EQUIVARIANT MAPS WITH NONZERO HOPF INVARIANT

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In this paper we will derive strong necessary conditions for the existence of a torus equivariant map $f: S^{4d-1} \to S^{2d}$ with nonzero Hopf invariant. These conditions are expressed in terms of the topological weight system, as defined by Wu-yi Hsiang [6], of the torus actions on S^{4d-1} and S^{2d}. They imply, for example:

COROLLARY. *Suppose a torus T acts almost effectively (that is, with discrete ineffective kernel) on either S^{4d-1} or S^{2d}. If an equivariant map $f: S^{4d-1} \to S^{2d}$ exists with nonzero Hopf invariant, then either:*

(i) $F(T,S^{4d-1}) \sim S^{4r-1}$ *and* $F(T,S^{2d}) \sim S^{2r}$, $0 \leq r \leq d$, *and, except when* $r = 0$ *and* $d = 1,2$, rank $T \leq d - r$,

or (ii) $F(T,S^{4d-1}) = \emptyset$ *and* $F(T,S^{2d}) \sim S^{2r}$ *or* $F(T,S^{4d-1}) \sim S^{2r-1}$ *and* $F(T,S^{2d}) \sim S^{0}$, $0 < r \leq d$, *and* rank $T \leq 3$ *if* $r = 1,2$, rank $T \leq 2$ *if* $r \geq 3$.

Here $F(T,X)$ denotes the fixed point set of T acting on X and $X \sim Y$ means $H^*(X;Q) = H^*(Y;Q)$. All cohomology will be with rational coefficients.

1. When $f: S^{4d-1} \to S^{2d}$ has nonzero Hopf invariant, its mapping cone $M(f)$ will be a rational cohomology projective plane whose cohomology is generated by an element of degree $2d$; such a space will be called a $P^2(2d)$. When f is equivariant with respect to a torus T acting on S^{4d-1} and S^{2d}, $M(f)$ will inherit a T action. The cohomology structure of the possible fixed point sets of the T action on $M(f)$ are well known (see, for example, [2, p. 393]) and it follows:

PROPOSITION 1. *If T acts on S^{4d-1} and on S^{2d} and if $f: S^{4d-1} \to S^{2d}$ is equivariant with nonzero Hopf invariant, then either*

(i) $F(T,S^{4d-1}) \sim S^{2r-1}$ *and* $F(T,S^{2d}) \sim S^{2r}$, $0 \leq r \leq d$,

or (ii) $F(T,S^{4d-1}) = \emptyset$ *and* $F(T,S^{2d}) \sim S^{2r}$ *or* $F(T,S^{4d-1}) \sim S^{2r-1}$ *and* $F(T,S^{2d}) \sim S^{0}$, $0 < r \leq d$.

When case (i) of Proposition 1 occurs we will say the T actions are of type (i); otherwise the T actions are of type (ii). Case (i) occurs when $F(T,M(f)) \sim P^2(2r)$ and case (ii) occurs when $F(T,M(f)) \sim \mathrm{pt} + S^{2r}$; these are the only two possibilities.

If T acts on a sphere S^n with $F(T,S^n) \sim S^r$ ($r = -1$ when $F(T,S^n) = \emptyset$) a (topological) weight is a corank 1 subtorus H so that $F(H,S^n) \sim S^q$ with $q > r$. Its multiplicity is $(q - r)/2$ which is always integer. The Borel formula states that the sum of the multiplicities of the weights is exactly $(n - r)/2$; in particular the collection of weights is finite. A local weight H can be identified with an element $\omega \in H^2(B_T)$ which is defined up to multiplication by a nonzero rational constant. We simply choose ω to be any generator of the kernel of the restriction

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map $H^*(B_T) \to H^*(B_H)$. This relationship will be written $H = \omega^{\perp}$. In complex linear representation theory, a linear weight can be thought of as an element of $H^2(B_T;Z)$; when T acts on S^n linearly, the linear weight system coincides with the topological weight system when linear weights which are rational multiples of each other are identified as the same topological weight.

