QUASICONFORMAL VARIATION OF THE GREEN'S FUNCTION

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In this paper we study the variation of the Green's function of a Riemann surface due to a perturbation of the conformal structure. The perturbations considered are parametrized by the complex dilatation of a quasiconformal map.

The basic idea in our approach is to consider the Green's function of a Fuchsian group. Results concerning Riemann surfaces are obtained via uniformization. The advantage of this point of view is that we can make use of powerful results about normalized quasiconformal maps of the unit disc. The main problem becomes to justify the validity of term by term estimates in the series defining the Green's function.

Variations of domain functionals, in particular variations of the Green's function, have been extensively studied. For previous results we refer to Sontag [7] and the references listed there.

Section 1 contains some basic results and definitions used in the paper. In section 2 we treat the Fuchsian group case. Section 3 gives applications to Riemann surfaces.

Results in this paper are based on part of the author's 1975 Stony Brook dissertation.

Finally, we would like to point out that we do not know how to generalize the main varational formula for infinitely generated groups or, equivalently, for arbitrary (not necessarily finite) hyperbolic Riemann surfaces.

1. PRELIMINARIES

1.1 A Fuchsian group Γ acting on the unit disc U is said to be of convergence type if

$$\sum_{\gamma\in\Gamma} (1 - |\gamma(z)|) < \infty, \tag{1.1}$$

uniformly for z in a compact subset of U. This condition is equivalent (Tsuji [8, p. 522]) to the existence of a Green's function on the Riemann surface U/Γ. Actually, we can write explicitly the Green's function as a function on U invariant under Γ. Namely

$$G(z,a) = \sum_{\gamma} \log \left| \frac{z - \gamma(a)}{1 - \overline{\gamma(a)}\, z} \right|. \tag{1.2}$$

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Condition (1.1) assures that (1.2) converges uniformly on compact subsets of $U - \Gamma a$.

Given a Riemann surface S which admits a Green's function, we can represent it by (1.2) via uniformization.

1.2 For a detailed presentation of quasiconformal mappings we refer to Ahlfors [1] and Lehto-Virtanen [4]. In this section we will recall some basic results used later on in the paper.

It is well known that given $\mu \in L^\infty(U, \mathbb{C})$, $\|\mu\|_\infty < 1$, there exists a unique quasiconformal map $w_\mu \colon U \to U$ fixing 0, 1 and having complex dilatation μ. Further, there exists a derivative

$$\dot{w}_\nu[\mu](z) = \lim_{s\to 0} \frac{w_{\nu+s\mu}(z) - w_\nu(z)}{s} \tag{1.3}$$

and

$$\dot{w}_\nu[\mu](z) = -\frac{1}{\pi}\int_{\mathbb{C}} R(w_\nu(t), w_\nu(z))(w_\nu)_t(t)^2\mu(t)\,du\,dv, \qquad t = u + iv. \tag{1.4}$$

Here

$$R(t,z) = \frac{z(z-1)}{t(t-1)(t-z)} \quad \text{and} \quad \nu(1/\bar{t}) = \overline{\nu(t)}\,t^2/\bar{t}^2 \tag{1.5}$$

For these results see Ahlfors [1, Chapter V].

For a quasiconformal map w with complex dilatation μ we define $K = \dfrac{1 + \|\mu\|}{1 - \|\mu\|}$. The following result, due to Mori (see Lehto-Virtanen [4, p. 66] or Ahlfors [1, p. 47]), will be useful: suppose that our map w is normalized by $w(0) = 0$, then for every pair of points z_1, z_2 with $|z_1| \leq 1$, $|z_2| \leq 1$

$$|w(z_1) - w(z_2)| \leq 16|z_1 - z_2|^{1/K}. \tag{1.6}$$

1.3 The set of Beltrami coefficients $M(\Gamma)$, with respect to a Fuchsian group Γ, is defined as the unit ball in the Banach space of all $\mu \in L^\infty(U, \mathbb{C})$ such that

$$\mu \circ \gamma\, \overline{\gamma'}/\gamma' = \mu. \tag{1.7}$$

It is not hard to verify that $\mu \in M(\Gamma)$ if and only if $w_\mu \circ \gamma \circ w_\mu^{-1}$ is a Möbius transformation for all $\gamma \in \Gamma$.

