A GROWTH CONDITION FOR CLASS $\mathscr{A}$

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The original notes of this material and preliminary work in several other topics are being held in the Gerald R. MacLane collection of the Mathematics Library at Purdue University.

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1. INTRODUCTION

Some years ago I introduced the class $\mathscr{A}$ of functions $f(z)$ which are holomorphic in the unit disc $D = \{|z| < 1\}$ and possess asymptotic values at a dense set on ∂D ([8]). Let $M(r) = \max_\theta |f(re^{i\theta})|$. Then I proved ([8, Theorem 14]) that $f \in \mathscr{A}$ if

$$\int_0^1 (1-r)\log M(r)\,dr < \infty. \tag{1.1}$$

More generally, $f \in \mathscr{A}$ if there exists a dense set Θ in $[0,2\pi)$ such that

$$\int_0^1 (1-r)\log^+|f(re^{i\theta})|\,dr < \infty \qquad (\theta \in \Theta), \tag{1.2}$$

and the arguments of [8] use only (1.2); note that (1.2) is compatible with arbitrarily large $M(r)$.

Several years later, R. Hornblower [5] significantly improved the condition (1.1) by proving that $f \in \mathscr{A}$ if

$$\int_0^1 \log^+\log^+ M(r)\,dr < \infty. \tag{1.3}$$

Further, Hornblower showed that (1.3) is almost sharp in the sense that there exist functions not in $\mathscr{A}$ for which, corresponding to each $\varepsilon > 0$,

$$\log^+\log^+ M(r) < \frac{\varepsilon}{(1-r)\log\dfrac{1}{1-r}} \qquad (r_\varepsilon < r < 1).$$

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Hornblower's proof of the sufficiency of (1.3) is short, but depends on a difficult lemma of N. Levinson [7, p. 135]; for this purpose Levinson's result plays a significant role in the proof of the major theorem in Chapter 8 of [7]. This latter theorem admits independent proofs from N. Sjöberg ([11], Theorème 3), A. Beurling ([2, Lemma 1]) and Y. Domar ([3, Theorem 3]).

I would like to give another approach to these ideas; it uses very different and basic ideas. As an application of these methods, we present as Theorem 4 a new proof of the Sjöberg-Levinson-Beurling theorem.

Nothing is lost by considering the problem as one in subharmonic functions. Let E be the box

$$E = \{0 < x < 1, -1 < y < 1\}. \tag{1.4}$$

The class [E] consists of those real-valued functions U(z) defined on E such that

(a) U is subharmonic in E,

(b) there exists a sequence of crosscuts $\{\gamma_n\}$ of E, where each γ_n joins a point of $\{0 < x < 1, y = -1\}$ to a point of $\{0 < x < 1, y = 1\}$, such that

$$\gamma_n \to \{x = 0; -1 < \operatorname{Im} z < 1\}$$

and such that

$$U(z) \le M < \infty \qquad (z \in \gamma_n, n = 1, 2, \ldots). \tag{1.5}$$

(Here $\gamma_n \to \{x = 0; -1 < \operatorname{Im} z < 1\}$ means that for each $\varepsilon > 0$,

$$\gamma_n \subset E \cap \{0 < x < \varepsilon\}, \qquad n > n_0(\varepsilon)).$$

Let $\lambda(t)$ be a positive function for $0 < t \le 1$ which decreases as t increases (in general, $\lambda(0) = +\infty$). Then a function U of [E] belongs to $[E,\lambda]$ if

$$\sup_{-1<y<1} U(x + iy) \le \lambda(x) \qquad (0 < x \le 1). \tag{1.6}$$

Finally, Λ is the class of such functions λ as above with the property that for each fixed $\varepsilon > 0$,

$$\limsup_{\substack{z \to i\tau \\ z \in E}} U(z) < \infty \qquad (-1 + \varepsilon < \tau < 1 - \varepsilon) \tag{1.7}$$

whenever $U \in [E,\lambda]$.

The basic result here is Theorem 2:

$$\int_0^1 \log^+ \lambda(t)\, dt < \infty \;\Rightarrow\; \lambda \in \Lambda. \tag{1.8}$$

That Hornblower's result follows from this is now standard. For suppose I is an arc on ∂E (say $I = \{-1 < \operatorname{Im} z < 1\}$) such that no point of I is the endpoint of an asymptotic path of f(z). Then ([8, Theorem 1]) there exist $M < \infty$ and a sequence $\{\gamma_n\}$ in E tending to I on which (1.5) holds for $U(z) = \log|f(z)|$. According to (1.3), (1.6) and (1.8), we have (1.7). Thus, Fatou's theorem produces many asymptotic paths which end at points of I.

Note from Carleman's principle that (1.5) and (1.7) imply that

$$\limsup_{\substack{z \to i\tau \\ z \in E}} U(z) \le M \qquad (-1 < \tau < 1) \tag{1.9}$$

where M is the constant of (1.5).

2. PRELIMINARY RESULTS

These results are suggested by those of ([6]; [10] (*cf.* [4, p. 86])) but seem to be new.

(A) *A theorem on multivalent functions*

LEMMA 1. *Let* $f(z) = u(z) + iv(z)$ *be holomorphic and non-constant in the unit disc* $D = \{|z| < 1\}$ *with* u *continuous on* ∂E. *Set* $u(e^{it}) = \phi(t)$ $(0 \le t < 2\pi)$ *and suppose there is a fixed integer* p *such that for each* λ $(-\infty < \lambda < \infty)$ *the closed subset*

$$A(\lambda) = \{e^{it} : \phi(t) = \lambda\} \qquad (-\infty < \lambda < \infty) \tag{2.1}$$

consists of at most 2p *components of* ∂D.

Then $w = f(z)$ *is at most* p*-valent in* D. *In particular, if* $p = 1$ *then* f(z) *is schlicht in* D.

Remark. The crucial aspect of this lemma is that the hypotheses concern only $u = \operatorname{Re}(f)$: no assumptions are made about v.

Proof. For $\min \phi(t) \le \lambda \le \max \phi(t)$ and $-\infty < \tau < \infty$, consider the number of solutions to the equation $f(z) = \lambda + i\tau$ for $z \in D$. We first suppose that λ is not a local extreme value of ϕ, that $A(\lambda)$ (*cf.* (2.1)) consists of $k \le 2p$ points (no arcs) and that if

$$L(\lambda) = \{z \in D : u(z) = \lambda\}, \tag{2.2}$$

then $f'(z) \ne 0$ for $z \in L(\lambda)$. To ensure these requirements, it is necessary to avoid an at most countable set of λ.

The closure of $L(\lambda)$ in $\{|z| \le 1\}$ consists of $q(= k/2)$ components, $\kappa_1, \ldots, \kappa_q$, each of which is a simple cross-cut of D. These κ_i divide D into $q + 1$ simply-connected domains $\{G_m\}$, and $A(\lambda)$ divides ∂D into 2q arcs $\{\Gamma_n^+\}$, $\{\Gamma_n^-\}$ in which $u \ge \lambda$ and $u \le \lambda$ respectively. We note

$$2q = k \leq 2p. \tag{2.3}$$

Consider a G_m in which, say, $u > \lambda$. Then G_m is bounded by a subset of the Γ^+ and these arcs alternate with s_m of the κ_i. The Cauchy-Riemann equations

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial s}, \tag{2.4}$$

(η = normal) now show that v is monotone along each κ_i. Hence f(z) assumes each complex number $\lambda + i\tau\,(-\infty < \tau < \infty)$ at most s_m times in $\bar{G}_m \cap D$. But $\sum s_m = q$ so the Lemma, in this case, is immediate from (2.3).

