EXTREMAL PROPERTIES OF A CLASS OF SLIT CONFORMAL MAPPINGS

W. E. Kirwan and Richard Pell

1. INTRODUCTION

Let U denote $\{z : |z| < 1\}$ and H(U) the space of functions analytic in U endowed with the topology of uniform convergence on compact subsets of U. It is well known that H(U) is a locally convex topological space.

We will be concerned with the set $S \subset H(U)$ consisting of functions f,

$$f(z) = z + a_2 z^2 + \ldots,$$

that are univalent on U, and with several subsets of S. We denote by A the collection of functions $f \in S$ that map U onto the complement of a single analytic slit γ which has an asymptotic direction at ∞ and which possesses the $\pi/4$ property; i.e., the angle between the radius vector and the tangent vector at any point on γ is in absolute value smaller than $\pi/4$. By σ we denote the collection of support points of S; i.e., functions $f \in S$ that satisfy

$$\operatorname{Re} L(f) = \max_{g \in S} \operatorname{Re} L(g)$$

for some continuous linear functional L on H(U) which is nonconstant on S. Finally by E(S) and $E(\overline{co}S)$ we denote the set of extreme points of S and the set of extreme points of the closure of the convex hull of S respectively.

There are various relations between these classes of functions. For example, $\sigma \subset A$ is a result due to Pfluger [8] and later Brickman and Wilken [2]. Further, $E(\overline{co}S) \subset E(S)$ by a general argument for compact subsets of locally convex spaces [4; p. 440]. Also, Brickman [1] proved the striking result that if $f \in E(S)$, then f maps U onto the complement of a single Jordan arc along which $|w|$ increases to ∞.

In [6] Hengartner and Schober proved that if $f \in A$, $f(z) = z + a_2 z^2 + \ldots$, then $|a_2| > 1$. In the present note, we show that in fact $|a_2| > \sqrt{2}$ for functions in A and this result is best possible. In particular, $|a_2| > \sqrt{2}$ holds for $f \in \sigma$. However, we are unable to show that this result is best possible for σ and in fact we have been unable to find an $f \in \sigma$ with $|a_2| < 1.77$.

By a general result for locally convex spaces (see [3; p. 231]), $\bar{\sigma} \supset E(\overline{co}S)$. It therefore follows from our result that if $f \in E(\overline{co}S)$, $|a_2| \geq \sqrt{2}$. Using this

Received October 31, 1977. Revision received March 31, 1978.
Research of first author supported in part by a National Science Foundation grant.

Michigan Math. J. 25 (1978).

fact we can produce a large collection of examples to show that Brickman's result [1] is a necessary but not sufficient condition for $f \in E(\overline{co}S)$.

Finally, we use the fact that $\bar{\sigma} \supset E(\overline{co}S)$ to obtain a refinement of Brickman's monotonic property of $C - f(U)$ for $f \in E(\overline{co}S)$.

2. THE MODULUS OF a_2

In order to prove our result on $|a_2|$ for $f \in A$, we need to use a result of Hengartner and Schober [6].

If $f \in A$, then $\mathbb{C} - f(U)$ is an analytic slit from some finite point to ∞. In addition, f has an analytic extension to $\bar{U}$ except for a pole of order 2 at some point ζ on $|z| = 1$. We denote by η the point on $|z| = 1$ that f maps to the finite tip of the slit. Using the $\pi/4$-property, Hengartner and Schober proved that

$$H(z) = \left(\left[\frac{f(z)}{zf'(z)}\right]^2 + \frac{z+\eta}{z-\eta}\right)\frac{(z-\eta)^2}{\eta z}$$

is analytic in $\bar{U}$ and $\operatorname{Re} H(z) > 0$ for $z \in U$. An easy consequence of this fact, as they pointed out, is the inequality

$$\operatorname{Re} a_2\eta < -1. \tag{2.1}$$

Indeed, (2.1) results from the inequality $\operatorname{Re} H(0) > 0$. This inequality plays an important role in the proof of our main theorem.