2. VARIATIONAL FORMULA FOR THE GREEN'S FUNCTION OF A FUCHSIAN GROUP

2.1 Let Γ be a Fuchsian group of convergence type acting on the unit disk U. Given a Beltrami coefficient $\mu \in M(\Gamma)$, $\Gamma_\mu = w_\mu \Gamma w_\mu^{-1}$ is again a Fuchsian group of convergence type. This follows from a result of Pfluger [5] on the invariance of the class of hyperbolic Riemann surfaces under quasiconformal mappings. Given $a \in U$ we have a function $M(\Gamma) \times (U - \Gamma a) \to \mathbb{R}$ given by $(\mu, z) \to G_\mu(w_\mu(z), w_\mu(a))$. Here G_μ denotes the Green's function for the group Γ_μ. We note that

$$G_\mu(w_\mu(z), w_\mu(a)) = \sum_{\gamma \in \Gamma} h(\mu, z, \gamma), \tag{2.1}$$

where

$$h(\mu, z, \gamma) = \log \left| \frac{w_\mu(z) - w_\mu(\gamma(a))}{1 - \overline{w_\mu(\gamma(a))}\, w_\mu(z)} \right|. \tag{2.2}$$

We now compute the derivative of $h(\cdot, z, \gamma)$ at ν in the direction μ. We denote it by $\dot{h}(\nu, z, \gamma)[\mu]$. Clearly

$$\begin{aligned} \dot{h}(\nu, z, \gamma)[\mu] = \operatorname{Re} \Bigg\{ & \frac{\dot{w}_\nu[\mu](z) - \dot{w}_\nu[\mu](\gamma(a))}{w_\nu(z) - w_\nu(\gamma(a))} \\ & + \frac{\overline{\dot{w}_\nu[\mu](\gamma(a))}\, w_\nu(z) + \overline{w_\nu(\gamma(a))}\, \dot{w}_\nu[\mu](z)}{1 - \overline{w_\nu(\gamma(a))}\, w_\nu(z)} \Bigg\}. \end{aligned} \tag{2.3}$$

Using formula (1.5) for the variation of a normalized quasiconformal mapping we obtain

$$\begin{aligned} \dot{h}(\nu, z, \gamma)[\mu] = \operatorname{Re} \Bigg\{ -\frac{1}{\pi} \int_{\mathbb{C}} \Bigg[& \frac{R(w_\nu(t), w_\nu(z)) - R(w_\nu(t), w_\nu(\gamma(a)))}{w_\nu(z) - w_\nu(\gamma(a))} \\ & - \frac{w_\nu(z)\, \overline{w_\nu(\gamma(a))}^{\,2} R(w_\nu(t), w_\nu(\gamma(1/\bar{a}))) - \overline{w_\nu(\gamma(a))}\, R(w_\nu(t), w_\nu(z))}{1 - \overline{w_\nu(\gamma(a))}\, w_\nu(z)} \Bigg] \\ & (w_\nu)_t(t)^2 \mu(t)\, du\, dv \Bigg\}, \qquad t = u + iv. \end{aligned} \tag{2.4}$$

A long elementary computation yields

$$\begin{aligned} \dot{h}(\nu, z, \gamma)[\mu] = -\frac{1}{2\pi} \int_{\mathbb{C}} & \left[\frac{1}{w_\nu(t) - w_\nu(\gamma(a))} - \frac{1}{w_\nu(t) - w_\nu(\gamma(1/\bar{a}))} \right] \\ \times & \left[\frac{1}{w_\nu(t) - w_\nu(z)} - \frac{1}{w_\nu(t) - w_\nu(1/\bar{z})} \right] (w_\nu)_t(t)^2 \mu(t)\, du\, dv. \end{aligned} \tag{2.5}$$

In the calculations, integrals which contain $\bar{t}$ are transformed by the change of variable $s = 1/\bar{t}$. In doing so, it should be recalled that $w_\nu(1/\bar{s}) = 1/\overline{w_\nu(s)}$ and $\gamma(1/\bar{s}) = 1/\overline{\gamma(s)}$.