Return now to the situation of a T-equivariant map $f: S^{4d-1} \to S^{2d}$ of nonzero Hopf invariant. If the actions are of type (i) with $r > 0$, the weight systems of the T actions on S^{4d-1} and S^{2d} are easily understood. In fact if $H \subseteq T$, the H-actions on S^{4d-1} and S^{2d} must be of type (i). It follows that

PROPOSITION 2. *If $f: S^{4d-1} \to S^{2d}$ is equivariant with nonzero Hopf invariant and if the T actions are of type* (i) *with $r > 0$, then the weights of the T action on S^{4d-1} coincide with the weights of the T action on S^{2d-1}; each weight appearing on S^{4d-1} with twice the multiplicity it appears on S^{2d}.*

Actions of type (ii) are studied in Section 2. Section 3 deals with corollaries, including that of the introduction. Section 4 gives examples.

2. Let $f: S^{4d-1} \to S^{2d}$ be equivariant with nonzero Hopf invariant and with the T actions of type (ii). Let H be a corank 1 subtorus of T. If $F(H,S^{4d-1}) \sim S^{4k-1}$ and $F(H,S^{2d}) \sim S^{2k}$, then $r \leq k \leq d$ and we call H a weight of type ω with multiplicity $2k$. If $F(H,S^{4d-1}) = \emptyset$ and $F(H,S^{2d}) \sim S^{2k}$ or if $F(H,S^{4d-1}) \sim S^{2k-1}$ and $F(H,S^{2d}) \sim S^0$ with $r < k \leq d$, we call H a weight of type ν with multiplicity $k - r$.

Let $F(T,M(f)) = F^1 + F^2$ with $F^1 \sim$ pt and $F^2 \sim S^{2r}$. In terms of local weights as defined by Hsiang, a weight of type ω with multiplicity $2k$ is a local weight at F^1 with multiplicity $2k$ and at F^2 with multiplicity $2k - r$. A local weight of type ν with multiplicity $k - r$ is a local weight at F^2 with multiplicity $k - r$.

THEOREM 3. *Suppose $f: S^{4d-1} \to S^{2d}$ is equivariant with nonzero Hopf invariant with the T actions of type* (ii). *Let $\{\omega_i\}$ be the weights of type ω and let $2d_i$ be the multiplicity of ω_i. Let $\{\nu_j\}$ be the weights of type ν with k_j the multiplicity of ν_j. Then there exist nonzero rational numbers c_i so that*

$$\sum_i c_i \prod_{k \neq i} \omega_k^r = \prod_j \nu_j^{k_j}$$

(1) and

$$\sum_{m \neq i} c_m \prod_{k \neq i,m} \omega_k^r \text{ is divisible by } \omega_i^{d_i - r} \qquad \text{for each } i.$$

(These equations take place in $H^*(B_T)$ which is polynomial on rank T *number of generators.)*

Example. Let T^2 act linearly on S^{15} by $4\theta + 2z_1 + 2z_2 + z_1^a z_2^b + z_1^c z_2^d$, $abcd \neq 0$, and on S^8 by $\theta + 2z_1 + 2z_2$ where θ is a one (real) dimensional trivial representation and z_1 and z_2 are the obvious representations of T^2 on $C = \mathbb{R}^2$. Then $F(T^2,S^{15}) = S^3$ and $F(T^2,S^8) = S^0$, so $r = 2$.

In $H^*(B_T) = Q[t_1,t_2]$ the local weights are t_1, t_2, $at_1 + bt_2$, and $ct_1 + dt_2$. $F(t_1^\perp, S^{15}) = S^7$ and $F(t_1^\perp, S^8) = S^4$, so t_1- and similarly t_2- is a local weight of type ω with multiplicity 4.

$$F((at_1 + bt_2)^\perp, S^{15}) = S^5 \quad \text{and} \quad F((at_1 + bt_2)^\perp, S^8) = S^0,$$

so $at_1 + bt_2-$ and similarly $ct_1 + dt_2-$ is a local weight of type ν with multiplicity 1.