Finally, we can easily remove the initial restrictions on λ, for if f assumes $w = \lambda_0 + i\tau_0$ more than $p + 1$ times in E, it must assume $w = \lambda_1 + i\tau_1$ at least $p + 1$ times, where now λ_1 satisfies the restrictions given at the beginning of the proof.

(B) *On boundary behavior of conjugate harmonic functions*

Let a and b, $a < b$, be real numbers, and $\phi(x)$ be a continuous real function such that

$$|\phi(x)| < M \qquad (a < x < b), \tag{2.5}$$

$$\phi(x) = 0 \qquad (x \leq a;\ x \geq b). \tag{2.6}$$

The function

$$\begin{aligned} u(z) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2}\phi(t)\,dt \\ &= \frac{1}{\pi}\int_a^b \frac{y}{(x-t)^2 + y^2}\phi(t)\,dt \qquad (\mathrm{Im}\, z > 0) \end{aligned} \tag{2.7}$$

is harmonic in the upper half-plane H and $u(x + iy) \to \phi(x)$ at points of continuity of ϕ. A function conjugate to u in H is

$$v(z) = \frac{1}{\pi}\int_a^b \frac{x-t}{(x-t)^2 + y^2}\phi(t)\,dt. \tag{2.8}$$

The boundary values of v are formally given by the Hilbert transform (Cauchy principal value)

$$\psi(x) = \lim_{\varepsilon \downarrow 0}\left\{\int_a^{x-\varepsilon} + \int_{x+\varepsilon}^b\right\}\frac{\phi(t)}{x-t}\,dt \qquad (a < x < b). \tag{2.9}$$

The next result makes this very explicit.

LEMMA 2. *Let* $\phi(x)$ *be continuous on* $(-\infty,\infty)$ *and satisfy* (2.5), (2.6). *Suppose that to each* $\eta > 0$ *may be associated* $\varepsilon_0 > 0$ *such that*

$$\frac{1}{\pi}\int_{x-\varepsilon}^{x+\varepsilon}\left|\frac{\phi(t)-\phi(x)}{t-x}\right| dt < \eta \qquad (-\infty < x < \infty, \varepsilon < \varepsilon_0). \tag{2.10}$$

Then if v *and* ψ *are defined according to* (2.8) *and* (2.9), *it follows that* $\psi(x)$ *is continuous for* $-\infty < x < \infty$ *and*

$$\lim_{y\to 0} v(x+iy) - \psi(x) = 0 \qquad (-\infty < x < \infty) \tag{2.11}$$

uniformly. In particular, the function

$$f(z) = u(z) + iv(z) \qquad (z \in H)$$

can be extended continuously to the real axis by

$$f(x) = \phi(x) + i\psi(x) \qquad (-\infty < x < \infty). \tag{2.12}$$

Proof. We first show that the limit in (2.9) exists uniformly in x as $\varepsilon \to 0$. Let $h > \varepsilon_1 > \varepsilon_2$. Then

$$\left|\left\{\int_{x-\varepsilon_1}^{x-\varepsilon_2} + \int_{x+\varepsilon_2}^{x+\varepsilon_1}\right\}\frac{\phi(t)}{x-t}dt\right| = \left|\left\{\int_{x-\varepsilon_1}^{x-\varepsilon_2} + \int_{x-\varepsilon_2}^{x-\varepsilon_1}\right\}\frac{\phi(t)-\phi(x)}{t-x}dt\right|$$

$$\le \int_{x-\varepsilon_1}^{x+\varepsilon_1}\left|\frac{\phi(t)-\phi(x)}{t-x}\right| dt = o(1) \qquad (\varepsilon_1 \to 0)$$

with desired uniformity. We also have that

$$v(x+iy) - \psi(x) = \frac{1}{\pi}\int_a^b \frac{y^2\phi(t)}{(t-x)[(t-x)^2+y^2]}dt. \tag{2.13}$$

We now prove (2.11). According to (2.5) and (2.13),

$$\left\{\int_a^{x-y^{1/2}} + \int_{x+y^{1/2}}^{b}\right\}\frac{y^2|\phi(t)|}{|t-x|[(t-x)^2+y^2]}dt \tag{2.14}$$

$$\le 2My^2\int_{y^{1/2}}^{\infty}\frac{du}{u(u^2+y^2)} \le 2My^2\int_{y^{1/2}}^{\infty}u^{-3}du = My.$$

Next, the odd parity of the relevant kernels and (2.10) yield that

$$\left| \int_{x-y^{1/2}}^{x+y^{1/2}} \frac{y^2\phi(t)}{(t-x)[(t-x)^2+y^2]}\,dt \right| = \left| \int_{x-y^{1/2}}^{x+y^{1/2}} \frac{y^2[\phi(t)-\phi(x)]}{(t-x)[(t-x)^2+y^2]}\,dt \right|$$
$$\leq \int_{x-y^{1/2}}^{x+y^{1/2}} \left| \frac{\phi(t)-\phi(x)}{t-x} \right| dt = o(1) \quad (y \to 0),$$

uniformly in x. This and (2.14) show that the continuous functions $v(x+iy)$ ($y > 0$ and fixed, $-\infty < x < \infty$) tend uniformly to $\psi(x)$ as $y \to 0$. Thus (2.11) holds and ψ is continuous. The remaining assertions of the Lemma concern the convergence of u to ϕ, and are standard.

In our application in section 3 of Lemma 2, we shall encounter the following situation: $a = -2$, $b = 4$, and ϕ is concave having derivatives existing off a finite t-set T with

$$(2.15) \qquad \begin{array}{lll} \phi(t) \equiv 0 & & (t < 0) \\ \phi(0) = 0, & \phi(1) = c, & (0 < c < 1) \\ \phi'(t) > 0, & \phi''(t) \leq 0 & (0 < t < 1,\ t \notin T) \\ \phi(1+t) = \phi(1-t) & & (t \geq 0). \end{array}$$

From the concavity of ϕ it is then clear that for $\varepsilon < 1$

$$(2.16) \qquad \int_{x-\varepsilon}^{x+\varepsilon} \left| \frac{\phi(t)-\phi(x)}{t-x} \right| dt \leq 2 \int_0^{\varepsilon} \frac{\phi(t)}{t}\,dt \qquad (-1 \leq x \leq 2).$$

COROLLARY. *Let ϕ satisfy* (2.5), (2.6) (*with* $a = -2$, $b = 4$), (2.15) *and*

$$(2.17) \qquad \int_0^1 \frac{\phi(t)}{t}\,dt < \infty.$$

Then (2.11) *holds, Lemma* 2 *applies and* v *is continuous on the closed half plane* $\bar{H}$.

For our final observation, let ϕ be as above and for $\eta > 0$ define a 'truncated' ϕ by

$$(2.18) \qquad \phi_\eta(t) = \min(\phi(t), \eta).$$

Then (2.15) holds for ϕ_η and (2.17) is satisfied for ϕ_η if and only if (1.17) is satisfied for ϕ.

LEMMA 3. *Let ϕ satisfy* (2.15), (2.17) *and let ϕ_η be given by* (2.18). *Define* u_η, v_η *by* (2.7) *and* (2.8) *with* $a = -2$, $b = 4$ *and ϕ_η in place of ϕ. Then v_η is continuous in the closed half-plane $\bar{H}$ and vanishes at ∞. Further, given $h > 0$, there exists $\eta_0(h)$ such that*

$$(2.19) \qquad |v_\eta(z)| < h, \qquad z \in \bar{H},\ 0 < \eta < \eta_0(h).$$

Proof. Lemma 2 shows that v_η is continuous on $[-1,3]$. Also, (2.15) and the reflection principle give that $v_\eta(x)$ $(= \psi_\eta(x))$ is continuous at every point in the real axis outside $[0,2]$. An inspection of (2.8), (2.9) and (2.11) shows that $v(\infty) = 0$ and that v is monotone on $(-\infty,0)$ and $(2,\infty)$. It thus suffices to show that

$$|\psi_\eta(x)| < h \qquad (0 \le x \le 2) \tag{2.20}$$

for sufficiently small η.