2.2 In this section we will show that the series

$$\sum_{\gamma} \dot{h}(\nu, z, \gamma)[\mu] \tag{2.6}$$

converges absolutely and uniformly in a neighborhood of $\nu = 0$ and for z in a compact subset of $U - \Gamma a$.

Denote by $J_{w_\nu}(t)$ the Jacobian of the map w_ν, we have

$$|(w_\nu)_t(t)|^2 \left(1 - \frac{|(w_\nu)_{\bar{t}}(t)|^2}{|(w_\nu)_t(t)|^2}\right) = J_{w_\nu}(t),$$

therefore

$$|(w_\nu)_t(t)|^2 \leq \frac{J_{w_\nu}(t)}{1 - \|\nu\|^2}. \tag{2.7}$$

From (2.5) and (2.7) we obtain

$$|\dot{h}(\nu, z, \gamma)[\mu]| \leq \frac{1}{2\pi} \frac{\|\mu\|}{1 - \|\nu\|^2} \int_{\mathbf{C}} \left| \frac{1}{w_\nu(t) - w_\nu(\gamma(a))} - \frac{1}{w_\nu(t) - w_\nu(\gamma(1/\bar{a}))} \right| \left| \frac{1}{w_\nu(t) - w_\nu(z)} - \frac{1}{w_\nu(t) - w_\nu(1/\bar{z})} \right| J_{w_\nu}(t)\, du\, dv,$$

by change of integration variable

$$|\dot{h}(\nu, z, \gamma)[\mu]| \leq \frac{1}{2\pi} \frac{\|\mu\|}{1 - \|\nu\|^2} \int_{\mathbf{C}} \left| \frac{1}{t - w_\nu(\gamma(a))} - \frac{1}{t - w_\nu(\gamma(1/\bar{a}))} \right| \left| \frac{1}{t - w_\nu(z)} - \frac{1}{t - w_\nu(1/\bar{z})} \right| du\, dv. \tag{2.8}$$

To estimate integral (2.8) we have to study

$$I(\nu, z, \gamma) = \int_{\mathbf{C}} \left| \frac{1}{t - w_\nu(\gamma(a))} - \frac{1}{t - w_\nu(\gamma(1/\bar{a}))} \right| \left| \frac{1}{t - w_\nu(z)} \right| du\, dv$$

and $I(\nu, 1/\bar{z}, \gamma)$. It will be sufficient to consider $I(\nu, z, \gamma)$.

First note that

$$I(\nu, z, \gamma) = \frac{1 - |w_\nu(\gamma(a))|^2}{|w_\nu(\gamma(a))|} \int_{\mathbf{C}} \frac{du\, dv}{|t - w_\nu(\gamma(a))|\,|t - w_\nu(\gamma(1/\bar{a}))|\,|t - w_\nu(z)|}. \tag{2.9}$$

We will make use of the following lemmas:

LEMMA. *Given* $\varepsilon > 0$, $1 - |w_\nu(\gamma(a))| < \varepsilon$ *for all* $\nu, \|\nu\|$ *small, and all* $\gamma \in \Gamma$ *except a finite number* $N(\varepsilon)$.

Proof. Using estimate (1.6) one sees that there exists $\delta > 0$ such that for every $\|\nu\|$ small

$$w_\nu\{z: 1-\delta < |z| \le 1\} \subseteq \{z: 1-\varepsilon < |z| \le 1\}.$$

Now, the limit set of Γ is contained in the unit circle $\{z: |z| = 1\}$. Therefore, except for a finite number $N(\varepsilon)$ of elements $\gamma \in \Gamma$, $1 - \delta < |\gamma(a)| < 1$.

LEMMA. *Let $K \subseteq \mathbb{C}$ compact. Then for $a, b \in K$, $0 < |a-b|$ small, there exist positive constants α, $\beta \in \mathbb{R}$ such that*

$$\int_K \frac{du\,dv}{|t-a|\,|t-b|} \le \alpha + \beta \log \frac{1}{|a-b|}.$$

Proof. Let R be the radius of circle centered at 0 and containing K. Set $t - a = (a-b)s$, $s = x + iy$.