THEOREM. *Let* $f \in A$, $f(z) = z + a_2z^2 + \dots$. *Then* $|a_2| > \sqrt{2}$ *and this result is best possible.*

Proof. For $f \in A$, $\mathbb{C} - f(U)$ is a slit domain and there is a Loewner chain

$$f(z,t) = e^t\left[z + \sum_{n=2}^{\infty} a_n(t)z^n\right] \qquad (0 \le t < \infty) \tag{2.2}$$

with $f(z,0) = f(z)$ and $f(z,t_1)$ subordinate to $f(z,t_2)$ if $0 \le t_1 < t_2 < \infty$ (see [9; p. 157]). Clearly $e^{-t}f(z,t)$ belongs to A for each t, $0 \le t < \infty$. For a fixed $f \in A$, consider the problem $\inf_{0\le t<\infty} |a_2(t)|$. Since the curve $\mathbb{C} - f(U)$ has an asymptotic direction at ∞, it is well known (and easy to show; see e.g. [7; p. 176]) that $\lim_{t\to\infty} e^{-t}f(z,t)$ is a Koebe function $z(1-xz)^{-2}$, $|x| = 1$. Thus for $f \in A$, $\lim_{t\to\infty} |a_2(t)| = 2$. It follows that there is a $t_0 < \infty$ with $|a_2(t_0)| = \inf_{0\le t<\infty} |a_2(t)|$. We may assume without loss of generality that $a_2(t_0) > 0$ (note that $|a_2(t_0)| > 1$ by [6]). Indeed, this may be achieved by a suitable rotation of f, $e^{i\varphi}f(e^{-i\varphi}z)$, which does not affect $|a_2|$.

From the minimal property of $a_2(t_0)$ we have

$$\frac{\partial}{\partial t}\log|a_2(t)|\bigg|_{t=t_0} = \operatorname{Re}\frac{a_2'(t_0)}{a_2(t_0)} \ge 0 \tag{2.3}$$

with inequality possible if $t_0 = 0$. On the other hand, by the Loewner equation [9; p. 163] we have

$$\frac{\partial f(z,t)}{\partial t} = zf'(z,t)\frac{1+\overline{\eta(t)}z}{1-\overline{\eta(t)}z}, \tag{2.4}$$

where $\eta(t)$ is the point on $|z| = 1$ that $f(z,t)$ maps to the finite tip of the slit $f(e^{i\theta}, t)$, $0 \leq \theta \leq 2\pi$.

Using (2.2) and comparing the coefficients of z^2 in (2.4), we obtain

$$a_2(t) + a_2'(t) = 2a_2(t) + 2\overline{\eta(t)} \qquad (0 \leq t < \infty).$$

Thus

$$a_2'(t) = a_2(t) + 2\overline{\eta(t)} \qquad (0 \leq t < \infty).$$

But for $t = t_0$, (2.3) implies $\mathrm{Re}\, a_2'(t_0) \geq 0$. Hence

$$a_2(t_0) = \mathrm{Re}\, a_2(t_0) \geq -2\,\mathrm{Re}\,\overline{\eta(t_0)} = -2\,\mathrm{Re}\,\eta(t_0). \tag{2.5}$$

We now apply the inequality (2.1) to conclude that

$$a_2(t_0)\,\mathrm{Re}\,\eta(t_0) = \mathrm{Re}\, a_2(t_0)\eta(t_0) < -1,$$

or

$$-2\,\mathrm{Re}\,\eta(t_0) > \frac{2}{a_2(t_0)}. \tag{2.6}$$

Combining (2.5) and (2.6) we obtain $[a_2(t_0)]^2 > 2$. Since $|a_2| = |a_2(0)| \geq a_2(t_0)$ the proof that $|a_2| > \sqrt{2}$ for $f \in A$ is complete.

It remains to show that the inequality is best possible. Consider the mapping defined by

$$f_\lambda(z) = \frac{z}{(1-z)^{2\cos\lambda e^{i\lambda}}} \qquad (0 < \lambda < \pi/2). \tag{2.7}$$

$f_\lambda \in S$ and maps U conformally onto the complement of a logarithmic spiral, s_λ, that is analytic and has the property that the angle between the radius vector to a point on s_λ and the tangent vector at that point (measured from the radius vector to the tangent vector) is identically equal to $-\lambda$. Indeed,

$$\mathrm{Re}\,\{e^{i\lambda} zf_\lambda'(z)/f_\lambda(z)\} > 0 \qquad \text{for } |z| < 1.$$

Thus $f_\lambda(z)$ belongs to the class of so-called spirallike functions introduced by L. Špaček (see [9; p. 171]). The fact that s_λ is a logarithmic spiral follows from the identity $\operatorname{Re}\{e^{i\lambda} z f'_\lambda(z)/f_\lambda(z)\} = 0$ for $|z| = 1$.

