A STRONGER INVARIANT FOR HOMOLOGY THEORY

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1. INTRODUCTION

In this paper we show that in any homology theory which satisfies the Eilenberg-Steenrod axioms, the homology groups for compact polyhedral pairs satisfy an invariance much stronger than homotopy type invariance; it is called m-homotopy type invariance. The simplest example is the torus T^2 and the wedge of spheres $S^2 \vee S^1 \vee S^1$, which do not have the same homotopy type but do have the same m-homotopy type; therefore, they must have the same homology groups. This is a special case of Theorem 3.8, which begins to classify spaces by m-homotopy type.

The proof uses certain multiple valued functions which we have called m-functions. An m-function is finite valued, and each point of its graph is assigned a multiplicity which is an element of a fixed ring. The multiplicities satisfy an additivity condition which insures that locally as well as globally, the multiplicity is conserved with respect to variations in the domain variable.

M-functions were used in [5] to describe the intersections of two smooth simple closed curves in general position in the plane. As one curve undergoes a homotopy, intersections appear and disappear; one gets a weighted multiple valued function which associates with each homotopy parameter value a finite number of intersections, each labeled $+1$, -1 or zero depending on the orientation of the intersection.

This situation occurs again in studying fixed points, for one is looking for intersections of the graph of a function $f: X \to X$ with the diagonal of the space $X \times X$. Given a homotopy $f_t: X \to X$ one obtains an m-function $g: I \to X$ in which the points of $g(t)$ are the fixed points of f_t and their multiplicities are the degrees of the fixed points.

One can construct m-functions that are fundamentally different from any continuous function. For example, as part of the m-homotopy equivalence mentioned above we have an m-function from S^2 to T^2 that can be described as follows. If one puts a two-sphere in the (hollow) interior of a torus, there is a projection from T^2 onto S^2. The inverse of this projection is an m-function; a graph point has multiplicity $+1$, -1, or zero depending on how the radial ray from the sphere center intersects the torus at the point. Unlike any continuous function $S^2 \to T^2$ this m-function has degree one and is not null-homotopic. Another difference is that m-functions do not behave well under products; diagrams involving products may not commute, and there is no cup product in m-homology.

It is not difficult to do homology with m-functions [1]. The m-homology theory, together with some applications to fixed points of continuous functions, is also

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given in [6]. The innovation consists in defining a singular simplex to be an m-function rather than a continuous function. The multiplicity plays the role of the coefficient ring in ordinary homology. This is an extension of the usual theory since the product of a continuous function and a coefficient α can be regarded as an m-function of multiplicity α. One obtains the Eilenberg-Steenrod axioms, but with the added result that m-homology is an m-homotopy invariant. Since homology groups for compact polyhedra are unique, it follows that their homology and m-homology groups are the same. More significantly, the usual homology groups must be m-homotopy type invariants. The purpose of this paper is to show that m-homotopy type invariance is indeed stronger than homotopy type invariance.

The author found, subsequent to the publication of [6], that G. Darbo gave a theory of weighted maps which are the same as m-functions in an alternative form. In [1], [2] and [3] he developed the homology theory, did coincidence theory over manifolds, and proved a Lefschetz fixed point theorem for weighted maps in the category of compact ANR's. In particular he showed that homology groups are m-homotopy type invariants. However, his papers do not consider the strength of this invariance.

2. M-FUNCTIONS

Suppose that $\tilde{f}: X \times Y \to R$ is a (standard) function, where X and Y are Hausdorff spaces and R is a ring with identity and without zero divisors. Then $\tilde{f}$ defines a multiple-valued function $f': X \to Y$ by $f' = c\ell\{(x,y) \in X \times Y: \tilde{f}(x,y) \neq 0\}$. Suppose also that $\tilde{f}$ satisfies the conditions:

(1) for all $x \in X$, $f'(x)$ is a finite or empty subset of Y;

(2) if $f'(x') = \{y_1, y_2, \ldots, y_n\}$ there exist disjoint neighborhoods $\hat{V}_i(y_i)$ such that for any neighborhoods $V_i(y_i) \subset \hat{V}_i$ there is a neighborhood $U(x')$ satisfying:

(a) $\sum_{y \in V_i} \tilde{f}(x,y) = \tilde{f}(x', y_i)$ for $x \in U$, $i = 1, 2, \ldots n$

(b) $\tilde{f}(x,y) = 0$ for $x \in U$ and $y \in \left[Y - \bigcup_{i=1}^{n} V_i \right]$;

(3) if $f'(x') = \emptyset$ there exists a neighborhood $U(x')$ such that $\tilde{f}(x,y) = 0$ for all $x \in U$, $y \in Y$.

Definition 2.1. Under the conditions above, f is the *defining function* of the multiple valued function $f: X \to Y \times R$ given by

$$f = \{(x,(y,r)): y \in f'(x) \text{ and } \tilde{f}(x,y) = r\},$$

and f is called an *m-function* from X to Y (the ring R is usually fixed and dropped from the notation).

If $\tilde{f}(x,y) = r$, then r is called the multiplicity of (x,y). The *multiplicity of* f at x is $m_x(f) = \sum_{y \in Y} \tilde{f}(x,y)$. It is constant on each component of X and we say that f has multiplicity m(f) if X is connected. Note that f′ and f may have a proper subset of X as domain. In fact, if $\tilde{f}(x,y) = 0$ for all x and y then f defines the empty m-function.