If an equivariant map $f\colon S^{15} \to S^8$ exists with nonzero Hopf invariant, Theorem 3 asserts that there are constants c_1, c_2 both nonzero so that

$$c_1 t_2^2 + c_2 t_1^2 = (at_1 + bt_2)(ct_1 + dt_2) = act_1^2 + (ad + bc)t_1 t_2 + bdt_2^2.$$

Thus f can only exist when $ad + bc = 0$.

The remainder of this section is devoted to the proof of Theorem 3. Let

$$X = M(f) \sim P^2(2d),$$

$F = F(T,X) = F^1 + F^2$, $F^1 \sim \text{pt}$, $F^2 \sim S^{2r}$, and let

$$i^*\colon H^*(X_T) \to H^*(F_T) = H^*(F) \otimes H^*(B_T)$$

be induced by inclusion. If $\mu \in H^2(B_T)$, let $F(\mu) = F(\mu^\perp, X)$ and let

$$h^*\colon H^*(X_T) \to H^*(F(\mu)_T) \quad \text{and} \quad j^*\colon H^*(F(\mu)_T) \to H^*(F_T)$$

be induced by inclusions. h^* and j^* depend on μ, of course, but the context will always be sufficiently clear to remove the ambiguities.

The Serre spectral sequence of $X \to X_T \to B_T$ degenerates for dimension reasons and it follows that $H^*(X_T)$ is generated as a $H^*(B_T)$-module by 1, x, x^2 for an appropriate $x \in H^{2d}(X_T)$, see for example [6, pp. 50–51]. Let $y \in H^{2r}(F^2)$ generate; then by subtracting from x a suitable element of $H^*(B_T)$, we can assume that $i^*(x) = (\Omega, \Gamma y) \in H^*(B_T) \oplus (H^*(B_T) \otimes H^*(F^2)) = H^*(F_T^1) \oplus H^*(F_T^2) = H^*(F_T)$, where Ω and Γ are in $H^*(B_T)$. Then $i^*(x^2(x - \Omega)) = 0$ and it follows that $x^2(x - \Omega) = 0$. Thus $H^*(X_T) = H^*(B_T)[x]/x^2(x - \Omega)$. Since $i^*(x^2) = (\Omega^2, 0)$ and $i^*(x(x - \Omega)) = (0, -\Gamma\Omega y)$, using [5, Theorem 3.6] we deduce

$$\Omega^2 = q_1 \prod_i \omega_i^{2d_i}$$

$$\Gamma\Omega = q_2 \left(\prod_i \omega_i^{2d_i - r}\right)\left(\prod_j \nu_j^{k_j}\right), \qquad q_1, q_2 \text{ some rational constants.}$$

By multiplying x and y by suitable nonzero rational constants, we can assume

$$\Omega = \prod_i \omega_i^{d_i} \quad \text{and} \quad \Gamma = \left(\prod_i \omega_i^{d_i - r}\right)\left(\prod_j \nu_j^{k_j}\right).$$

For each $\mu \in H^2(B_T)$, $F(\mu)_T = F(\mu)_{T/\mu^\perp} \times B_{\mu^\perp}$ with the inclusion

$$j\colon F \times B_{T/\mu^\perp} \times B_{\mu^\perp} = F \times B_T \to F(\mu)_T$$