$$\begin{aligned}\int_K \frac{du\,dv}{|t-a|\,|t-b|} &\le \int_{|t|\le R} \frac{du\,dv}{|t-a|\,|t-b|} \le \int_{|s|\le (2R/|a-b|)} \frac{dx\,dy}{|s|\,|s+1|}\\ &\le \int_{|s|\le 3/2} \frac{dx\,dy}{|s|\,|s+1|} + (\text{const.}) \int_{3/2\le|s|\le 2R/|a-b|} \frac{dx\,dy}{|s|^2}\\ &= \alpha + \beta \operatorname{Log} \frac{1}{|a-b|}.\end{aligned}$$

LEMMA. *Let $K \subseteq \mathbb{C}$ compact. Then $g(c) = \displaystyle\int_K \frac{du\,dv}{|t-c|}$, $c \in \mathbb{C}$ is continuous.*

Proof. Let $c, c' \in K$. Then

$$\begin{aligned}|g(c) - g(c')| &\le \int_K \frac{\big||t-c'| - |t-c|\big|}{|t-c|\,|t-c'|}\, du\,dv \le |c-c'| \int_K \frac{du\,dv}{|t-c|\,|t-c'|}\\ &\le |c-c'|\left(\alpha + \beta \log \frac{1}{|c-c'|}\right) \to 0, \quad \text{as } c \to c'.\end{aligned}$$

If $c \notin K$, $\displaystyle\int_K \frac{du\,dv}{|t-c|\,|t-c'|}$ is bounded.

LEMMA. *Let K_1, $K_2 \subseteq \mathbb{C}$ be disjoint compact sets. Then there exist positive constants α, $\beta \in \mathbb{R}$ such that*

$$\int_{\mathbb{C}} \frac{du\,dv}{|t-a|\,|t-b|\,|t-c|} \le \alpha + \beta \log \frac{1}{|a-b|}$$

for all $c \in K_1$, and $a, b \in K_2$, $0 < |a-b|$ small.

Proof. Take $\tilde{K}_1, \tilde{K}_2 \subseteq \mathbb{C}$ compact sets so that $K_i \subseteq \tilde{K}_i$, distance $(\partial\tilde{K}_i, K_i) > 0$, $i = 1,2$, and $\tilde{K}_1 \cap \tilde{K}_2 = \emptyset$.

$$\int_{\mathbb{C}} \frac{du\,dv}{|t-a|\,|t-b|\,|t-c|} = \int_{\mathbb{C}-(\tilde{K}_1 \cup \tilde{K}_2)} \frac{du\,dv}{|t-a|\,|t-b|\,|t-c|}$$
$$+ \int_{\tilde{K}_1} \frac{du\,dv}{|t-a|\,|t-b|\,|t-c|} + \int_{\tilde{K}_2} \frac{du\,dv}{|t-a|\,|t-b|\,|t-c|}$$
$$\leq A + B\int_{\tilde{K}_1} \frac{du\,dv}{|t-c|} + C\int_{\tilde{K}_2} \frac{du\,dv}{|t-a|\,|t-b|} \leq \alpha + \beta \log \frac{1}{|a-b|}.$$

From (2.9) and the above lemmas we obtain, for z in a compact subset of U and for almost all $\gamma \in \Gamma$,

$$I(\nu,z,\gamma) \leq \frac{1-|w_\nu(\gamma(a))|^2}{|w_\nu(\gamma(a))|}\left(M_1 + M_2 \log \frac{|w_\nu(\gamma(a))|}{1-|w_\nu(\gamma(a))|^2}\right), \qquad M_1 > 0, M_2 > 0,$$

or, observing that $|w_\nu(\gamma(a))|$ is uniformly bounded away from 0 and uniformly bounded by 1, the equivalent estimate

$$I(\nu,z,\gamma) \leq (1-|w_\nu(\gamma(a))|)(A_1 - A_2 \log(1-|w_\nu(\gamma(a))|)), \quad A_1 > 0, A_2 > 0. \tag{2.11}$$