The image of a point of X under an m-function is a number of points of Y, each tagged with a multiplicity from R. As x varies, its images move around in Y. Two or more points may coalesce, each contributing its multiplicity to the point of coalescence. Two points of equal and opposite multiplicity may coalesce at a point of zero multiplicity and then disappear. A point of zero multiplicity may appear and then divide into a number of points whose multiplicities sum to zero.

If $f: X \to Y \times R$ and $g: Y \to Z \times R$ are m-functions, their *composition* $g \circ f: X \to Z \times R$ is given by the defining function $\tilde{g} \tilde{\circ} \tilde{f}(x,z) = \sum_{y \in Y} \tilde{f}(x,y)\tilde{g}(y,z)$. The composition of m-functions is an m-function, composition is associative, and $m(g \circ f) = m(f)m(g)$. We can regard a continuous function $f: X \to Y$ as an m-function of multiplicity one, and the identity function $1_X: X \to X$ serves as an identity for m-functions. Thus Hausdorff spaces and m-functions over the ring R form a category M_R.

If $f,g: X \to Y \times R$ are m-functions, we define their *sum* $(f + g): X \to Y \times R$ by $\tilde{f} \tilde{+} \tilde{g}(x,y) = \tilde{f}(x,y) + \tilde{g}(x,y)$. If $a \in R$, we define the *product* m-function $(af): X \to Y \times R$ by $\tilde{a}\tilde{f}(x,y) = a\tilde{f}(x,y)$. We have $m(f+g) = m(f) + m(g)$ and $m(af) = am(f)$. With this addition and multiplication, the set $\mathrm{Hom}_R(X,Y)$ of m-functions from X to Y over R forms an R-module. The zero element under addition is the empty m-function $0: X \to Y \times R$ defined by $\tilde{0}(x,y) = 0$ for all $x \in X$, $y \in Y$. An m-function on pairs of Hausdorff spaces, used in the next section, is defined as follows: $h: (X,A) \to (Y,B) \times R$ is an m-function on pairs if $h'(A) \subset B$.

3. M-HOMOTOPY

We consider the category of pairs of Hausdorff spaces and m-functions; the pair $(X \times I, A \times I)$ is denoted by $(X,A) \times I$. Suppose that $X' \subset X$ and that two m-functions $f_0, f_1: (X,A) \to (Y,B)$ agree on X' ($f_0|X' = f_1|X'$).

Definition 3.1. We say that f_0 is m-*homotopic* to f_1 relative to X' ($f_0 \sim_m f_1 \operatorname{rel} X'$) if there exists an m-function $F: (X,A) \times I \to (Y,B)$ with $F(x,0) = f_0(x)$, $F(x,1) = f_1(x)$, and $F|X' \times \{t\} = f_0|X' = f_1|X'$ for $t \in [0,1]$.

PROPOSITION 3.2. *M-homotopy relative to* X' *is an equivalence relation on the set of* m-*functions from* (X,A) *to* (Y,B).

The proof is conventional, requiring only a slight modification to deal with m-functions.

PROPOSITION 3.3. *Compositions of* m-*homotopic maps are* m-*homotopic.*

Proof. We assume that we have m-homotopic m-functions

$$\begin{cases} f_0 \sim_m f_1 & \text{by } F\colon (X,A) \times I \to (Y,B) \\ g_0 \sim_m g_1 & \text{by } G\colon (Y,B) \times I \to (Z,C) \end{cases}$$

Define the m-function $p_2\colon (X,A) \times I \to I$ by $\tilde{p}_2\colon (X,A) \times I \times I \to R$, given by

$$\tilde{p}_2(x,t,s) = \begin{cases} 1 & t = s \\ 0 & t \neq s \end{cases}$$

The required m-homotopy $g_0 \circ f_0 \sim_m g_1 \circ f_1$ is then the composition

$$(X,A) \times I \xrightarrow{F \times p_2} (Y,B) \times I \xrightarrow{G} (Z,C)$$

or $H = G \circ (F \times p_2)$. We find that

$$\begin{aligned} \tilde{H}(x,t,z) &= \sum_{y,s} \tilde{F}(x,t,y)\, \tilde{p}_2(x,t,s)\, \tilde{G}(y,s,z) \\ &= \sum_{y} \tilde{F}(x,t,y)\, \tilde{G}(y,t,z). \end{aligned}$$

Substitution of $t = 0$ and $t = 1$ gives the desired result.

PROPOSITION 3.4. *If* $f_0 \sim_m f_1\colon (X,A) \to (Y,B)$ *and* $g_0 \sim_m g_1\colon (X,A) \to (Y,B)$, *then* $(f_0 + g_0) \sim_m (f_1 + g_1)$. *Also* $af_0 \sim_m af_1$ *for* $a \in R$.