To prove (2.20), let $0 \le x \le 2$. Then (2.16) gives

$$\psi_\eta(x) = \frac{1}{\pi}\left\{\int_{-2}^{x-1} + \int_{x+1}^{4}\right\}\frac{\phi_\eta(t)}{x-t}\,dt + \text{P.V.}\left\{\frac{1}{\pi}\int_{x-1}^{x+1}\frac{\phi_\eta(t)-\phi_\eta(x)}{x-t}\,dt\right\},$$

$$|\psi_\eta(\xi)| \le \frac{4}{\pi}\eta + \frac{2}{\pi}\int_0^1 \frac{\phi_\eta(t)}{t}\,dt.$$

Now $\phi_\eta(t) \downarrow 0$ a.e. for $0 \le t \le 1$ as $\eta \to 0$. This with (2.17) allows Lebesgue's dominated convergence theorem to be applied, so (2.20) is assured by choosing a suitable $\eta_0 > 0$ and then taking $\eta < \eta_0$.

Remark. Necessary and sufficient conditions that ψ be continuous can be obtained from [12, p. 180, Ex. 5].

SECTION 3

(A) *The Hornblower theorem*

THEOREM 1. *Suppose that* $\lambda(t)$ *is positive and non-increasing on* $(0,1]$. *Suppose further that there exists a non-decreasing function* $\phi(t)$, $0 \le t \le 1$, *with*

$$\phi(0) = 0, \qquad \phi(1) < 1 \tag{3.1}$$

and

$$\int_0^1 \frac{\phi(t)}{t}\,dt < \infty, \qquad \int_0^1 \lambda(\phi(t))\,dt < \infty. \tag{3.2}$$

Then $\lambda(t) \in \Lambda$, *where* Λ *has been defined in* (1.7).

Proof. We first impose the additional conditions that $\phi \in C^2(0,1]$ and that (2.15) holds for $0 \le t \le 1$. By defining $\phi(t) = \phi(2-t)$ $(1 \le t \le 2)$ we then satisfy (2.15) for all t. Next, truncate ϕ with (small) η, as in (2.18), and observe that the resulting function $\phi_\eta(t)$ satisfies (3.1) and (3.2) as well as the hypotheses of Lemma 1 with $p = 1$.

Define $u(z)$ and $v(z)$ by (2.7) and (2.8) with $a = -2$, $b = 4$ and set

$$f(z) = u(z) + iv(z) \qquad (z \in H).$$

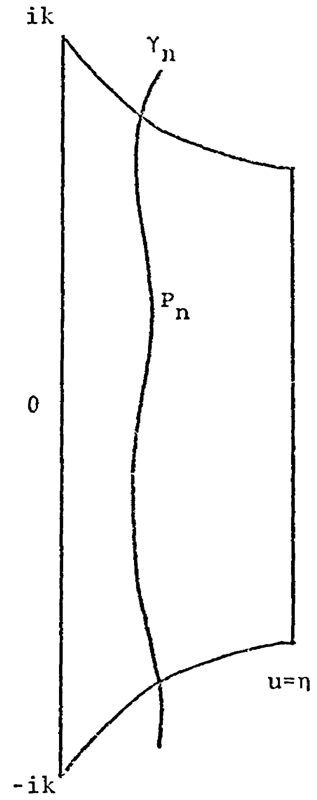

Figure 1

Since v is monotone on the intervals on which ϕ_η is constant, it follows from Lemmas 1 and 2 that $w = f(z)$ maps H one-to-one onto a symmetric Jordan domain P as indicated in Figure 1.

Recall the definitions E, [E], [E,λ], Λ from (1.4)-(1.7). To prove (1.7) when (1.5) holds, we may take $\tau = 0$ in (1.7), and choose η so small that

$$P \subset \{|v| < h < 1\} \cap E;$$

this is possible according to Lemma 3. We also assume that

$$\lambda(t) \geq \lambda(1) > M, \tag{3.3}$$

where M appears in (1.5); otherwise λ may be replaced by $\max(M,\lambda(t))$.

Now let $\{\gamma_n\}$ be a sequence on which $U(w) \leq M$, such that $\gamma_n \to [-ih, ih]$. Then for $n > n_0$, γ_n contains a cross-cut γ_n' of P, which divides P into two components; let P_n be that one which contains a segment of the arc $\{u = \eta\}$. From (3.3) and Carleman's principle of domain extension it is clear that

$$U(w) \leq \Phi_n(w) \qquad (w \in P_n,\ n > n_0) \tag{3.4}$$

where Φ_n is bounded and harmonic in P with boundary values $\lambda(u)$ on $\partial P \cap \partial P_n$ and M on the remainder of ∂P.

The inverse of f, $z = f^{-1}(w)$, maps P onto H and carries U(w) and $\Phi_n(w)$ into T(z) and $\Psi_n(z)$ respectively, and P_n onto a Jordan domain $G_n \subset H$ such that ∂G_n contains the interval $[\varepsilon_n, 2-\varepsilon_n']$ of the real axis. One sees using (2.12) and (2.15) that $\varepsilon_n \downarrow 0$, $\varepsilon_n' \downarrow 0$ and $G_n \uparrow H$ as $n \to \infty$, and (3.4) becomes

$$T(z) \leq \Psi_n(z) \qquad (z \in G_n, n \geq n_0). \tag{3.5}$$

Now Ψ_n is a bounded harmonic function, and so

$$\Psi_n(z) = \frac{1}{\pi}\int_{\varepsilon_n}^{2-\varepsilon_n'} \frac{y\lambda(\phi(t))}{(t-x)^2+y^2}\,dt + M\omega_n(z), \tag{3.6}$$

where ω_n is the harmonic measure at z of the complement of $[\varepsilon_n, 2-\varepsilon_n']$ relative to the real axis. It follows at once from (3.5) and (3.6) that

$$\begin{aligned} T(z) \leq \Psi_n(z) &\leq \frac{1}{\pi}\int_0^2 \frac{y\lambda(\phi(t))}{(t-x)^2+y^2}\,dt + M \\ &\equiv \Psi(z) + M \qquad (z \in H). \end{aligned} \tag{3.7}$$

Since $\phi(t) = \phi(2-t)$ $(0 \leq t \leq 2)$, we see from (3.6) and the second assumption of (3.2) that $\Psi(z)$, as just defined in (3.7), is harmonic in H, and bounded outside any neighborhood of $z = 0$ or $z = 2$.

We reinterpret (3.7) in P. Then Ψ corresponds to the positive harmonic function $\Phi(w)$ in P with continuous boundary values 0 on $\partial P \cap \{u = 0\} = (-ik, ik)$ (*cf.* Fig. 1). Thus (3.7) becomes

$$U(w) \leq \Phi(w) + M \qquad (w \in D), \tag{3.8}$$

so that

$$\limsup_{\substack{w\to 0\\ w \in P}} U(w) \leq M$$

and $\lambda(t) \in \Lambda$. This proves the Theorem in the special case that (2.15) is satisfied.

To complete the proof we show

LEMMA 4. *Let* $\lambda(t)$ *and* $\phi(t)$ *satisfy the hypotheses of Theorem* 1. *Then there exists* $\Phi(t)$ *which satisfies the same hypotheses and* (2.15); *in fact, we may arrange*

$$\phi(t) \leq \Phi(t) \qquad (0 \leq t \leq 1). \tag{3.9}$$

Proof. Since ϕ is non-decreasing and $\phi(0) = 0$, (3.2) shows that ϕ is continuous at $t = 0$. We define $\phi(t) \equiv \phi(1) < 1$ for $t \geq 1$ (this is a different extension than used in the proof of the special case of Theorem 1 above, but no confusion should arise).