For f_λ defined by (2.7),

$$f_\lambda(z) = z + 2\cos\lambda e^{i\lambda} z^2 + \dots . \tag{2.8}$$

We will show that for any $\lambda < \pi/4$, f_λ may be approximated uniformly on compact subsets of U by functions in A. For $\lambda = \pi/4$ we have from (2.8) that $|a_2| = \sqrt{2}$. It will then follow that the theorem is best possible.

For a fixed $\lambda < \pi/4$ we consider a curve τ_λ constructed from s_λ in the following way. τ_λ begins at the finite tip of s_λ and follows s_λ to a point P of large modulus. From P, τ_λ follows the tangent line of s_λ at P on to ∞. τ_λ is a C^1 curve and the absolute value of the angle between the radius vector to any point on the curve and the tangent vector does not exceed λ. Let g_λ map U onto the complement of τ_λ with $g_\lambda(0) = 0$ and $g'_\lambda(0) > 0$. Clearly as $|P| \to \infty$, $g_\lambda \to f_\lambda$ in H(U). Thus in order to complete the proof, it suffices to show that we can approximate g_λ in H(U) by functions in A.

Let $\tau_\lambda : z = \alpha(t)$, $0 < m \le t < \infty$. Since τ_λ is a ray beyond the point P, we may assume $\alpha(t)$ is of the form $At + B$ if t is large. Also, as noted above, $\alpha'(t)$ is continuous on $m \le t < \infty$ and $\alpha'(t) \equiv A$ for large t. For $\varepsilon > 0$, we may choose an analytic function b(t) such that

$$|b(t) - \alpha'(1/t)| < \varepsilon t^2 \qquad (0 \le t \le 1/m). \tag{2.9}$$

Indeed $\alpha'(1/t)$ has a continuous extension to $t = 0$ (the value at $t = 0$ we denote by $\alpha'(\infty)$). By the Weierstrass theorem, there exists a polynomial p(t) such that

$$\left| p(t) - \frac{\alpha'(1/t) - \alpha'(\infty)}{t^2} \right| < \varepsilon \qquad \left(0 \le t \le \frac{1}{m}\right),$$

and we can set $b(t) = t^2 p(t) + \alpha'(\infty)$. Let $\beta'(1/t) = b(t)$. From (2.9) we have

$$|\beta'(t) - \alpha'(t)| < \varepsilon \frac{1}{t^2} \qquad (m \le t < \infty).$$

Set $\beta(t) = \int_m^t \beta'(t)\,dt + \alpha(m)$ and let $\kappa_\lambda : z = \beta(t)$, $m \le t < \infty$. Then κ_λ is an analytic curve with an asymptotic direction at ∞. Indeed, $\lim_{t\to\infty} \arg \beta'(t) = \lim_{t\to\infty} \arg \alpha'(t)$. Now, $\arg[\alpha'(t)/\alpha(t)]$ measures the angle between the radius vector and the tangent vector at the point $\alpha(t)$ on τ_λ (and this angle in absolute value does not exceed λ). Clearly, then, since $\lambda < \pi/4$ and $\left| \frac{\alpha'(t)}{\alpha(t)} - \frac{\beta'(t)}{\beta(t)} \right|$ is small if ε is small, the angle between the radius vector and the tangent vector at any point of κ_λ does not exceed $\pi/4$ in absolute value. Let h_λ map U conformally onto the complement of κ_λ with $h_\lambda(0) = 0$, $h'_\lambda(0) > 0$. Then $\omega = h_\lambda(z)/h'_\lambda(0)$ defines a function in A. If we now let $\varepsilon \to 0$, it follows by an application of the Carathéodory Kernel Theorem [9, p. 29] that $h_\lambda \to g_\lambda$ in H(U). This completes the proof.

As noted earlier, since $\sigma \subset A$, $|a_2| > \sqrt{2}$ holds for $f \in \sigma$. However, we cannot show this inequality is best possible in σ. By considering the linear functional

$$L_r(g) = g(-r) \qquad \text{for fixed } r,\ 0 < r < 1, \tag{2.10}$$

we can produce an $f \in \sigma$ with $1.77 < |a_2| < 1.774$. Indeed, the extremal function f for the problem $\max_{g \in S} \operatorname{Re} L_r(g)$ satisfies a Schiffer differential equation of the form

$$\left(\frac{zf'(z)}{f(z)}\right)^2 \left(\frac{f^2(-r)}{f(z) - f(-r)}\right) = -Re^{i\phi} \frac{(z - e^{i\phi})^2}{(z + r)(z + 1/r)},$$

where $f(-r) = Re^{i\phi}$. If we compare the coefficients of z in this equation, we obtain

$$a_2 = -\frac{e^{-i\phi}}{2R} - e^{-i\phi} - \frac{1}{2}\left(r + \frac{1}{r}\right).$$

Using a result of Grunsky [5] (see also [9; p. 196]) on the determination of the values for $g(z)/z$, $g \in S$, one can numerically determine $Re^{i\phi}$ for fixed r and hence the smallest value of $|a_2|$ as r varies over the interval (0, 1). This smallest value lies between 1.77 and 1.774.