Proof. We are given two m-homotopies

$$F, G\colon (X,A) \times I \to (Y,B).$$

The required m-homotopy is

$$F + G\colon (X,A) \times I \to (Y,B),$$

$$\begin{aligned} \text{for } (F+G)|(X,A) \times \{0\} &= F|(X,A) \times \{0\} + G|(X,A) \times \{0\} \\ &= f_0 + g_0, \end{aligned}$$

and a similar formula holds for $t = 1$. The second statement in the proposition is obvious.

As a corollary to these propositions we have

THEOREM 3.5. *There is a category* MH_R *whose objects are pairs of Hausdorff spaces and whose morphisms are* m-*homotopy classes of* m-*functions over a ring* R. *Further, as in* M_R *the hom-sets of* MH_R *form an* R-*module.*

Definition 3.6. An m-function f: $(X,A) \to (Y,B)$ is an m-*homotopy equivalence* if [f] is an equivalence in MH_R. Two spaces have the same m-*homotopy type* if there is an m-homotopy equivalence between them.

Clearly, if $h_0 \sim_m h_1$, then $m(h_0) = m(h_1)$, for they are both restrictions of the same m-function. Therefore $h \sim_m 1_X$ implies that $m(h) = 1$. Consequently, if g is an m-homotopy inverse of f, then $g \circ f \sim_m 1_X$ and $m(g) = 1/m(f)$; note that $1/m(f)$ must then be an element of the ring R.

To illustrate the notion of m-homotopy as well as to provide a result that will be useful later we consider certain maps of spheres. Suppose that S^n is the standard n-sphere with unit radius, centered at the origin in E^{n+1}. The north and south poles are denoted by N and S; the coordinate axis containing these poles, with N at +1, is the z-axis. The closed northern and southern hemispheres are denoted by S^n_+ and S^n_-. We define certain functions. r: $S^n \to S^n$ is reflection in the equatorial n-flat; the image of x is obtained by changing the sign of the z-coordinate of x. c_x: $S^n \to S^n$ is the constant map which takes all points of S^n onto the point x.

PROPOSITION 3.7. $1_{S^n} + r \sim_m 2c_N$.

Here we regard these functions as m-functions; the addition and multiplication are in the R-module $\mathrm{Hom}_R(S^n, S^n)$. The integer 2 is the sum of two units in the ring.

Proof. We first show that $2c_N \sim_m c_N + c_S$ by an m-homotopy Φ: $S^n \times I \to S^n$. To define Φ we choose a path α: $I \to S^n$ with $\alpha(0) = N$, $\alpha(1) = S$. The defining function of Φ is then

$$\tilde{\Phi}: S^n \times I \times S^n \to R$$

$$\tilde{\Phi}(x,t,y) = \begin{cases} 2 & \text{if } y = N, t = 0 \\ 1 & \text{if } y = N\ t > 0 \\ 1 & \text{if } y = \alpha'(t), t > 0 \\ 0 & \text{otherwise} \end{cases}$$

Second, by an ordinary homotopy one has $c_N \sim f$, $c_S \sim g$ where

$$\begin{cases} f \,|\, S^n_+ = 1 \,|\, S^n_+ \\ f \,|\, S^n_- = r \,|\, S^n_- \end{cases} \quad \text{and} \quad \begin{cases} g \,|\, S^n_+ = r \,|\, S^n_+ \\ g \,|\, S^n_- = 1 \,|\, S^n_- \end{cases}.$$

Therefore $2c_N \sim_m f + g$, regarding f and g as m-functions, and by their definitions, $f + g = 1_{S^n} + r$.

To perform surgery on an n-manifold one customarily excises a subset homeomorphic to $S^{i-1} \times D^{n-i+1}$ and attaches a copy of $D^i \times S^{n-i}$ $(i = 1, 2, \ldots, n)$; these two sets have homeomorphic boundaries and the attaching map is essentially the identity. We shall say that we have added an i-handle by *conservative surgery* if the excised set $s^{i-1} \times D^{n-i+1}$ is contained in a disc D^n embedded in the manifold.

THEOREM 3.8. *If the manifold* $\hat{M}^n$ *is obtained from* M^n *by adding an* i*-handle by conservative surgery then* $\hat{M}^n \sim_m M^n \vee S^i \vee S^{n-i} = M'$.

Proof. We shall construct $\hat{M}$ from M by removing a disc, and then glueing on three discs in a certain way to replace it. The example to bear in mind below