If $\phi_1(t) = t^{-1}\int_t^{2t} \phi(u)\,du$ $(t > 0)$, then ϕ_1 is continuous, nondecreasing, and $\phi_1(0) = 0$, $\phi_1(t) = \phi(1) < 1$ for $t \geq 1$, and

$$\phi(t) \leq \phi_1(t) \leq \phi(2t) \qquad (t > 0). \tag{3.10}$$

The right inequality of (3.10) shows that ϕ_1 satisfies the first condition in (3.2), and ϕ_1 satisfies the second condition since λ is monotone and the left inequality of (3.10) holds.

Similarly, if $\phi_2(t) = t^{-1}\int_t^{2t} \phi_1(u)\,du$ we achieve a ϕ_2 which satisfies (3.1) and (3.2) with

$$\phi_2(t) \in C\,[0,1], \quad \phi_2(t) \in C^1(0,1], \quad \phi_2'(t) \geq 0 \qquad (0 < t \leq 1)$$
$$\text{and} \quad \phi(t) \leq \phi_2(t).$$

Now if ε is so small that $\phi_2(1) + \varepsilon < 1$, we set $\phi_3(t) = \phi_2(t) + \varepsilon t^2$. Consequently,

$$\phi_3 \in C\,[0,1], \quad \phi_3 \in C^1(0,1], \quad \phi_3'(t) > 0 \qquad (0 < t \leq 1] \tag{3.11}$$

and $\phi(t) \leq \phi_3(t)$.

Thus, by replacing ϕ by ϕ_3 and then dropping the subscript 3, we need only prove the Lemma in the case that ϕ satisfies (3.11). It is necessary to rig Φ in terms of ϕ so that Φ comes out concave down:

$$\Phi'(t) = \int_t^1 \frac{\phi'(u)}{u}\,du > 0 \qquad (0 < t \leq 1) \tag{3.12}$$

and

$$\Phi(t) = \int_0^t \Phi'(u)\,du = \int_0^t du \int_u^1 \frac{\phi'(s)}{s}\,ds$$
$$+ \int_0^1 \frac{\phi'(s)}{s}\,ds \int_0^{\min(s,t)} du = \phi(t) + t\int_t^1 \frac{\phi'(u)}{u}\,du \tag{3.13}$$
$$= \phi(t) + t\Phi'(t) \qquad (0 < t \leq 1).$$

That this makes sense, (that is that Φ' is integrable on $[0,1]$) follows from (3.13) with $t = 1$ and Fubini's theorem. Then $\Phi(0) = 0$, $\Phi \in C\,[0,1]$, and Φ' is the derivative of Φ on $[0,1]$. Integrating the penultimate expression of (3.13) by parts gives

$$\Phi(t) = t\phi(1) + t\int_t^1 \frac{\phi(u)}{u^2}\,du \qquad (0 < t \leq 1). \tag{3.14}$$

We now check that Φ satisfies the more stringent conditions imposed at the beginning of the proof of Theorem 1. Definition (3.12) shows that $\Phi \in C^2(0,1]$ and $\Phi''(t) = -t^{-1}\phi'(t) < 0$ $(0 < t \leq 1)$. Next, (3.12) and (3.13) give

$$\phi(t) \leq \Phi(t) \leq \phi(t) + \int_t^1 \phi'(u)\,du = \phi(1) < 1 \tag{3.15}$$

(to get $\Phi'(1) > 0$, we may add εt to Φ for small ε). Thus Φ satisfies (3.1), is in $C^2(0,1]$, and satisfies (2.15) for $0 \leq t \leq 1$. The second condition of (3.2) is a consequence of the left inequality of (3.15), and the first bound of (3.2) follows from (3.14):

$$\int_0^1 \frac{\Phi(t)}{t}\,dt = \phi(1) + \int_0^1 dt \int_t^1 \frac{\phi(u)}{u^2}\,du$$

$$= \phi(1) + \int_0^1 \frac{\phi(u)}{u}\,du < \infty$$

This completes the proof of Lemma 4.

(B) *Remarks*

We observe the sources of the conditions (3.2). The second ensures that the Poisson integral $\Psi(z)$ in (3.7) exists: if $\int_0^1 \lambda(\phi(t))\,dt = +\infty$, then it is clear from (3.6) that $\Psi_n(z) \to +\infty$ as $n \to \infty$ and the argument is up the chimney. The first condition allows use of Lemma 3. But this condition is not accidental; it is vital to the argument. For if $\int_0^1 t^{-1}\phi(t)\,dt = \infty$, the principal value integral $\psi(\xi)$ (*cf.* (2.9)) has values $\mp\infty$ at $\xi = 0,2$ and so v is unbounded. Thus no matter how small η is used to truncate ϕ, the domain P obtained as the image of H extends from $v = -\infty$ to $v = +\infty$ and can't be confined to E.

We also want to motivate the definition (3.12) of Φ in terms of ϕ $(= \phi_3)$. To obtain a function Φ with Φ' decreasing and $\int_0^t \Phi'\,du \geq \int_0^t \phi\,du$, we take the graph of ϕ' and rearrange so that $\phi'(t)\,dt = -t\,d\Phi'(t)$. This procedure gives a function Φ with Φ' decreasing. It does not always give the smallest dominant Φ which is concave down (for $\phi(t) = t$, we get $\Phi(t) = t + t\log(1/t)$) but is close in extreme cases and certainly serves our purpose. There appears to be no *simple* way of laying hands on the smallest dominant that is concave down.

(C) *A reformulation of* (3.2)

It is possible to replace (3.2) by an equivalent condition which does not involve ϕ; this is the condition of Hornblower, and gives another interpretation of condition (3.16) below.

THEOREM 2. *Suppose that* $\lambda(t) \geq 1$ *and that* $\lambda(t)$ *is nonincreasing for* $0 < t \leq 1$. *Then there exists a* ϕ *in accord with the hypotheses of Theorem* 1 *if and only if*

$$\int_0^1 \log \lambda(t)\, dt < \infty. \tag{3.16}$$

Proof. Let us first replace (3.2) by a simpler condition, which does not involve composition. Suppose that (3.2) holds for a ϕ which also satisfies (2.15) (this is no loss of generality, according to Lemma 4). Let h be the inverse of ϕ on $[0,c]$, with $c = \phi(1)$. Then (3.2) becomes

$$\int_0^c t\frac{h'(t)}{h(t)}\, dt < \infty, \quad \int_0^c \lambda(t) h'(t)\, dt < \infty. \tag{3.17}$$

Conversely, if $\lambda(t)$ is a positive decreasing function on $[0,1]$, and there exists an absolutely continuous function $h(t) \geq 0$ $(0 \leq t \leq c)$ with $h'(t) > 0$ $(0 < t < c)$ for some $c > 0$ such that (3.17) holds, then $\lambda \in \Lambda$. In fact, we may let ϕ be the inverse of h.