Now, given $z \in U$, define $z^* = z/|z|$. Using Mori's theorem (section 1.2)

$$1-|w_\nu(z)| \leq |w_\nu(z^*) - w_\nu(z)| \leq 16|z^*-z|^{1/K} = 16(1-|z|)^{1/K},$$

and considering the inverse map we obtain the double estimate

$$((1-|z|)/16)^K \leq 1-|w_\nu(z)| \leq 16(1-|z|)^{1/K} \tag{2.12}$$

Recall that $K = \dfrac{1+\|\nu\|}{1-\|\nu\|}$. Using (2.11) and (2.12)

$$\begin{aligned} I(\nu,z,\gamma) &\leq (1-|\gamma(a)|)^{1/K}(A - B\log(1-|\gamma(a)|)) \\ &= (1-|\gamma(a)|)^{(1/K)-\delta}(1-|\gamma(a)|)^{\delta}(A - B\log(1-|\gamma(a)|)) \\ I(\nu,z,\gamma) &\leq K_\delta(1-|\gamma(a)|)^{(1/K)-\delta} \end{aligned} \tag{2.13}$$

Beardon [2] has shown that for every finitely generated Fuchsian group of the second kind there is $t_0 < 1$ such that

$$\sum_{\gamma\in\Gamma} (1-|\gamma(a)|)^t < \infty, \qquad \text{for } t > t_0.$$

Choose δ and ε sufficiently small so that $1/K - \delta > t_0$, for $\|\nu\| < \varepsilon$. This proves the statement at the beginning of the section, with the additional hypothesis on Γ to be finitely generated.

2.3 Denote by $\dot{G}(z,a)[\mu]$ the derivative of the map $\nu \to G_\nu(w_\nu(z), w_\nu(a))$ at $\nu = 0$ in the direction μ. The preceeding section shows that $\dot{G}(z,a)[\mu]$ exists and

$$\begin{aligned}\dot{G}(z,a)[\mu] &= -\frac{1}{2\pi}\sum_{\gamma\in\Gamma}\int_{\mathbf{C}}\left[\frac{1}{t-\gamma(a)} - \frac{1}{t-\gamma(1/\bar{a})}\right] \\ &\quad\times\left[\frac{1}{t-z} - \frac{1}{t-1/\bar{z}}\right]\mu(t)\,du\,dv\end{aligned} \tag{2.14}$$

Let ω be a fundamental region of Γ in $\mathbf{C}$. Then

$$\begin{aligned}\dot{G}(z,a)[\mu] &= -\frac{1}{2\pi}\sum_{\beta\in\Gamma}\sum_{\gamma\in\Gamma}\int_{\beta(\omega)}\left[\frac{1}{t-\gamma(a)} - \frac{1}{t-\gamma(1/\bar{a})}\right] \\ &\quad\times\left[\frac{1}{t-z} - \frac{1}{t-1/\bar{z}}\right]\mu(t)\,du\,dv,\end{aligned}$$

by a change of variable in the integral we obtain

$$\begin{aligned}\dot{G}(z,a)[\mu] &= -\frac{1}{2\pi}\sum_{\beta,\gamma\in\Gamma}\int_{\omega}\left[\frac{1}{\beta(t)-\gamma(a)} - \frac{1}{\beta(t)-\gamma(1/\bar{a})}\right] \\ &\quad\times\left[\frac{1}{\beta(t)-z} - \frac{1}{\beta(t)-1/\bar{z}}\right]\mu\circ\beta(t)\,|\beta'(t)|^2\,du\,dv \\ &= -\frac{1}{2\pi}\sum_{\beta,\gamma\in\Gamma}\int_{\omega}\left[\frac{1}{\gamma^{-1}\circ\beta(t)-a} - \frac{1}{\gamma^{-1}\circ\beta(t)-1/\bar{a}}\right](\gamma^{-1}\circ\beta)'(t) \\ &\quad\times\left[\frac{1}{\beta(t)-z} - \frac{1}{\beta(t)-1/\bar{z}}\right]\beta'(t)\,\mu(t)\,du\,dv \\ &= -\frac{1}{2\pi}\int_{\omega}\left[\sum_{\gamma\in\Gamma}\left(\frac{1}{\gamma(t)-a} - \frac{1}{\gamma(t)-1/\bar{a}}\right)\gamma'(t)\right] \\ &\quad\times\left[\sum_{\beta\in\Gamma}\left(\frac{1}{\beta(t)-z} - \frac{1}{\beta(t)-1/\bar{z}}\right)\beta'(t)\right]\mu(t)\,du\,dv\end{aligned}$$

$$\dot{G}(z,a)[\mu] = -\frac{2}{\pi}\int_{\omega}\frac{\partial G}{\partial t}(t,a)\frac{\partial G}{\partial t}(t,z)\mu(t)\,du\,dv.$$

$$\dot{G}(z,a)[\mu] = -\frac{4}{\pi}\operatorname{Re}\int_{\bar{\omega}}\frac{\partial G}{\partial t}(t,a)\frac{\partial G}{\partial t}(t,z)\mu(t)\,du\,dv \tag{2.15}$$

Here we have $\bar{\omega} = \omega \cap U$.