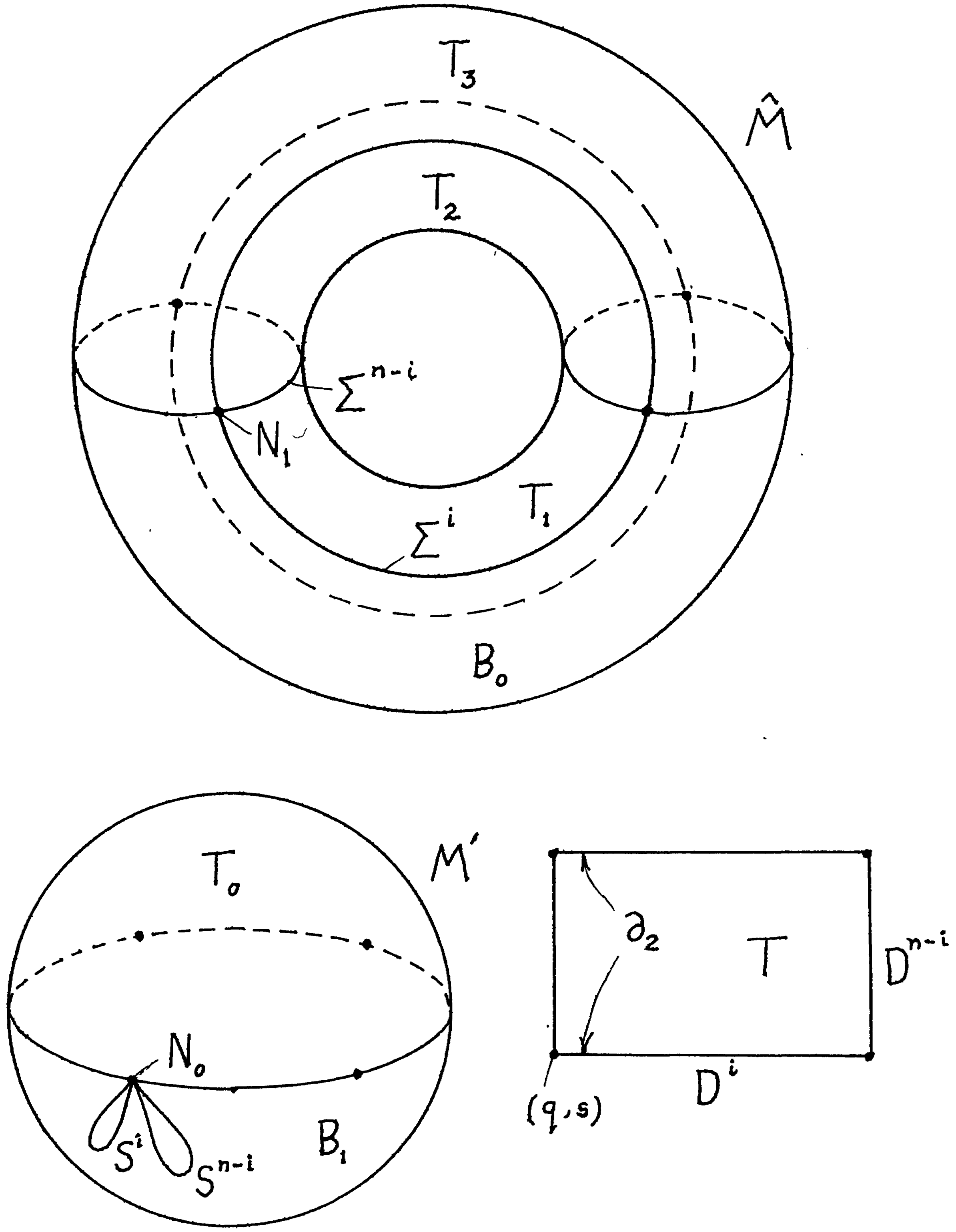

Figure 1 $\hat{M} \sim_m M'$

is the case where M is a 2-sphere, $\hat{M}$ is a torus, and S^i, S^{n-i}, Σ^1 and Σ^{n-i} are all circles. Here the "bottom," B, is simply a disc. In the general two-dimensional case B is a 2-manifold with a disc cut out.

We first remove a disc $T = D^i \times D^{n-1}$ from M to obtain $B = \mathrm{cl}(M - T)$. We use the notation

$$\begin{aligned}\partial B &= \partial T = (S^{i-1} \times D^{n-i}) \cup (D^i \times S^{n-i-1})\\ \partial_1 B &= \partial_1 T = S^{i-1} \times D^{n-i}\\ \partial_2 B &= \partial_2 T = D^i \times S^{n-i-1}.\end{aligned}$$

We now construct two copies, B_0 and B_1, of B, and four copies T_0, T_1, T_2, T_3 of T. We define $\hat{M}$ by glueing together some of these copies by

$$\hat{M} = (B_0 + T_1 + T_2 + T_3)/R_1$$

where the numerator represents the disjoint union and R_1 is the equivalence relation generated by the pointwise identifications

$$\partial_1 T_1 = \partial_1 B_0, \quad \partial_2 T_1 = \partial_2 T_2, \quad \partial_1 T_2 = \partial_1 T_3, \quad \partial_2 T_3 = \partial_2 B_0.$$

The original manifold is described by $M = (B_1 + T_0)/R_0$ where R_0 is the equivalence relation generated by $\partial_1 B_1 = \partial_1 T_0$, $\partial_2 B_1 = \partial_2 T_0$, which simply glues together a copy of the original manifold.