Suppose that (3.17) holds for an h as just described. Then since h and $\log h$ are absolutely continuous on any interval $[\varepsilon, c]$ $(\varepsilon > 0)$, we find that

$$\int_\varepsilon^c t\frac{h'(t)}{h(t)}\, dt = c \log h(c) + \varepsilon \log\frac{1}{h(\varepsilon)} + \int_\varepsilon^c \log\frac{1}{h(t)}\, dt;$$

also h increases and $h(0) \geq 0$, so it is routine to see from this that

$$\int_0^c t\frac{h'(t)}{h(t)}\, dt < \infty \iff \int_0^c \log\frac{1}{h(t)} < \infty. \tag{3.18}$$

We now show that (3.16) and (3.17) are equivalent. Suppose that (3.16) holds and that λ is also in $C^1(0,1]$. Then if we take

$$h(t) = \frac{t}{\lambda(t)}, \quad h'(t) = \frac{1}{\lambda(t)} - \frac{t\lambda'(t)}{\lambda^2(t)} > 0, \tag{3.19}$$

the conditions (3.17) are satisfied and coalesce into one (this is the motivation for the choice (3.19)): $\int_0^1 \left(1 - t\frac{\lambda'(t)}{\lambda(t)}\right) dt < \infty$. That this condition is (3.16) was shown in the discussion of (3.18). If λ is not C^1, one uses a double average

$$\lambda^*(t) = 2/t \int_{t/2}^t (2/u) \int_{u/2}^u \lambda(s)\, ds \qquad (0 < t, s < 1) \tag{3.20}$$

instead. Since $\lambda(t) \le \lambda^*(t) \le \lambda(t/4)$ we see that λ^* satisfies (3.17) if λ does, and since $\lambda^* \in \Lambda$, so is λ.

Conversely, when (3.17) is satisfied,

$$O(1) \ge \int_0^t \lambda(u)\, h'(u)\, du \ge \lambda(t) \int_0^t h'(u)\, du$$
$$= \lambda(t)\, h(t),$$

and so

$$\int_\varepsilon^1 \log \lambda(t)\, dt \le \int_\varepsilon^1 \log \frac{O(1)}{h(t)}\, dt$$
$$\le O(1) + \int_\varepsilon^1 \log \frac{1}{h(t)}\, dt = O(1),$$

as we saw in (3.18). Thus (3.17) implies (3.16).

4. THE GEOMETRY INVOLVED

(A) *Cusps and the Dirichlet problem*

The key to Theorem 1 is the geometry of the region P which is shown in Figure 1 of section 3. Let us consider this situation in more detail. Let P be a bounded simply-connected domain contained in $\{\mathrm{Re}(w) > 0\}$ with $0 \in \partial P$. We suppose there is a continuous function $v = \psi(u)$ with $\psi(0) = 0$ that is analytic for $0 < u \le \delta$ $(\delta > 0)$, so that for some $0 < \varepsilon \le \delta$

$$P \cap \{\max(u,v) \le \varepsilon\} = \{w;\ 0 < u \le \varepsilon, \psi(u) < v \le \varepsilon\}. \tag{4.1}$$

We then say that the curve γ: $\{w;\ v = \psi(u)\}$ is a *positive cusp* for P at 0. If P lies below such a γ, then γ is a *negative cusp* at 0. (This means that v is replaced by $-v$ in (4.1)).

Associated to a positive or negative cusp γ is a real valued function $g(t,\gamma)$, obtained as follows (the motivation for this is in our proof of Theorem 1). If, say, γ is a positive cusp, let $w = f(z)$ map H onto P with $f(0) = 0$, so that

$$\gamma \cap \{0 < u \le \varepsilon\}$$

is the image of some interval $0 < x \le \delta_1$. Then if $\phi(x) = \mathrm{Re}(f(x + i0))$, $0 < x \le \delta_1$, let $g(t,\gamma)$ $(= g(t,\gamma+))$ be the function inverse to ϕ; *cf.* the proof of Theorem 1. Thus, g is defined on some interval $0 < t \le \varepsilon$. If γ is a negative cusp, $g(t,\gamma)$ $(= g(t,\gamma-))$ is defined similarly with respect to a conformal map from the lower half-plane to P with $f(0) = 0$.

The solvability of the Dirichlet problem in P with boundary values $\lambda(u)$ on γ and M ($<\infty$) on $\partial P - \gamma$ reduces to the condition

$$\int_0 \lambda(\xi)\, d_\xi\, \omega(w,\xi) < +\infty \qquad (w \in P) \tag{4.2}$$

(*cf.* [9, p. 28]), where ω is the harmonic measure of γ in P. The general idea is that the sharper the cusps on P, the smaller is ω. In our proof of Theorem 1 it seemed necessary to construct P in terms of λ. However, it is natural to investigate this question more directly.

We had better describe condition (4.2) more accurately: it is that the *Poisson solution* of the stated Dirichlet problem exist. The trouble is that this Dirichlet problem—in which the behavior of the solution at the boundary point 0 is undefined—will always have solutions. But the only solution which will serve our purpose is that given by the Poisson integral (and that need not always exist). Since the boundary values here are non-negative in any event, it is routine to see that (4.2) is equivalent to the existence of a *nonnegative* solution to this Dirichlet problem; it will have an infinite number of such solutions, if it has any, and one will be the Poisson solution. Note that the existence of the Poisson integral requires only the second condition

$$\int_0^1 \lambda(\phi(t))\, dt < \infty \tag{4.3}$$

of (3.2).

The question is purely local.

LEMMA 5. *Let* P_1 *and* P_2 *be simply-connected domains in the right half-plane such that* (4.1) *holds for both domains for some* $\varepsilon > 0$ *and the same function* $\psi(u)$. *Then given* $\lambda(u)$, *the integrals in* (4.2) *converge or diverge together for* P_1 *and* P_2.

Remark. Thus γ is here a positive cusp at 0 for both regions; the same is true when γ is a negative cusp.

Proof. Let $w = f_j(z)$ map H ($=\{y > 0\}$) onto P_j with $f_j(0) = 0$. By the reflection principle, $f_2^{-1}(f_1(z))$ is holomorphic near $z = 0$ and maps an upper half-neighborhood of the origin onto another such neighborhood. Thus

$$s(t) = f_2^{-1}(f_1(z)) = c_1 z + c_2 z^2 + \ldots, \qquad c_1 > 0,\ c_1 \text{ real}. \tag{4.4}$$

As in the proofs of Theorems 1 and 2, let $\phi_j(t) = \mathrm{Re}(f_j(t))$ and let $g_j(t)$ be the inverse of ϕ_j on $[0,\delta]$ for some $\delta > 0$. Then (4.3) is equivalent to

$$\int_0^\delta \lambda(t)(g_j)'(t)\, dt < \infty \qquad (j = 1, 2) \tag{4.5}$$

as was observed in the proof of Theorem 2 (*cf.* (3.17)).

Now $f_1 = f_2 \circ s$, $\phi_1 = \phi_2 \circ s$, $g_2 = s \circ g_1$, $(g_2^+)' = (s' \circ g_1)(g_1)'$. But s' is bounded away from 0 and ∞, so the equivalence of the conditions (4.5) is clear.

Definition. Let U be subharmonic in $R = \{u > 0\}$ and let γ be a positive cusp which ends at the point $w = ia$. Then U is *compatible* with γ provided there exists $p(u)$, continuous and nonnegative on γ, such that

$$U(w) \le p(u) \qquad (w \in \gamma); \tag{4.6}$$

$$\int_0^\delta p(t)(g)'(t, \gamma+)\, dt < \infty \tag{4.7}$$

for some $\delta > 0$. That U be compatible with a negative cusp is given by replacing $g'(t,\gamma^+)$ by $g'(t,\gamma^-)$ in (4.7).

The function U possesses a *dense set of compatible cusps* on the imaginary axis provided: there exist sets of cusps $\{\gamma_m^+\}$ and $\{\gamma_m^-\}$ respectively positive and negative, each of which is compatible with U, such that the sets $\{ia_m^+\}$, $\{ia_m^-\}$ are each dense on the imaginary axis.

It is now trivial to prove

THEOREM 3. *Let* U *be subharmonic in* $\{u > 0\}$, *possessed of a dense set of compatible cusps on the imaginary axis with*

$$U(w) < M \qquad (w \in \gamma_m) \tag{4.8}$$

where the arcs γ_m *in the right half-plane tend to an arc* γ *on the imaginary axis. Then*

$$\limsup_{w \to i\tau} U(w) \le M \tag{4.9}$$

for any $i\tau$ *interior to* γ.

