E and it follows from Lemma 2.2 that $P(\mathbf{C};f) < \infty$, where $\mathbf{C}$ is the complex plane. Use now Lemma 2.3 to get $f(z) \in \mathscr{S}(B)$, where B is the GCD of the characteristic system of A. Thus we proved that $\mathscr{S}(E;A) \subset \mathscr{S}(E;B)$. To complete the proof of Theorem 1.4 it remains to note that $\mathscr{S}(B) \subset \mathscr{S}(A_0, A_1, A_2, ..., A_n) \subset \mathscr{S}(A)$.

THEOREM 2.2. *Suppose A is a P^*-operator and E an extended fundamental strip of A. If at least one of the difference operators A_i associated to A contains only one term, then no properly meromorphic function belongs to $\mathscr{S}(E;A)$.*

Proof. Consider the characteristic system $\{A_i(t)(i = 0,1,2,...,n)\}$ of A. Its GCD $B(t) \equiv 1$. Thus theorem 1.4 shows that $\mathscr{S}(E;A) = \mathscr{S}(B)$, where B is the identity operator. No properly meromorphic function belongs to $\mathscr{S}(B)$.

The distance d between the two boundary lines of a strip Π will be called the *size* of Π. We will also call this distance d the *L-size* of both the difference operator B and its characteristic function $B(t)$, if Π is a fundamental strip of B and is parallel to L. The L-size of B and $B(t)$ is zero, if B contains only one term.

Consider now the differential-difference operator A and the difference operators A_i $(i = 0,1,...,n)$ associated to A. If at least one of the A_i has an L-size (i.e., has a fundamental strip parallel to L), then we introduce also the L-size of A defining it as the smallest of the L-sizes of the operators A_i (evidently, of those A_i which have an L-size).

We agree also to say that the number $0 \le s \le \infty$ is the *L-regularity size* of a meromorphic function $f(z)$, if s is the supremum of the sizes of all strips S parallel to L such that $f(z)$ is regular in S.

THEOREM 2.3. *Suppose A is a P^*-operator, E an extended fundamental strip of A, and $f(z)$ is a properly meromorphic function belonging to $\mathscr{S}(E;A)$. The L-size of A is, for every line L for which this size exists, not less than the L-regularity size of $f(z)$. These two sizes may be equal only if some member $A_i(t)$ of the characteristic system $\{A_i(t)(i = 0,1,2,...,n)\}$ of A is the* GCD *of this system.*

Proof. Let $P(t)$ and $Q(t)$ be exponential polynomials (they may be considered as characteristic functions of some difference operators) and let $Q(t)$ be a divisor of $P(t)$. Note the following properties of this pair: (1) If $P(t)$ has an L-size, then $Q(t)$ has also an L-size; (2) the L-size of $P(t)$ is not less than the L-size of $Q(t)$; and (3) these L-sizes are equal if and only the ratio $P(t)/Q(t)$ is a monomial of the form $C \exp(\gamma t)$ with C and γ constant. These properties are easily proved, if we assume (without loss of generality) that L is the imaginary Y axis and have in mind that a polynomial $C(t) = \sum_{k=1}^{p} C_k \exp(\gamma_k t)$ has a Y-size if and only if the set

$$\{\mathrm{Re}\gamma_k (k = 1, 2, \ldots, p)\}$$

contains a single maximal and a single minimal element, and that the difference between these two extremal elements is the Y-size of $C(t)$.

Suppose now that $B(t)$ is the GCD of the characteristic system

$$\{A_i(t)\,(i = 0, 1, 2, \ldots, n)\}$$

of A. If A has an L-size, then B also has an L-size and this size is not more than the L-size of A. These sizes are equal if and only if $A_i(t) = B(t) \cdot C \exp(\gamma t)$ at least for one i, i.e., if and only if $A_i(t)$ is itself the GCD of the characteristic system.

Suppose that $f(z)$ is a properly meromorphic function and $f(z) \in \mathscr{S}(E; A)$ Theorem 1.4 shows that $f(z) \in \mathscr{S}(B)$ where B is the difference operator corresponding to $B(t)$. It follows now from Theorem 1.2 that the maximal possible L-regularity size of $f(z)$ is equal to the L-size of B and the proof is complete.

3. OPERATORS WITH REAL STEPS

All difference and differential-difference operators of this section are supposed to have only real steps. In particular, all steps α_{ij} of the operator $A = A(D)$ are assumed to be real. In this special case the operator A has the following property:

The principal part of the function $A[f(z)]$ ($f(z)$ is a meromorphic function) at the point z_0 depends only on the principal parts of $f(z)$ at poles belonging to the line A passing through z_0 parallel to the real axis (but does not depend on the poles of $f(z)$ lying outside this line A).

This property enables us to extend the previous results for the space $\mathscr{S}(E; A)$ to the space $\mathscr{S}(A)$. Furthermore these results may be extended to the space $\mathscr{S}_\Lambda(A)$ which consists of all functions $f(z)$ meromorphic on the line Λ (Λ is parallel to the real axis) such that $A[f(z)]$ is regular on Λ. For example, we will show that the equalities

$$\mathscr{S}_\Lambda(A) = \mathscr{S}_\Lambda(B) = \mathscr{S}_\Lambda(A_0, A_1, A_2, \ldots, A_n), \tag{3.1}$$

where $\mathscr{S}_\Lambda(A_0,A_1,A_2,...,A_n) = \mathscr{S}_\Lambda(A_0) \cap \mathscr{S}_\Lambda(A_1) \cap ... \cap \mathscr{S}_\Lambda(A_n)$, may be added to Theorem 1.1.