breaking up as the product of the inclusion $F \times B_{T/\mu^\perp} \to F(\mu)_{T/\mu^\perp}$ and the identity on $B_{\mu^\perp}$. It follows, again from [5, p. 316], that for each ω_i we can find a generator $x_i \in H^{2d_i}(F(\omega_i)_T)$ so that $j^*(x_i) = (\omega_i^{d_i}, c_i\omega_i^{d_i-r}y)$ for some nonzero rational c_i. Let $h^*(x) = \alpha_i x_i^2 + \beta_i x_i + \gamma_i$ where $\alpha_i, \beta_i, \gamma_i \in H^*(B_T)$. Then

$$\alpha_i \omega_i^{2d_i} + \beta_i \omega_i^{d_i} + \gamma_i = \Omega$$

$$c_i \beta_i \omega_i^{d_i - r} = \Gamma \quad \text{and} \quad \gamma_i = 0.$$

Accordingly

$$\alpha_i \omega_i^{d_i} + \frac{1}{c_i}\left(\prod_{k \neq i} \omega_k^{d_k - r}\right)\left(\prod_j \nu_j^{k_j}\right) = \prod_{k \neq i} \omega_k^{d_k}$$

Letting $\alpha_i = \delta_i \prod_{k \neq i} \omega_k^{d_k - r}$

$$c_i \delta_i \omega_i^{d_i} + \prod_j \nu_j^{k_j} = c_i \prod_{k \neq i} \omega_k^r \tag{2}$$

We claim that (2) is possible only if

$$\sum_i c_i \prod_{k \neq i} \omega_k^r = \prod_j \nu_j^{k_j} \quad \text{and}$$

(2)

$$c_i \delta_i \omega_i^{d_i - r} = -\sum_{m \neq i} c_m \prod_{k \neq m,i} \omega_k^r \qquad \text{for each } i,$$

which is the assertion of the theorem.

To prove the claim, let s be the number of ω_i. Let c_i, ω_i be thought of as fixed and let $A_i \in H^{2(s-2)r}(B_T)$ and $B \in H^{2(s-1)r}(B_T)$ be any solution to

$$A_i \omega_i^r + B = c_i \prod_{k \neq i} \omega_k^r \qquad i = 1, \ldots, s. \tag{3}$$

If A_i', B' is another solution to (3), then

$$(A_i - A_i')\,\omega_i^r + (B - B') = 0 \qquad i = 1, \ldots, s.$$

Therefore $B - B'$ is divisible by $\omega_1^r, \ldots, \omega_s^r$ and since $H^*(B_T)$ is a polynomial ring, a degree argument yields $B = B'$, so $A_i = A_i'$. Now

$$B = \sum_m c_m \prod_{k \neq m} \omega_k^r \quad \text{and} \quad A_i = -\sum_{m \neq i} c_m \prod_{k \neq m, i} \omega_k^r$$

is one solution to (3), so (2) follows.

Remark 4. The proof of Theorem 3 remains true for the T-equivariant map $f\colon F(H, S^{4d-1}) \to F(H, S^{2d})$ for any subtorus H whose actions are of type (i). Accordingly, equations (1) hold when $\{\omega_i\}$ and $\{\nu_j\}$ are the weights of type ω and ν respectively which are zero on H, that is $H \subseteq \omega_i^\perp$ and $H \subseteq \nu_j^\perp$.

3. COROLLARY 5. *Suppose an equivariant $f\colon S^{4d-1} \to S^{2d}$ of nonzero Hopf invariant exists where the T actions are of type* (ii). *If ν_0 is a weight of type ν with multiplicity k_0 and ω_0 is a weight of type ω with multiplicity $2d_0$, then the linear span $L(\nu_0, \omega_0)$ of ν_0 and ω_0 has at least r weights of type ν and* $\max(d_0 - r + 2, k_0 + 1)$ *weights of type ω. If ω_0 and ω_1 are two weights of type ω, there is a weight ν_0 of type ν in $L(\omega_0, \omega_1)$, so $L(\omega_0, \omega_1) = L(\omega_0, \nu_0)$.*