Proof. The proof imitates that used in Theorem 1; to account for the elimination of (3.1), we have instituted a dense set of cusps. Thus to prove (4.9) with, say, $\tau = 0$, we replace the situation of Figure 1 by introducing a region P with a negative cusp ending in the interval $(0, i\varepsilon]$, and a positive cusp ending in the interval $[-i\varepsilon, 0)$.

(B) *Examples*

When the context makes clear whether γ is a positive or negative cusp we write γ instead of γ^+ or γ^-.

Example 1. *Simple angle.* Let P be the region $\left\{\pi\left(\frac{1}{2} - \alpha\right) < \arg w < \frac{\pi}{2}\right\}$ where $0 < \alpha < 1$; the positive cusp γ is the ray $\left\{\arg w = \pi\left(\frac{1}{2} - \alpha\right)\right\}$. Then $f(z) = ie^{-i\alpha\pi} z^\alpha$ maps H to P so that $\phi(x) \sim cx^\alpha$, $g(t) \sim c' t^{1/\alpha}$ $(c, c' > 0)$ and the integrability condition of (4.7) becomes

$$\int_0^\delta p(t)\, t^{(1-\alpha)/\alpha}\, dt < \varepsilon. \tag{4.10}$$

The special case $\alpha = 1/2$ corresponds to that derived in [8, Theorem 12].

Note that if $p(t)$ is a positive decreasing function for which (4.10) holds, then

$$p(t) \le At^{-(1/\alpha)} \qquad (x > 0) \tag{4.11}$$

for some constant A; this is much stronger than (3.16). (Conversely, when (4.11) is satisfied we obtain (4.10) with a slightly larger exponent). To get (4.11) from (4.10), choose $\delta > 0$ with

$$\int_0^\delta p(t)\, t^{(1-\alpha)/\alpha}\, dt < \varepsilon.$$

Then if $t < \delta$, we have $p(t)\displaystyle\int_0^t u^{(1-\alpha)/\alpha}\, du < \varepsilon$ so that $p(t) \le \alpha^{-1}\varepsilon t^{-(1/\alpha)}$, which gives (4.11).

Example 2. *Cusp formed by tangent circle.* Consider now the region P which is obtained from the first quadrant of the w-plane by deleting the semi-circle $\{|w - a/2| \le a/2, \operatorname{Im}(w) > 0\}$, where $a > 0$. The arc

$$\{|w - a/2| = a/2, \operatorname{Im}(w) > 0\}$$

is a positive cusp for P. The required map $f\colon H \to P$ behaves near the origin (*cf.* Lemma 5) like

$$w = \frac{i\pi a}{\log \dfrac{1}{z} + i\pi} = \frac{i\pi a}{\log \dfrac{1}{r} + i(\pi - \theta)} \qquad (0 < \theta < \pi).$$

Thus we have essentially

$$\phi(x) = \frac{a\pi^2}{(\log x)^2 + \pi^2} \qquad (x > 0);$$

$$g(t) = e^{-\pi\sqrt{((a-t)/t)}} \qquad (0 < t < a).$$

Since

$$g'(t) = \frac{1}{2}\sqrt{a}\,\pi t^{-3/2} e^{-\pi\sqrt{(a/t)}}\{1 + O(\sqrt{t})\} \qquad (t \to 0)$$

the integrability condition in (4.7) becomes

$$\int_0 p(t)\, t^{-3/2} e^{-\pi\sqrt{(a/t)}}\, dt < \infty;$$

thus if we were to restrict ourselves only to circular cusps, then the allowable functions $\lambda(t)$ would be those for which

$$\int_0 \lambda(t)\, e^{-c/\sqrt{(t)}}\, dt < \infty \qquad \text{for some } c > 0. \tag{4.12}$$

The argument used at the end of Example 1 readily adapts to show that (4.12) implies (and is equivalent to) $\lambda(t) = O(e^{k/\sqrt{(t)}})$ for some $k > 0$.

Example 3. *Exponential cusp.* We can get rather close to Theorem 2 by using cusps which tend rapidly to the imaginary axis. Thus the map $f\colon H \to P$ is obtained by composition, with

$$z_1 = \left(\log\frac{1}{z} + i\pi\right) = \left(\log\frac{1}{r} + i(\pi - \theta)\right),$$

$$z_2 = \log z_1,$$

$$w = \frac{ic}{z_2} = \frac{ic}{\log\left(\log\dfrac{1}{z} + i\pi\right)};$$

here a and c are positive constants.

The boundary values for small positive x are given by $x \to (\phi(x), \psi(x))$ where

$$\phi(x) = \frac{c \tan^{-1}\pi\xi}{[\log(\xi^{-2} + \pi^2)]^2 + [\tan^{-1}(\pi\xi)]^2}$$

with $\xi = \left(\log\dfrac{1}{x}\right)^{-1}$, so $\xi \to 0$ as $x \to 0$. A direct computation shows that

$$\frac{d\phi}{dx} = \frac{c\pi\xi^2}{x\left(\log\dfrac{1}{\xi}\right)^2}\left[1 + O\left(\frac{1}{\log\dfrac{1}{\xi}}\right)\right] \qquad (\xi \to 0). \tag{4.13}$$

Let $t = \phi(x)$ so $x = g(t)$ with g as above; then

$$\log\frac{1}{g(t)} = \frac{c\pi[1 + \eta_1(t)]}{4t\left(\log\dfrac{1}{t}\right)^2} \tag{4.14}$$

with

$$\eta_1(t) = O\left(\frac{\log\log\frac{1}{t}}{\log\frac{1}{t}}\right) \qquad (t \to 0). \tag{4.15}$$

Since $g'(t) = (\phi'(x))^{-1}$, (4.13) and (4.14) yield that

$$g'(t) = \frac{c\pi(1+\eta_2(t))}{16t^2\left(\log\frac{1}{t}\right)^2}\exp\left\{\frac{-c\pi(1+\eta_1(t))}{4t\left(\log\frac{1}{t}\right)^2}\right\} \qquad (t \to 0)$$

where η_1 and η_2 satisfy (4.15).

This bound on g' (with c arbitrary and greater than 0) shows that (4.5) will hold for all functions $p(t)$ with

$$\log p(t) = O\left(\frac{1}{t\left(\log\frac{1}{t}\right)^2}\right) \qquad (t \to 0); \tag{4.16}$$

(4.16) is rather close to the bound of Theorem 2.

The question now arises as to whether Theorem 3 covers significantly more functions than Theorems 1 and 2. We construct an example to show that there is a significant difference.

LEMMA 6. *Let* $\{x_n\}_1^\infty$ *be a countable set of distinct real numbers provided with neighborhoods*

$$I_{p,n} = (x_n - \delta_{p,n}, x_n + \delta_{p,n}) \qquad p = 1, 2, \ldots; 1 \le n \le p, \tag{4.17}$$

where, for fixed n, $\delta_{p,n}$ *decreases as a function of* p. *Set*

$$G_p = \bigcup_{n=1}^{p} I_{p,n}, \tag{4.18}$$

an open set containing $\{x_1, \ldots, x_p\}$.

Then the $\delta_{p,n}$ *may be so chosen that*

$$\xi \in G_p \textit{ for all } p \ge p_0(\xi) \Rightarrow \xi = x_m \textit{ for some } m. \tag{4.19}$$

Proof. Let $\rho_n = \min|x_i - x_j|(1 \le i < j \le n)$; then $\rho_n > 0$. We take

$$\delta_p = \delta_{p,n} = \min(\rho_{p+1}/2, p^{-1}),$$

noting that ρ_n decreases as a function of n and that $\delta_p \downarrow 0$ as $p \to \infty$. It is clear that $I_{p,1}, \ldots, I_{p,p}$ are disjoint and $G_p \cap I_{p+1,p+1} = \emptyset$. Thus if $\xi \in G_p$, $\xi \in I_{p,m}$ for a unique m; if also $\xi \in G_{p+1}$, ξ can only belong to $I_{p+1,m}$ or $I_{p+1,p+1}$. However, $I_{p,m} \cap I_{p+1,p+1} = \emptyset$, and so $\xi \in I_{p+1,m}$ for the same m, and this is true for all $p \geq p_0(\xi)$. We conclude that $\xi = x_m$.