Proof of Theorem 1.1. Suppose that $f(z) \in \mathscr{S}_\Lambda(A)$ and $\{\lambda_i\}$ is the set of all poles of $f(z)$ on the line Λ. This set is either finite or $\lambda_i \to \infty$ as $i \to \infty$. It follows from Mittag-Leffler's theorem that there exists a function $\varphi(z)$ meromorphic in the complex plane such that (1). $\varphi(z)$ is regular outside Λ and (2). $f(z) - \varphi(z)$ is regular on Λ. Thus for an arbitrary differential-difference operator Q the two relations $\varphi(z) \in \mathscr{S}(Q)$ and $f(z) \in \mathscr{S}_\Lambda(Q)$ *are equivalent:*

$$f(z) \in \mathscr{S}_\Lambda(Q) \Leftrightarrow \varphi(z) \in \mathscr{S}(Q). \tag{3.2}$$

Suppose now that A is a P-operator and E is an extended fundamental strip of A parallel to the imaginary axis. The function $\varphi(z)$ has in E only a finite number of poles since all these poles belong to the bounded half-closed segment $\Lambda \cap E$. Thus $\varphi(z) \in \mathscr{S}(E;A)$. Theorem 1.4 shows that $\varphi(z) \in \mathscr{S}(B)$ and (3.2) that $f(z) \in \mathscr{S}_\Lambda(B)$. We have proved that $\mathscr{S}_\Lambda(A) \subset \mathscr{S}_\Lambda(B)$. This inclusion, together with the evident ones $\mathscr{S}_\Lambda(B) \subset \mathscr{S}_\Lambda(A_0,A_1,A_2,...,A_n) \subset \mathscr{S}_\Lambda(A)$, gives (3.1).

Let $f(z) \in \mathscr{S}(A)$. It implies $f(z) \in \mathscr{S}_\Lambda(A)$ for every line Λ parallel to the real axis. Thus $f(z) \in \mathscr{S}_\Lambda(B)$ for every Λ and consequently $f(z) \in \mathscr{S}(B)$.

We have proved that $\mathscr{S}(A) \subset \mathscr{S}(B)$. We complete the proof of Theorem 1.1 using the obvious inclusion $\mathscr{S}(B) \subset \mathscr{S}(A_0,A_1,A_2,...,A_n) \subset \mathscr{S}(A)$.

We are going to state without proof some other results about the spaces $\mathscr{S}(A)$ and $\mathscr{S}_\Lambda(A)$. We omit the proofs as they are almost identical to the proof of Theorem 1.1.

THEOREM 3.1. *The space $\mathscr{S}(A)$ ($\mathscr{S}_\Lambda(A)$) contains no properly meromorphic function (no properly meromorphic function on Λ) if A is a P-operator and at least one of the difference operators A_i $(i = 0,1,2,...\ n)$ associated to A contains only one term.*

LEMMA 3.1. *Let $\{A_i(t)\ (i = 0,1,2,...,n)\}$ be the characteristic system of the operator A, $B(t)$ the* GCD *of this system, and B the difference operator corresponding to $B(t)$. If $f(z) \in \mathscr{S}_\Lambda(A)$ and $P(\Lambda;f) < \infty$, then $f(z) \in \mathscr{S}_\Lambda(B)$.*

Let A be a P-operator and E an extended fundamental strip of A. The half-closed segment $I = E \cap \Lambda$ will be called an *extended fundamental segment of A.*

LEMMA 3.2. *Suppose A is a P-operator and I, $I \subset \Lambda$, an extended fundamental segment of A. If $f(z) \in \mathscr{S}_\Lambda(A)$, then $P(\Lambda;f) = P(I;f)$.*

Let the steps $\beta_1 < \beta_2 < ... < \beta_p$ of the difference operator B (see (8)) be arranged in increasing order. The number $\beta_p - \beta_1$ we call the *size* of both the operator B and its characteristic function $B(t)$ (note that $\beta_p - \beta_1$ is also the Y-size of B, if Y is the imaginary axis; it is also the greatest of the L-sizes of B). We introduce also the *size* of the differential-difference operator A identifying it with the smallest of the sizes of the difference operators A_i $(i = 0,1,2,...,n)$ associated to A.

We need one more notion to state the next theorem. Let $f(z)$ be a function meromorphic on the line Λ. The number $c > 0$ is the *regularity size* of $f(z)$ on

Λ if c is the supremum of the lengths of all segments Σ such that $\Sigma \subset \Lambda$ and $f(z)$ is regular on Σ.

THEOREM 3.2. *Let A be a P-operator, $\{A_i(t)\ (i = 0, 1, 2, \ldots, n)\}$ its characteristic system, and $f(z) \in \mathscr{S}_\Lambda(A)$. If $f(z)$ is properly meromorphic on Λ, then the regularity size of $f(z)$ on Λ is not more than the size of A. Equality may hold between these two sizes only if some member $A_i(t)$ of the characteristic system of A is the* GCD *of this system.*

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