We must define the various maps used to construct m-homotopy equivalences. We first write $T_j = D_j^i \times D_j^{n-i}$ $(j = 0,1,2,3)$. Selecting a point

$$(q, s) \in \partial D^1 \times \partial D^{n-i} \subset \partial T = \partial B,$$

we have corresponding points $(q_j, s_j) \in \partial T_j$. From this point on we shall regard B_0, T_1, T_2, T_3 as subsets of $\hat{M}$ and B_1, T_0 as subsets of M, joined along their boundaries by the equivalence relations described above. Following this convention $(q_1, s_1) = (q_2, s_2) = (q_3, s_3) \in \hat{M}$. We now choose certain discs in $\hat{M}$ to be regarded as hemispheres:

$$\Sigma_W^i = D_1^i \times \{s_1\}, \Sigma_E^i = D_2^i \times \{s_2\} = D_3^i \times \{s_3\},$$

$$\Sigma_E^{n-i} = \{q_3\} \times D_3^{n-i}, \quad \Sigma_W^{n-i} = \{q_1\} \times D_1^{n-i} = \{q_2\} \times D_2^{n-i},$$

where the equalities on the right stem from the glueing of $\hat{M}$. The subscripts are chosen to suggest eastern and western hemispheres of spheres which we denote by

$$\begin{aligned}\Sigma^i &= \Sigma_E^i \cup \Sigma_W^i\\ \Sigma^{n-i} &= \Sigma_E^{n-i} \cup \Sigma_W^{n-i},\end{aligned}$$

both with north pole at $(q_1, s_1) = N_1$.

The manifold $M \vee S^i \vee S^{n-i} = M'$ is formed by attaching S^i and S^{n-i} to M at the point $(q_0, s_0) = N_0$, which is taken to be the north pole of both spheres. We select homeomorphisms

$$h_1: S^i \to \Sigma^i,\ h_2: S^{n-i} \to \Sigma^{n-i},$$

taking N_0 to N_1, and use these maps to define eastern and western hemispheres of S^i, S^{n-i} (*e.g.* $S^i_E = h_1^{-1}(\Sigma^i_E)$).

We now define the following functions.

(a) There are identity-like homeomorphisms

$$t_{jk}: T_j \to T_k \qquad (j,k = 0,1,2,3)$$
$$b_{jk}: B_j \to B_k \qquad (j,k = 0,1)$$

which are the obvious maps (identity if $j = k$) between copies of T and copies of B. Clearly $t_{jk} = t_{kj}^{-1}$, $b_{jk} = b_{kj}^{-1}$.

(b) There are projections

$$p_{j1}: T_j \to D^i_j \times \{s_j\} = \begin{cases} \Sigma^i_E & (j = 2,3) \\ \Sigma^i_W & (j = 1) \end{cases}$$

$$p_{j2}: T_j \to \{q_j\} \times D^{n-i}_j = \begin{cases} \Sigma^{n-i}_E & (j = 3) \\ \Sigma^{n-i}_W & (j = 1,2). \end{cases}$$

(c) There exist retractions

$$p_{01}: B_0 \to D^i_1 \times \{s_1\} = \Sigma^i_W$$
$$p_{02}: B_0 \to \{q_3\} \times D^{n-i}_3 = \Sigma^{n-i}_E.$$

The first may be obtained by retracting $\hat{M}$ onto the disc T_1 and following this by the projection p_{11}.

(d) From (b) and (c) we have maps

$$p_1: \hat{M} \to \Sigma^i;\ p_1 | B_0 = p_{01},\ p_1 | T_j = P_{j1} \qquad (j = 1,2,3)$$
$$p_2: \hat{M} \to \Sigma^{n-i};\ p_2 | B_0 = p_{02},\ p_2 | T_j = p_{j2} \qquad (j = 1,2,3)$$

If $\hat{M}$ is the torus, p_1 and p_2 are merely projections onto longitudinal and meridional circles.

(e) There is a reflection

$$r: \Sigma^i \to \Sigma^i.$$

This is the map described above in Proposition 3.7 but now it is a reflection which interchanges Σ^i_W with Σ^i_E.

From this point on, all of these maps will be regarded as m-functions of multiplicity one, and will often be represented by arrows; for example

$$t_{12} = (T_1 \to T_2),$$

$$b_{00} + p_{01} = (B_0 \to B_0 + \Sigma^i_W) = (B_0 \to B_0) + (B_0 \to \Sigma^i_W).$$

An m-function described by an arrow may have a sum of terms on the right-hand side with coefficients (± 1). The m-function is the sum of the indicated m-functions, with the multiplicity of each summand represented by its coefficient. We shall describe m-functions on $\hat{M}$, M' and M by their restrictions to the subsets $B_0, T_1, T_2, T_3 \subset \hat{M}$, B_1, T_0, S^i, $S^{n-i} \subset M'$, and B_1, $T_0 \subset M$. In particular we shall need

$$\pi = \begin{bmatrix} B_0 \to B_1 \\ T_1 \to T_0 \\ T_2 \to T_0 \\ T_3 \to T_0 \end{bmatrix} : \hat{M} \to M \subset M'$$

$$\beta = \begin{bmatrix} B_1 \to B_0 \\ T_0 \to T_1 - T_2 + T_3 \end{bmatrix} : M \to \hat{M}.$$