While this Lemma is subject to some generalization, it is important that the hypothesis of (4.19) cannot be relaxed to: $\xi \in G_p$ for infinitely many p. For example, let $\{x_n\}$ be dense in [0,1], and sets $I_{p,n}$ be given. Then there is always a $\xi \neq x_n$ for any n which is in infinitely many G_p. For example, let $p_1 = 1$. Then I_{p_1,p_1} contains a point x_{q_1}, $q_1 > p_1$. Then there is a closed interval $J_1 \subset G_{q_1}$ such that $x_1, \ldots, x_{q_1} \notin J_1$. We repeat this argument and find a closed interval $J_2 \subset G_{q_2}$, $\bar{J}_2 \subset J_1$, and $x_1, \ldots, x_{q_1}, \ldots, x_{q_2} \notin J_2$, etc. We then can choose $\xi = \bigcap_1^\infty J_k$.

Example 4. *To show that Theorem* 3 *is more general than Theorems* 1 *and* 2. We shall not attempt the most general sort of example, with an arbitrary set of cusps, supposed to be compatible, given. Our function will be constructed in the upper half-plane H of the z-plane, and the cusps will end on the x-axis.

Thus, let $\{c_n\}$ be a countable dense subset of the reals, and let β, $0 < \beta < \pi/2$, be fixed. We will have two linear cusps, say γ_{2n-1} and γ_{2n}, terminate at each c_n. Let γ_{2n-1} be an arc of $\arg(z - c_n) = \beta$, and γ_{2n} be an arc of $\arg(z - c_n) = \pi - \beta$, and agree that these arcs include their endpoint in H, but not on ∂H (*i.e.*, c_n). The lengths of the γ_n are of some concern, and are specified in (4.30) and (4.31) below.

We will construct a function f(z), holomorphic in H, such that

$$|f(z)| < 1 \qquad (z \in \gamma_m, m \geq 1). \tag{4.20}$$

Thus the subharmonic function

$$U(z) = \log|f(z)| \tag{4.21}$$

(which will turn out to be harmonic here!) will be compatible with the cusps $\{\gamma_m\}$ (*cf.* (4.7)) and

(4.22) *the only linear cusps with which* U(z) *is compatible are those in direction* β *or* $\pi - \beta$ *which end at some* c_n;

finally, for any interval $[x_1, x_2]$, the function

$$\lambda(t) = \sup_{\substack{x_1 \leq x \leq x_2 \\ t \leq y \leq 1}} U(z) \qquad (0 < t < 1) \tag{4.23}$$

satisfies

$$\int_0^1 \log\lambda(t)\,dt = +\infty. \tag{4.24}$$

Hence U(z) does not fall under the purview of Theorem 2.

We start with intertwined sequences $\{a_n\}$, $\{a_n^*\}$, $\{b_n\}$, $\{b_n^*\}$ with

$$(4.25) \qquad 1 > b_1^* > \dots > a_{n-1}^* > b_n^* > b_n > a_n > a_n^* > b_{n+1}^* > \dots > 0,$$

and require that $a_n \to 0$. The numbers in (4.25) are given at the outset and held fixed. Next, E_n, E_n^* are the open strips

$$(4.26) \qquad E_n = \{z; a_n < y < b_n\}, \quad E_n^* = \{z; a_n^* < y < b_n^*\} \qquad (n \geq 1),$$

and we define

$$(4.27) \qquad E = \bigcup E_n, \quad E^* = \bigcup E_n^*.$$

With

$$(4.28) \qquad \delta_n = b_n - a_n,$$

choose a positive sequence $\{\mu_n\}$, with

$$(4.29) \qquad \log \mu_n \geq \delta_n^{-1}, \quad \delta_n \mu_n a_n^{1/n} > 1 \qquad (n \geq 1).$$

We now specify the lengths of the γ_m: $\gamma_1, \gamma_2 \dots$ are to be short enough so that

$$(4.30) \qquad \gamma_p \cap \gamma_q = \emptyset \qquad (p \neq q),$$

$$(4.31) \qquad \gamma_m \cap E_{m-1}^* = \emptyset \qquad (m > 1).$$

Next, let γ_m^e denote the full ray obtained by prolonging γ_m throughout H, and embed $\gamma_m^e \cap E_n$ in a parallelogram $P_{m,n}$. If for $a_n < y_0 < b_n$, $\gamma_m^e \cap \{\mathrm{Im}(z) = y_0\}$ is the point (x_0, y_0), with $x_0 = x_0(m, y_0)$, then we have

$$P_{m,n} = \{z; |x - x_0| < h_n(m, y_0), a_n < y_0 < b_n\}.$$

The sequence h_n is to approach zero so rapidly that the following three conditions are met: when n is odd we must have, in the language of Lemma 6, that the choice $\delta_{p,m} = \delta_p = h_{2p-1}$ $(p = 1, 2, \dots)$ is adequate to obtain (4.19) using the $\{c_m\}$ in place of the $\{x_n\}$; for even n, the choice $\delta_{p,m} = h_{2p}$ must similarly satisfy (4.19). Finally, we must suppose the $\{h_n\}$ so small that if

$$\gamma\colon \arg(z - \xi) = h, \qquad \xi \text{ real}, 0 < h < \pi,$$

is any ray in H, and if both $|h - \alpha| \geq n^{-1}$, $|h - (\pi - \alpha)| > n^{-1}$, then

(4.32) *the segment* $\gamma \cap E_n$ *has at most half its length in* $P_n = \bigcup_{k=1}^{n} P_{k,n}$.

It is easy to arrange (4.32). For example, if γ is a ray of inclination $\beta \pm n^{-1}$, then the horizontal separation of γ and γ_k^e will exceed h_n outside a vertical interval of height $K_n h_n$ where K_n depends only on n and β. Thus, we need only choose h_n so that $n \cdot 2h_n < \delta_n / K_n$.

The final preliminary is to choose, corresponding to each γ_m, a pair of open domains D_m, D_m^* so that ($^-$ below refers to closure relative to H)

$$(4.33)\qquad \begin{aligned} &\gamma_m \subset D_m \subset D_m^*, \bar{D}_m \subset D_m^* && \\ &\bar{D}_m^* \cap \bar{D}_p^* = \emptyset && m \neq p \\ &D_m^* \cap E_n = \emptyset && m > n \\ &D_m^* \cap E_n \subset P_{m,n} && m \leq n; \end{aligned}$$

it is clear from (4.30) and (4.31) that this is possible.

With this preparation we may apply, for example, the recent theorem of N. Arakelyan [1] or more *ad hoc* methods, and obtain a function $f_1(z)$ holomorphic in H with

$$|f_1(z) - \mu_n - 2| < 1, \qquad (z \in \bar{E}_n, n \geq 1).$$

Again, there exists $f_2(z)$, holomorphic in H, such that

$$|f_2(z) - f_1(z)| < 1 \qquad \left(z \in \bigcup_1^\infty \bar{D}_m\right)$$

$$|f_2(z)| < 1 \qquad \left(z \in H - \bigcup_1^\infty D_m^*.\right)$$

Now consider $f_3(z) = f_1(z) - f_2(z)$ and $f(z) = e^{f_3(z)}$ $(z \in H)$; then

$$U(z) = \log|f(z)| = \operatorname{Re}(f_3(z))$$

is *harmonic* in H,

$$(4.34)\qquad |U(z)| < 1 \qquad \left(z \in \bigcup \bar{D}_m\right),$$

$$(4.35)\qquad U(z) \geq \mu_n \qquad \left(z \in E_n - \bigcup_m D_m^*, n \geq 1\right).$$

Since $\gamma_m \subset D_m$, (4.34) shows that $U(z)$ is compatible with the cusps $\{\gamma_m\}$.