Now, we consider the function $\nu \to G_\nu(z,a)$ for z, a fixed and $\|\nu\|$ small. Using (2.15) and the chain rule we can compute the differential $d_0 G(z,a)[\mu]$ at $\nu = 0$

$$
\begin{aligned}
d_0 G(z,a)[\mu] &= -\frac{4}{\pi}\operatorname{Re}\int_{\tilde\omega} \frac{\partial G}{\partial t}(t,a)\frac{\partial G}{\partial t}(t,z)\mu(t)\,du\,dv \\
&\quad - 2\operatorname{Re}\left\{\frac{\partial G}{\partial z}(z,\xi)\,\dot w[\mu](z) + \frac{\partial G}{\partial \xi}(z,\xi)\,\dot w[\mu](\xi)\right\}.
\end{aligned}
\tag{2.16}
$$

3. APPLICATIONS

3.1 Let S be a finite Riemann surface with boundary curves. It is known that these conditions are equivalent to the existence of a uniformization of S by a finitely generated Fuchsian group of the second kind. Let μ be a Beltrami differential on S and denote by S_μ the Riemann surface topologically equivalent to S with the conformal structure induced by μ. We have a commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{\;w_{\hat\mu}\;} & U \\
\rho \big\downarrow & & \big\downarrow \rho_\mu \\
S & \xrightarrow[\text{id}]{} & S_\mu
\end{array}
$$

where ρ is a universal covering map, $\hat\mu = \mu \circ \rho\, \overline{\rho'}/\rho'$ and $\rho_\mu(w_{\hat\mu}(z)) = \rho(z)$. It is easy to see that the map ρ_μ is holomorphic. If Γ is the covering group for ρ, $\Gamma_\mu = w_{\hat\mu}\Gamma w_{\hat\mu}^{-1}$ is the covering group for ρ_μ.

Fix $p, q \in S$ and $z, a \in U$ such that $\rho(z) = p$, $\rho(a) = q$. Then

$$G_{S_\mu}(p,q) = G_{\hat\mu}(w_{\hat\mu}(z), w_{\hat\mu}(a)).$$

To compute the differential of $\mu \to G_{S_\mu}(p,q)$ one observes that the map $\mu \to \hat\mu$ is linear and uses the results in section 2 to obtain

$$d_0 G(p,q)[\mu] = -\frac{4}{\pi}\operatorname{Re}\int_S \frac{\partial G}{\partial r}(r,p)\frac{\partial G}{\partial r}(r,q)\mu(r)\,dA_r. \tag{3.1}$$

3.2. Now we consider the variational problem for a plane domain. A similar formulation, for the simply connected case, can be found in Sontag [7]. Let Ω be a plane domain of finite connectivity. Assume η is a measurable, complex valued function on $\mathbb{C}$ depending on a complex parameter t as follows

$$\eta_t(z) = t\nu(z) + t\varepsilon_t(z),$$

ν and ε_t being bounded measurable functions with $\|\varepsilon_t\|_\infty \to 0$ as $t \to 0$. Denote by f_t the normalized quasiconformal map of $\mathbb{C} \cup \{\infty\}$ leaving 0, 1, ∞ fixed and with complex dilatation η_t. Using the same procedure as in section 3.1 we obtain

$$\lim_{t\to 0}\frac{1}{t}\,[G_t(f_t(z),f_t(a))-G(z,a)]=-\frac{4}{\pi}\operatorname{Re}\int_\Omega \frac{\partial G}{\partial w}(w,z)\frac{\partial G}{\partial w}(w,z)\nu(w)\,dA_w. \tag{3.2}$$

Using the chain rule

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t}\,[G_t(z,a)-G(z,a)] &= -\frac{4}{\pi}\operatorname{Re}\int_\Omega \frac{\partial G}{\partial w}(w,z)\frac{\partial G}{\partial w}(w,a)\nu(w)\,dA_w\\ &\quad -2\operatorname{Re}\left\{\frac{\partial G}{\partial z}(z,a)\dot f[\nu](z)+\frac{\partial G}{\partial z}(a,z)\dot f[\nu](a)\right\}\end{aligned} \tag{3.3}$$

In (3.2) and (3.3) G_t denotes the Green's function of $f_t(\Omega)$.