Proof. Let H be the connected component of 1 in $\nu_0^\perp \cap \omega_0^\perp$. Then a weight $\mu \in L(\nu_0, \omega_0)$ if and only if $H \subseteq \mu^\perp$. By Remark 4, if $\omega_0, \ldots, \omega_m$ and $\nu_0, \ldots, \nu_n$ are the weights of type ω and ν in $L(\nu_0, \omega_0)$

(4) $$\sum_{i=0}^{m} c_i \prod_{k \neq i} \omega_k^r = \prod_{j=0}^{n} \nu_j^{k_j}$$

(5) $$\sum_{j=1}^{m} c_j \prod_{k \neq 0, j} \omega_k^r \text{ is divisible by } \omega_0^{d_0 - r}$$

Write each weight ω_i (after multiplying by a suitable nonzero constant) in the form $\omega_i = \nu_0 + a_i \mu$, where $\mu \in \operatorname{Lin}(\omega_0, \nu_0)$ is not a weight of type ω. Then (4) implies

$$f(t) = \frac{\sum_{i=0}^{m} c_i \prod_{k \neq i} (t + a_k)^r}{\prod_i (t + a_i)^r} = \sum_{i=0}^{m} \frac{c_i}{(t + a_i)^r}$$

has a zero of order k_0 at $t = 0$. Accordingly

$$0 = f^{(k)}(0) = (-1)^k \frac{(r + k - 1)}{(r - 1)!} \sum_{i=0}^{m} c_i a_i^{-r-k}, \qquad 0 \leq k < k_0.$$

The Van der Monde determinant tells us the $k_0 \times (r + 1)$ matrix whose $(i + 1, j + 1)$th entry is a_i^{-j-r} has maximum possible rank. Since none of the c_i's is zero, $m + 1 \geq k_0 + 1$. Similarly equation (5) implies that $m \geq d_0 - r + 1$. Thus $m + 1 \geq \max(k_0 + 1, d_0 - r + 2)$ as claimed.

Let $K = \max_{0 \leq i \leq n} k_i$. Then (4) implies $m \cdot r \leq K(n+1)$. Since $m \geq K$, $n+1 \geq r$. Finally applying Remark 4 to the connected component of 1 in $\omega_0^\perp \cap \omega_1^\perp$, we see that $L(\omega_0, \omega_1)$ must contain some weight of type ν.

COROLLARY 6. *Let T act almost effectively on either S^{4d-1} or S^{2d} and suppose there is an equivariant $f: S^{4d-1} \to S^{2d}$ with nonzero Hopf invariant.*

(i) *If the actions of T are type* (i), *then* rank $T \leq d - r$, *except when $r = 0$ and $d = 1,2$. If $(d,r) = (1,0)$,* rank $T \leq 2$; *if $(d,r) = (2,0)$,* rank $T \leq 3$.

(ii) *If the actions of T are type* (ii), *then* rank $T \leq 3$ *if $r = 1$ or 2,* rank $T \leq 2$ *if $r \geq 3$.*

Proof. (ii) First of all, the weights of type ω and ν together span $H^2(B_T)$. For letting H be the connected component of 1 in $\bigcap_i \omega_i^\perp \cap \bigcap_j \nu_j^\perp$, we have by the Borel formula $\dim F(H, S^{4d-1}) = 4d - 1$ and $\dim F(H, S^{2d}) = 2d$. Using [1, pp. 13, 81], $F(H, S^{4d-1}) = S^{4d-1}$ and $F(H, S^{2d}) = S^{2d}$. Therefore $H = \{1\}$.

Now if rank $T \geq 3$, a theorem of Sylvester-Gallai ([7, p. 451]) implies that there are two weights ω_0 and ω_1 of type ω so that $L(\omega_0, \omega_1)$ has no other weight of type ω. Using Remark 4, there are rational constants c_0 and c_1 so that $c_1 \omega_0^r + c_0 \omega_1^r$ splits completely in $L(\omega_0, \omega_1)$. This is equivalent to $c_1 x^r + c_0$ splitting completely in $Q[x]$. This latter is only possible if $r = 1,2$.