Now let γ be a linear cusp *distinct* from each γ_m, $m \geq 1$. First we suppose γ is not in one of the directions β or $(\pi - \beta)$. Then (4.28), (4.32) and (4.33) imply for $n > n_0(\gamma)$ that

(4.36) $\gamma \cap \left[\bar{E}_n - \bigcup_1^\infty D_m^* \right]$ *contains a segment of length greater than* $\delta_n/2$.

If, however, γ is in the direction β, we need Lemma 6 and the choice of h_n made above, which ensures that γ will miss infinitely many of the parallelogramata $P_{2k-1,n}$ ($k = 1, 2, \ldots$). Thus to include this possibility, (4.36) is revised to

(4.37) $\gamma \cap \left[\bar{E}_n - \bigcup_1^\infty D_m^* \right]$ *contains a segment of length greater than or equal to* $\delta_n/2$ *for infinitely many* n;

of course, this set of n depends on γ. If γ has direction $(\pi - \alpha)$, (4.37) also holds.

Suppose $\gamma \notin \{\gamma_m\}$, γ a linear cusp. Then, according to (4.37) and the construction of $U(z)$, there exist infinitely many n such that $U(z) \geq \mu_n$ on a segment, in E_n, whose length is at least $\delta_n/2$; let us designate this infinite set of n by $Z(\gamma)$. Conditions (4.6) and (4.10) lead us to the following estimate:

$$\int_0^1 p(t)\, t^{\pi(\pi-\beta)/\beta}\, dt \geq \frac{1}{2} \sum_{Z(\gamma)} \delta_n \mu_n a_n^{\pi(\pi-\beta)/\beta}\, dt;$$

and the second condition of (4.29) shows that $\int_0^1 p(t)\, t^{\pi(\pi-\beta)/\beta}\, dt = \infty$ for the cusp γ.

This proves (4.22).

Finally, to prove (4.24), we note that the strip $\{x_1 < x < x_2, y > 0\}$ will contain infinitely many regions D_m^*; assume m, fixed, is one. Since the $\bar{D}_p$ are disjoint (see (4.33)) we may use (4.35) for $z \in \partial D_m^* \cap E_n$ and obtain, in the notation of (4.23), that $\lambda(t) \geq \mu_n$ $(a_n < t < b_n)$. Now (4.24) follows from this and the first condition of (4.29). This completes Example 4.

5. THE SJÖBERG-LEVINSON-BEURLING THEOREM

THEOREM 4. *Let* $M(u)$ *be a positive even function which decreases for increasing* $|u|$, *and tends to* ∞ *as* $u \to 0$, *but sufficiently slowly so that*

(5.1) $$\int_0^1 \log\log M(u)\, du < \infty.$$

If the function $f(w)$ $[w = u + iv]$, *holomorphic in the rectangle*

$$R = \{w;\ |u| \leq a,\ |v| \leq b\},$$

satisfies

(5.2) $$\log|f(w)| \leq M(u) \qquad (w = u + iv \in R),$$

then if $0 < \delta < 1$

(5.3) $$|f(w)| \leq C \qquad (|u| \leq a, |v| \leq b(1-\delta))$$

where C *depends only on* M(u), a, b *and* δ.

Remark. The relation between Theorem 4 and [5] was discussed in section 1.

Proof. Let $R_\delta = \{w; |u| \leq a, |v| \leq b(1-\delta)\}$, and assume, with no loss of generality, that $\delta < a/4$. We will produce a cusped region P_δ, which is symmetric with respect to both the u- and v-axes such that if $w \in R_\delta$, then either $w \in P_\delta$ or $|\mathrm{Re}(w)| \geq \delta$. Our main task is to produce a suitable auxiliary function k(u) such that k is continuous, $k(-u) = k(u)$, $k(u) > 0$ $(0 < |u| < \delta', \delta' > 0)$, $k(0) = 0$. Then for $0 \leq |u| \leq \delta_1$, we will let ∂P_δ consist of the arcs

$$(u, b - k(u)) \quad \text{and} \quad (u, -b + k(u)),$$

where δ_1 is so small that $\delta_1 < (\delta, \delta')$ and $k(u) < b\delta/2$ for $u \leq \delta_1$. We then connect the points $(\pm\delta_1, \pm(b - k(\delta_1)))$ to $(\pm\delta, \pm(b-\delta))$ by linear segments, and complete ∂P_δ by including the line segments $\{u = \pm\delta, -(1-\delta)b < v < (1-\delta)b\}$. Now k(u) for $|u| < \delta_1$ will not depend on a, b, δ but how we augment this portion of ∂P_δ to form the full boundary of P_δ will. It will be easy to see from our previous work that $|f(w)|$ is bounded by a constant, say C_1, for $w \in P_\delta \cap R_\delta$, so the constant C of (5.3) can be chosen as the maximum of C_1 and $M(\delta)$.

We now construct k(u), and show how it and the above specifications of P_δ imply that f is bounded in $P_\delta \cap R_\delta$.

We will map the upper half-plane H onto a region P of the upper half-plane in the w-plane which is symmetric about the v-axis and

$$P \cap \{\max(|u|, v) \leq \delta_2\} = \{w; |u| < \delta_2, k(u) < v \leq \delta_2\}$$

for some $\delta_2 > 0$ (*cf.* (4.1)). That is, we again use (2.7), (2.8), but now

$$\phi(x) = -\phi(-x) \qquad (0 \leq x \leq 1)$$

rather than being zero. The dominance argument used in the proof of Theorem 1, when applied to the subharmonic function $\log|f(w)|$, now shows that $|f|$ is bounded on $P_\delta \cap R_\delta$ by an expression which depends only on a, b, δ and M(x), once a function $\phi(x)$ can be found so that

(5.4) $$\phi(-x) = -\phi(x),\ \phi(x) \in C^2, \text{ concave down, with } \int_0^1 \frac{\phi(x)}{x}\,dx < \infty,$$

and such that the Dirichlet problem in H with boundary values

$$\log M(\phi(x)) \qquad (-1 < x < 1)$$

is solvable in H; *i.e.,* that

$$\int_0^1 \log \lambda(\phi(x))\,dx < \infty \tag{5.5}$$

where we have set $\lambda(t) = \log M(t)$. We recall from section 3 (*cf.* (3.20)) that it is possible to arrange that $\lambda(t) \in C^1$ and that if we set $h(t) = t/\lambda(t)$, as in (3.19), then $h'(t) > 0$ for $t > 0$.

Since ϕ will be odd, it suffices to construct $\phi(x)$ for $x > 0$. We write, for $0 < x < \delta_2$, δ_2 sufficiently small, $\phi(x) = t$ if and only if $x = g(t)$. Then, according to (3.17), it suffices to find a $g(t)$ which is concave up, such that (3.17) holds.

Choose $g(t) = h(t) = t/\lambda(t)$, where h is described above; this choice is motivated since then the two conditions (3.17) coalesce to one:

$$\int_0^1 t\,d[-\log \lambda(t)] < \infty. \tag{5.6}$$

Now

$$\int_\varepsilon^1 t\,d[-\log \lambda(t)] = -\log \lambda(1) + \varepsilon \log \lambda(\varepsilon) + \int_\varepsilon^1 \log \lambda(t)\,dt,$$

and so (5.1) implies (5.6); that is, we have produced a $g[\phi]$ which satisfies (3.17) [(5.4), (5.5)]. This completes the proof.

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