The latter is the first real m-function to appear in this section; it is triple-valued on T_0 and has multiplicity one. It is the key to the m-homotopy equivalence, for it is a "degree one" m-function of M onto $\hat{M}$ and no continuous function of this sort exists. We note that $\pi \circ \beta = 1_M$. It is easy but tiresome to verify that β is an m-function. Its defining function is:

$$x \in \operatorname{Int}(B_1), \tilde{\beta}(x,y) = \begin{cases} 1 & y = b_{10}(x) \\ 0 & \text{otherwise} \end{cases}$$

$$x \in \operatorname{Int}(T_0), \tilde{\beta}(x,y) = \begin{cases} 1 & y = t_{0j}(x), j = 1,3 \\ -1 & y = t_{02}(x) \\ 0 & \text{otherwise} \end{cases}$$

$$x \in \partial_1 T_0 - \partial_2 T_0, \tilde{\beta}(x,y) = \begin{cases} 1 & y = t_{01}(x) = b_{10}(x) \\ 0 & \text{otherwise} \end{cases}$$

$$x \in \partial_2 T_0 - \partial_1 T_0, \tilde{\beta}(x,y) = \begin{cases} 1 & y = t_{03}(x) = b_{10}(x) \\ 0 & \text{otherwise} \end{cases}$$

$$x \in \partial_1 T_0 \cap \partial_2 T_0, \tilde{\beta}(x,y) = \begin{cases} 1 & y = t_{01}(x) = t_{02}(x) = t_{03}(x) = b_{10}(x) \\ 0 & \text{otherwise} \end{cases}$$

The additivity condition must be verified. If $x = N_0$, its only image point is N_1, of multiplicity one. We can find neighborhoods U of N_0, $V = \beta'(U)$ of N_1 such that if x' is in $U \cap \operatorname{Int}(B_1)$ it has one image point in V of multiplicity one; if $x' \in \operatorname{Int}(T_0)$ it has three image points in V, two of multiplicity $+1$ and one of multiplicity -1; if $x' \in U \cap (\partial_1 T_0 - \partial_2 T_0)$ it has two image points in V, one in $\partial_1 T_1$ of multiplicity one, and one in $\partial_1 T_2 = \partial_1 T_3$ of multiplicity zero; if

$$x' \in U \cap (\partial_2 T_0 - \partial_1 T_0)$$

it has two image points in V, one in $\partial_2 T_3$ of multiplicity one, the other in $\partial_2 T_1 = \partial_2 T_2$ of multiplicity zero. In any case the sum of the multiplicities of the image points in V is one. Similar arguments hold for all possible locations of x, and it follows that β is an m-function.

We can now define the required m-homotopy equivalences $g\colon \hat{M} \to M'$, $h\colon M' \to \hat{M}$. They are given by

$$\begin{cases} g = \pi - h_1^{-1} r p_1 + h_2^{-1} p_2, \\ h | M = \beta, \quad h | S^i = h_1, \quad h | S^{n-i} = h_2, \end{cases}$$

Clearly g is an m-function of multiplicity one, for it is a sum of m-functions of suitable multiplicities. The m-function h, also of multiplicity one, is an extension of β over M'; the only point at which the domains of h_1, h_2 meet M is N_0, and it is routine to verify the m-function conditions there.

To demonstrate m-homotopy equivalence we first show that $h \circ g \sim_m 1_{\hat{M}}$. We have $h \circ g = \beta \circ \pi - rp_1 + p_2$ which we write

$$h \circ g = \begin{bmatrix} B_0 \to B_0 \\ T_1 \to T_1 - T_2 + T_3 \\ T_2 \to T_1 - T_2 + T_3 \\ T_3 \to T_1 - T_2 + T_3 \end{bmatrix} - rp_1 + p_2$$

Since $1_{\hat{M}} = [B_0 \to B_0, T_1 \to T_1, T_2 \to T_2, T_3 \to T_3]$, we have

$$h \circ g = 1_{\hat{M}} + \begin{bmatrix} B_0 \to \emptyset & \\ T_1 \to T_3 - T_2 & \\ T_2 \to T_3 - T_2 + T_1 - T_2 & \\ T_3 \to & T_1 - T_2 \end{bmatrix} - rp_1 + p_2.$$

Here the notation $[B_0 \to \emptyset]$ means the empty m-function 0, whose domain is B_0, whose range is $\hat{M}$, and whose graph is the empty set. We now define

$$\gamma = \begin{bmatrix} B_0 \to \emptyset \\ T_1 \to T_3 - T_2 \\ T_2 \to T_3 - T_2 \\ T_3 \to \emptyset \end{bmatrix} \qquad \delta = \begin{bmatrix} B_0 \to \emptyset \\ T_1 \to \emptyset \\ T_2 \to T_1 - T_2 \\ T_3 \to T_1 - T_2 \end{bmatrix},$$

which are both m-functions of multiplicity zero from $\hat{M}$ to $\hat{M}$. For example, under γ each point in $\mathrm{Int}(T_1 \cup T_2)$ has image points in T_3 and T_2 with multiplicities

+1 and −1; each point in $\partial(T_1 \cup T_2)$ has one image point in $\partial_1 T_2 \cap \partial_1 T_3$ with multiplicity zero.