If $r = 1,2$ another application of Sylvester-Gallai's theorem together with Corollary 4 shows that rank $T \leq 3$ ([8, p. 10]).

(i) As in (ii) above, the weights span $H^2(B_T)$. Using Proposition 2, if $r > 0$ the weights for the T action on S^{2d} span $H^2(B_T)$, so let $\omega_1, \ldots, \omega_s$, $s = \text{rank } T$, be a collection of linearly independent weights. Then

$$F(L(\omega_1, \ldots, \omega_s)^\perp) \subseteq F(L(\omega_1, \ldots, \omega_{s-1})^\perp) \subseteq \ldots \subseteq F(L(\omega_1, \omega_2)^\perp)$$
$$\subseteq F(\omega_1^\perp) \subseteq S^{2d}$$

is a sequence of spaces whose dimension, by the Borel formula, increases by at least 2 at each step. Hence $s \leq d - r$.

If $r = 0$, let μ be any weight for the T action on S^{2d}. Then applying (i) or (ii) to the $\mu^\perp$ actions on S^{4d-1} and S^{2d}, we see that (rank T) $- 1$ is at most equal to the larger of $d - 1$ and 3. Thus (i) is proven except when $d = 1,2$, or 3 and $r = 0$. These cases are easily handled separately.

4. This section deals with examples. The first two are summarized from [4]; the fourth is a variation of an example given there.

Example 7. Let $SO(2d)$ act on S^{2d} and on D^{2d} linearly in the usual way. The map $h: D^{2d} \to S^{2d} = D^{2d}/S^{2d-1}$ which collapses S^{2d-1} into a fixed point of S^{2d} is equivariant. Let $SO(2d)$ act on $D^{2d} \times D^{2d}$ by the diagonal action. The map $f: S^{4d-1} = \partial(D^{2d} \times D^{2d}) = S^{2d-1} \times D^{2d} \cup D^{2d} \times S^{2d-1} \to S^{2d}$ defined by $F(x,y) = h(x)$ on $D^{2d} \times S^{2d-1}$ and $f(x,y) = h(y)$ on $S^{2d-1} \times D^{2d}$ is equivariant and has Hopf invariant two. If T^d is a maximum torus of $SO(2d)$, its rank is d, and subtori of T^d can be found acting with all the possibilities for r of type (i) actions, and with the maximum ranks given by Corollary 6.

Example 8. If $0 \le r \le d$, there is a map $S^{4d-1} \to S^{2d}$ with nonzero Hopf invariant which is equivariant with respect to a linear free S^1 action on S^{4d-1} and a linear S^1 action on S^{2d} fixing S^{2r}.

If $0 < r \le d$, there is a map $S^{4d-1} \to S^{2d}$ with nonzero Hopf invariant which is equivariant with respect to a linear semifree S^1 action on S^{4d-1} fixing S^{2r-1} and a linear S^1 action on S^{2d} fixing S^0.

Example 9. Let $\omega_1, \ldots, \omega_d$ be distinct integers. Let $c_1, \ldots, c_d$ be integers so that $\sum c_i \omega_i^s = 0$, $0 \le s \le d-2$, and $\sum c_i \omega_i^{d-1} \ne 0$. The existence of such integers is guaranteed by the nonsingularity of the $d \times d$ matrix $(a_{ij}) = (\omega_i^j)$, $i = 1, \ldots, d$, $j = 0, \ldots, d-1$. Then $\sum_i c_i t^{\omega_i} = p(t)(t-1)^{d-1}$ where $p(1) \ne 0$.