3.3. We now derive as a special case of (3.1) a variational formula of Schiffer (see Schiffer and Spencer [6, p. 313]).

The Schiffer interior variation can be obtained as a quasiconformal variation as follows. Let S be a Riemann surface as in 3.1, $p_0\in S$ and γ a simple closed analytic curve bounding a cell which contains p_0; assume further that γ is contained in a single coordinate chart. Choose a local parameter z so that $z(p_0)=0$ and $z(\gamma)$ is the boundary ∂U of the unit disc. Let $r(z)$ be a function holomorphic in a neighborhood of ∂U. We have a representation $r(z)=\sum_{n=-\infty}^{\infty}a_nz^n$. Gardiner [3] has shown that the classical Schiffer variation is a quasi-conformal variation with $\mu_\varepsilon(p)=0$ for p outside γ and

$$\mu_\varepsilon(z)=\frac{\varepsilon\sum_{n=1}^{\infty}a_{-n}n\bar z^{n-1}}{1+\varepsilon\sum_{n=1}^{\infty}a_nnz^{n-1}}, \tag{3.4}$$

for $z=z(p)$, p inside γ.

We note that

$$\mu_\varepsilon(z)=\varepsilon\sum_{n=1}^{\infty}a_{-n}n\bar z^{n-1}+o(\varepsilon). \tag{3.5}$$

Stokes' theorem gives

$$\int_{|t|\le 1} n\bar t^{n-1}h(t)\,du\,dv=\frac{1}{2i}\int_{|t|=1}\bar t^n h(t)\,dt=\frac{1}{2i}\int_{|t|=1}\frac{h(t)}{t^n}\,dt, \tag{3.6}$$

for $h(t)$ holomorphic on $|t|\le 1$.

For p, q outside γ, $z = z(p)$, $a = z(q)$, from 3.1, 3.5, 3.6 we obtain

$$\begin{aligned} \dot{G}(p,q)[\mu_\varepsilon] &= -\frac{4}{\pi}\operatorname{Re}\frac{\varepsilon}{2i}\int_{|t|=1}\left(\sum_{n=1}^{\infty} a_{-n}t^{-n}\right)G_t(t,z)\,G_t(t,a)\,dt + o(\varepsilon) \\ &= -\operatorname{Re}\frac{2\varepsilon}{\pi i}\int_{\gamma} r(s)\,G_s(s,p)\,G_s(s,q)\,d\ell_s + o(\varepsilon) \end{aligned} \tag{3.7}$$

REFERENCES

1. L. V. Ahlfors, *Lectures on Quasiconformal Mapping.* Manuscript prepared with the assistance of Clifford J. Earle, Jr. Van Nostrand Mathematical Studies, No. 10. D. Van Nostrand Co., Inc., Toronto, Ont.-New York-London, 1966.
2. A. F. Beardon, *Inequalities for certain Fuchsian groups.* Acta Math. 127 (1971), 221–258.
3. F. P. Gardiner, *Schiffer's interior variation and quasiconformal mapping.* Duke Math. J. 42 (1972), 371–380.
4. O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane.* Springer-Verlag, New York-Heidelberg-Berlin, 1973.
5. A. Pfluger, *Sur une propriété de l'application quasiconforme d'une surface de Riemann ouverte.* C. R. Acad. Sci. Paris 227 (1948), 25–26.
6. M. M. Schiffer and D. C. Spencer, *Functionals of finite Riemann surfaces.* Princeton University Press, Princeton, N.J., 1954.
7. A. Sontag, *Variation of the Green's function due to quasiconformal distortion of the region.* Arch. Rational Mech. Anal. 59 (1973), 257–280.
8. M. Tsuji, *Potential theory in modern function theory.* Maruzen Co., Ltd., Tokyo, 1959.

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