We now have

$$h \circ g = 1_{\hat{M}} + \delta + \gamma - rp_1 + p_2,$$

and we claim that

$$\gamma \sim_m [B_0 \to \emptyset, T_1 \to \Sigma_E^{n-i} - \Sigma_W^{n-i}, T_2 \to \Sigma_E^{n-i} - \Sigma_W^{n-i}, T_3 \to \emptyset]$$

$$\delta \sim_m [B_0 \to \emptyset, T_1 \to \emptyset, T_2 \to \Sigma_W^i - \Sigma_E^i, T_3 \to \Sigma_W^i - \Sigma_E^i].$$

For the first of these there are obvious homotopies

$$\varphi_t : t_{33} \sim p_{32}, \qquad \Psi_t : t_{22} \sim p_{22}$$

obtained by taking strong deformation retractions of T_3 onto $\{q_3\} \times D_3^{n-i} = \Sigma_E^{n-i}$, T_2 onto $\{q_2\} \times D_2^{n-i} = \Sigma_W^{n-i}$. Then we can define the required m-homotopy Φ_t by

$$\Phi_t | (\mathrm{Int}(B_0) \cup \mathrm{Int}(T_3)) = 0, \quad \Phi_t | T_1 = t_{13}\varphi_t - t_{12}\Psi_t,$$
$$\Phi_t | T_2 = t_{23}\varphi_t - t_{22}\Psi_t.$$

A similar procedure works for δ.

Since

$$rp_1 = [B_0 \to \Sigma_E^i, T_1 \to \Sigma_E^i, T_2 \to \Sigma_W^i, T_3 \to \Sigma_W^i],$$
$$p_2 = [B_0 \to \Sigma_E^{n-i}, T_1 \to \Sigma_W^{n-i}, T_2 \to \Sigma_W^{n-i}, T_3 \to \Sigma_E^{n-i}],$$

we can put together these various results to find that

$$h \circ g \sim_m 1_{\hat{M}} + \left[\begin{array}{lllr} B_0 & \to & & -\Sigma_E^i + \Sigma_E^{n-i} \\ T_1 & \to & \Sigma_E^{n-i} - \Sigma_W^{n-i} & -\Sigma_E^i + \Sigma_W^{n-i} \\ T_2 & \to & \Sigma_E^{n-i} - \Sigma_W^{n-i} + \Sigma_W^i - \Sigma_E^i & -\Sigma_W^i + \Sigma_W^{n-i} \\ T_3 & \to & \Sigma_W^i - \Sigma_E^i & -\Sigma_W^i + \Sigma_E^{n-i} \end{array}\right.$$

There are several cancellations here, and we now need only show that

$$(\hat{M} \to \Sigma_E^{n-i} - \Sigma_E^i) \sim_m 0;$$

i.e., the remaining m-function is m-homotopic to the empty m-function. This m-function is a sum of two functions $(\hat{M} \to \Sigma_E^{n-i})$ and $(\hat{M} \to -\Sigma_E^i)$. Since the point N_1 is a strong deformation retraction of both Σ_E^i and Σ_E^{n-i} we see that the constant map $(\hat{M} \to N_1)$ is homotopic to both of these functions. Then

$$(\hat{M} \to \Sigma_E^{n-i} - \Sigma_E^i) \sim_m (\hat{M} \to N_1 - N_1) = 0$$

which shows that $h \circ g \sim_m 1_{\hat{M}}$ and concludes the first half of the proof.

Taking up $g \circ h$, we first write

$$d_0^i = \pi(\Sigma_W^i) = \pi(\Sigma_E^i) = D_0^i \times \{s_0\} \subset \partial_1 T_0,$$

$$d_0^{n-i} = \pi(\Sigma_E^{n-i}) = \pi(\Sigma_W^{n-i}) = \{q_0\} \times D_0^{n-i} \subset \partial_2 T_0.$$

We observe, for example, that $h_2^{-1} p_2(T_1) = S_W^{n-i}$. Using arrow notations we have

$$g \circ h = \begin{bmatrix} B_0 & \to B_1 - S_E^i + S_E^{n-i} \\ T_1 & \to T_0 - S_E^i + S_W^{n-i} \\ T_2 & \to T_0 - S_W^i + S_W^{n-i} \\ T_3 & \to T_0 - S_W^i + S_E^{n-i} \end{bmatrix} \circ \begin{bmatrix} T_0 \to T_1 - T_2 + T_3 \\ B_1 \to B_0 \\ S^i \to \Sigma^i \\ S^{n-i} \to \Sigma^{n-i} \end{bmatrix}$$

In taking the composition, the following maps will appear,

$$h_1^{-1} r p_1 \beta = \begin{bmatrix} T_0 \to S_E^i \\ D_1 \to S_E^i \end{bmatrix}, \qquad h_2^{-1} p_2 \beta = \begin{bmatrix} T_0 \to S_E^{n-i} \\ B_1 \to S_E^{n-i} \end{bmatrix},$$

$$\pi h_1 = \begin{bmatrix} S_E^i \to d_0^i \\ S_W^i \to d_0^i \end{bmatrix}, \qquad \pi h_2 = \begin{bmatrix} S_E^{n-i} \to d_0^{n-i} \\ S_W^{n-i} \to d_0^{n-i} \end{bmatrix}.$$

We find that

$$g \circ h = \begin{bmatrix} T_0 & \to & T_0 - S_E^i + S_E^{n-i} \\ B_1 & \to & B_1 - S_E^i + S_E^{n-i} \\ S_E^i & \to & d_0^i - S_W^i + N_0 \\ S_W^i & \to & d_0^i - S_E^i + N_0 \\ S_E^{n-i} & \to & d_0^{n-i} - N_0 + S_E^{n-i} \\ S_W^{n-i} & \to & d_0^{n-i} - N_0 + S_W^{n-1} \end{bmatrix}$$