Let $\varphi\colon S^1 \to U(d)$ be the map $\varphi(z) = \operatorname{diag}(z^{\omega_1}, \ldots, z^{\omega_d})$ where $\operatorname{diag}(a_1, \ldots, a_d)$ is the $d \times d$ diagonal matrix with entries $a_1, \ldots, a_d$. Let S^1 act on $U(d)$ by inner automorphism via φ, so $z \cdot A = \varphi(z) A \varphi(z^{-1})$. Let S^1 act on S^{2d-1} linearly and semifreely fixing S^1; denote this action by $z * x$ for $z \in S^1$ and $x \in S^{2d-1}$. Let $z \cdot x$ denote complex scalar multiplication by z on $x \in C^d$. By [3, Theorem 1], the map $g\colon S^1 \to U(d)$ defined by $g(z) = \operatorname{diag}(z^{c_1}, \ldots, z^{c_d})$ extends to an equivariant map, also denoted by g, $S^{2d-1} \to U(d)$ with $[g] \ne 0$ in $\prod_{2d-1}(U(d)) = Z$.

Now let $f\colon S^{2d-1} \times S^{2d-1} \to S^{2d-1}$ be given by $f(x,y) = g(x)y$. Then if $S^1 \times S^1$ acts on $S^{2d-1} \times S^{2d-1}$ by $(z_1, z_2)(x,y) = (z_1 * x, z_2 \varphi(z_1) y)$ and on S^{2d-1} by $(z_1, z_2)x = z_2 \varphi(z_1)x$ it follows that f is equivariant. Applying the Hopf construction gives a map $f\colon S^{4d-1} = S^{2d-1} * S^{2d-1} \to SS^{2d-1} = S^{2d}$ which is equivariant with respect to a $S^1 \times S^1$ action on S^{4d-1} fixing S^1 and an $S^1 \times S^1$ action on S^{2d} fixing S^0. Since $[g] \ne 0$ in $\prod_{2d-1}(U(d))$, the Hopf invariant of f is not zero.

Suppose there exist two sets of d integers whose first $d-1$ symmetric functions agree, but whose d-th symmetric functions disagree. If $d \le 10$, two such sets do exist but if $d > 10$ the existence of such a pair of sets is completely unknown. Using Theorem 1′ of [3], we can construct a map $S^{4d-1} \to S^{2d}$ of nonzero Hopf invariant which is equivariant with respect to a fixed point free $S^1 \times S^1$ action on S^{4d-1} and a $S^1 \times S^1$ action on S^{2d} fixing S^0. Such an example is constructed in Example 7, but in Example 7 all the subtori of T have actions which are of type (i). In this example, however, there is a $S^1 \subseteq S^1 \times S^1$ which fixes S^{2d-1} in S^{4d-1} and S^0 in S^{2d}.

Example 10. Let $T^4 = S^1 \times S^1 \times S^1 \times S^1$ and let $\varphi_1, \varphi_2, \varphi_3$ be maps $T^4 \to U(4)$ defined by

$$\varphi_1(z_1, z_2, z_3, z_4) = \operatorname{diag}(z_1 z_2 z_3 \bar z_4, z_1 z_2 \bar z_3 z_4, z_1 \bar z_2 z_3 z_4, \bar z_1 z_2 z_3 z_4)$$

$$\varphi_2(z_1, z_2, z_3, z_4) = \operatorname{diag}(z_1 \bar z_2 \bar z_3 z_4, \bar z_1 z_2 \bar z_3 z_4, \bar z_1 \bar z_2 z_3 z_4, z_1 z_2 z_3 z_4)$$

$$\varphi_3(z_1, z_2, z_3, z_4) = \operatorname{diag}(z_1^2, z_2^2, z_3^2, z_4^2)$$

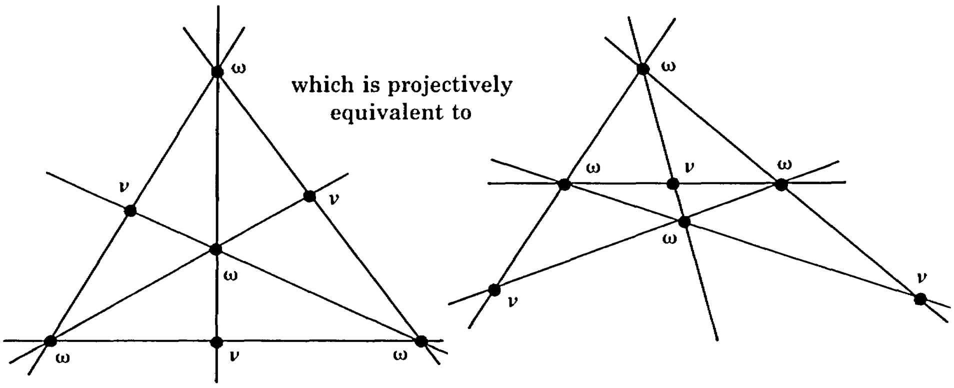
Let $f\colon S^7 \times S^7 \to S^7$ be Cayley multiplication which is given in terms of 4-tuples of complex numbers by

$$f\left(\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}, \begin{bmatrix} r'_1 \\ r'_2 \\ r'_3 \\ r'_4 \end{bmatrix}\right) = \begin{bmatrix} r_1 r'_1 - r_2 \bar{r}'_2 - \bar{r}'_3 r_3 - r'_4 \bar{r}_4 \\ r_2 \bar{r}'_1 + r_1 r'_2 - \bar{r}'_3 r_4 + r'_4 \bar{r}_3 \\ r'_3 r_1 - r'_4 \bar{r}_2 + r_3 \bar{r}'_1 - r_4 \bar{r}'_2 \\ r'_3 r_2 - r'_4 \bar{r}_1 + r_4 r'_1 - r_3 r'_2 \end{bmatrix}$$

Then f is equivariant with respect to the T^4 actions on $S^7 \times S^7$ and on S^7 given by $z \cdot (x,y) = (\varphi_1(z)x, \varphi_2(z)y)$ and $z \cdot x = \varphi_3(z)x$. Accordingly, the Hopf construction yields an equivariant map $S^{15} \to S^8$ with Hopf invariant 1.

Taking various subtori of T will yield actions of type (ii) with $r = 1,2,$ or 3. For example, the subtorus $H = \{(z_1, z_2, z_3, z_4)\colon z_1 z_2 z_3 \bar{z}_4 = 1\}$ has $F(H,S^{15}) = S^1$ and $F(H,S^8) = S^0$. Thus rank $T = 3$ is possible in (ii) of Corollary 6.

Since the weights of a torus action are elements of $H^2(B_T;Q)$ defined only up to multiplication by a nonzero rational, they are naturally considered as points in a projective space over the rationals of dimension rank $T - 1$. The weights of type ν and ω for the H actions of Example 10 have the following configuration in the projective plane.



(4)

Figure 1.

For any torus action of type (ii), the weights Ω of type ω and N of type ν satisfy:

1. The sets Ω and N are disjoint, finite, and not all colinear.
2. Given $\omega_1, \omega_2 \in \Omega$ there is a weight of type ν in $L(\omega_1, \omega_2)$.
3. Given $\omega_1 \in \Omega$ and $\nu \in N$, there is a $\omega_2 \neq \omega_1$ of type ω in $L(\omega_1, \nu_1)$.

It is known [8] that there are no such configurations of points Ω and N in a projective 3 space. In the projective plane, the only configuration with rational coordinates known is the configuration (4). There is a configuration constructed by Bredon from a regular pentagon with real coordinates. Should (4) be the only possible configuration in the projective plane over the rationals, Corollary 5 can be used to strengthen (ii) of Corollary 6 and arrive at: Under the hypotheses of Corollary 6, (ii)$'$ If the T actions are of type (ii) then rank $T \leq 2$ with the sole exception of $r = 1$, $d = 4$, rank $T = 3$, which is constructible as in Example 10.

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