Since N_0 is a strong deformation retract of both S_E^i and of S_E^{n-i}, the top two lines are m-homotopic *rel* N_0 to the map

$$\begin{bmatrix} T_0 \to T_0 - N_0 + N_0 \\ B_1 \to B_1 - N_0 + N_0 \end{bmatrix} = 1_M$$

Since N_0 is a strong deformation retract of d_0^i and d_0^{n-i} the middle two and bottom two lines are m-homotopic *rel* N_0 respectively to

$$\begin{bmatrix} S^i_E \to 2N_0 - S^i_W \\ S^i_W \to 2N_0 - S^i_E \end{bmatrix} = 2(S^i \to N_0) - r1_{S^i},$$

$$\begin{bmatrix} S^{n-i}_E \to N_0 - N_0 + S^{n-i}_E \\ S^{n-i}_W \to N_0 - N_0 + S^{n-i}_W \end{bmatrix} = 1_{S^{n-i}}$$

But we have shown in 3.7 that the first of these is m-homotopic to 1_{S^i}. The above m-homotopies can be carried out independently because they are all *rel* N_0, and N_0 is the point that separates M′ into the pertinent subsets.

COROLLARY 3.9. *Each surface has the same* m-*homotopy type as the wedge of either the two-sphere or the projective plane or the Klein bottle with an even number of one-spheres.*

Proof. Any surface can be obtained by adding handles through conservative surgery to one of these surfaces. Theorem 3.8 shows that adding a handle to a 2-manifold changes the m-homotopy type in the same way as attaching two circles at a point.

4. SINGULAR HOMOLOGY AND M-HOMOLOGY

We obtain singular m-homology from singular homology by allowing a singular simplex to be an m-function over R rather than a continuous function. We start with Δ^n, the standard geometric n-simplex, and the standard boundary maps $\partial_i: \Delta^{n-1} \to \Delta^n$ $(i = 0, 1, \ldots, n)$, which will be regarded as m-functions of multiplicity one. An *n-simplex* in a Hausdorff space X is an m-function $\sigma^n: \Delta^n \to X$ (over R). The *chain complex* $C_n(X) = \mathrm{Hom}_R(\Delta^n, X)$ is the R-module of all n-simplices in X if $n \geq 0$; it is zero if $n < 0$. Each chain, being a finite sum of m-functions, is itself an m-function and a singular m-homology simplex rather than a formal sum of singular simplices as in the usual theory. The *boundary map* $\partial: C_n(X) \to C_{n-1}(X)$ is defined by

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \sigma \circ \partial_i;$$

the boundary of any n-simplex is a single m-homology (n − 1)-simplex. The proof that $\partial\partial = 0$ is the same as for ordinary simplicial homology, so $\{C_n(x), \partial\}$ is indeed a chain complex. We put $Z_n = \mathrm{Ker}\,\partial$, $B_n = \mathrm{Im}\,\partial$ and define $H_n(X) = Z_n/B_n$ as the n^{th} m-homology R-module of X. In [6] we proved the following.

THEOREM 4.1. M-*homology over a ring* R *is a functor from* M_R *to the category of graded* R-*modules; it satisfies the Eilenberg-Steenrod axioms with homotopy replaced by* m-*homotopy.*

The proof of this theorem is almost entirely conventional. An m-function $f: X \to Y$ induces an R-module homomorphism $f_*: H_n(X) \to H_n(Y)$ just as in the usual theory, and this yields the functor. There are two departures from convention in proving the Eilenberg-Steenrod axioms. The first is that the usual proof yields the fact that m-homology is an m-homotopy invariant instead of merely a homotopy invariant; that is, m-homotopic m-functions induce the same R-module homomorphisms. The second occurs in the proof of the Excision Theorem which requires the result, mentioned in section 1, that components of m-functions are also m-functions.

THEOREM 4.2. *In any Eilenberg-Steenrod homology theory* H *in the category* CP *of compact polyhedra and continuous functions, the homology groups are* m-*homotopy-type invariants.*

Proof. We consider m-homology in the category CP; singular simplices are m-functions, but maps between spaces are continuous functions regarded as m-functions of multiplicity one. Since a homotopy can be regarded as an m-homotopy, this theory is a homology theory in the sense of Eilenberg and Steenrod [4]. In particular, it satisfies the homotopy invariance axiom. By uniqueness, the homology R-modules are the same as for H. Now we consider m-homology in the category CPM of compact polyhedra and m-functions over R. The calculation of m-homology R-modules for any space in CPM is the same as in CP; it is not affected by the introduction of additional morphisms between spaces. Therefore, the m-homology R-modules of any compact polyhedron are the same as its homology R-modules. Since the m-homology R-modules are m-homotopy type invariants, the proof is complete.

Theorem 4.1 shows that M_R is in some ways a better category for homology than TOP. We obtain a stronger invariant and have cycles that are morphisms in the category rather than formal linear combinations of them. Theorem 3.8 shows that the invariant is indeed stronger. Worosz [5] has worked out the corresponding m-cohomology theory, but the ring structure in cohomology is not preserved by m-functions.

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