In [6] Hengartner and Schober proved that $|a_3| > 3/8$ if $f \in A$,

$$f(z) = z + a_2 z^2 + \dots .$$

Using the previous theorem and the area theorem, we can improve this estimate considerably.

COROLLARY. *Let* $f \in A$, $f(z) = z + a_2 z^2 + \dots$, *then* $|a_3| > 1$.

Proof. By the area theorem, $|a_3 - a_2^2| \le 1$. On the other hand if $f \in A$, $|a_2| > \sqrt{2}$ and hence $1 < |a_2|^2 - 1 \le |a_3|$.

3. THE CLOSURE OF A

In this section we give a characterization of $\bar{A}$ (the closure of A in H(U)) in terms of the boundary behavior of the members of $\bar{A}$. First, however, we need to discuss some preliminaries.

Let γ: $z = z(t)$ $(t_0 \le t \le t_1)$ denote a regular $(z'(t) \ne 0)$ C^1 curve. For $t_0 \le t_2 \le t_1$, $\arg[z'(t_2)/z(t_2)]$ is the angle between the radius vector and the tangent vector at the point $z(t_2)$ on γ. If

$$|\arg[z'(t)/z(t)]| < \pi/2 \tag{3.1}$$

for all t, then $|z(t)|$ is a strictly increasing function. Indeed,

$$\frac{d}{dt}\log|z(t)| = \operatorname{Re}\frac{z'(t)}{z(t)} > 0$$

under the assumption of (3.1).

Next we note that given a point z_0 in $\mathbb{C}$ and a value λ ($|\lambda| < \pi/2$ for our purposes), the conditions

$$\arg\frac{z'(t)}{z(t)} = \lambda, \qquad z(t_0) = z_0 \tag{3.2}$$

determine a unique (up to parameterization) arc starting at the point z_0. The arc determined by (3.2) is a logarithmic spiral and, of course, the angle between the radius vector and tangent vector (measured from the radius vector to the tangent vector) is λ. For reference we call the arc defined by (3.2) the *λ-spiral emanating from* z_0.

In the proof of the theorem of this section we will need to construct a specific type of neighborhood. Given a point $z_0 = r_0 e^{i\theta_0}$ in $\mathbb{C}$ we consider the curvilinear quadrilateral defined by

$$\{z : z = re^{i\theta}, r_1 < r < r_2, \theta_1 < \theta < \theta_2\}, \tag{3.3}$$

where $r_1 < r_0 < r_2, \theta_1 < \theta_0 < \theta_2$ and $r_2 - r_1$ and $\theta_2 - \theta_1$ are small. Construct the $(-\lambda)$-spiral $(0 < \lambda < \pi/4)$ emanating from $r_1 e^{i\theta_1}$. This spiral winds (with increasing modulus) in a clockwise direction about the origin. Follow this spiral until it intersects the circle $|z| = r_2$. Denote the arc of the spiral so determined by γ_1. Now construct a λ-spiral emanating from $r_1 e^{i\theta_2}$. This spiral winds (with increasing modulus) in a counterclockwise direction about the origin. Follow it until it intersects the circle $|z| = r_2$. Denote the arc of the spiral so determined by γ_2. We denote by H_1 the Jordan domain bounded by the circular arc $|z| = r_1$, $\theta_1 \le \theta \le \theta_2$, γ_1, γ_2 and the arc of $|z| = r_2$ joining the endpoints of γ_1 and γ_2 (see Fig. 1). Next construct the λ-spiral emanating from a point $r_1 e^{i\theta_3}$ that passes through $r_2 e^{i\theta_1}$. Necessarily $\theta_3 < \theta_1$ but $\theta_1 - \theta_3$ is small. Denote by γ_3 that portion of this arc that joins $r_1 e^{i\theta_3}$ and $r_2 e^{i\theta_1}$. Similarly construct the $(-\lambda)$-spiral emanating from a point $r_1 e^{i\theta_4}$ that passes through $r_2 e^{i\theta_2}$. Here $\theta_4 > \theta_2$ but $\theta_4 - \theta_2$ is small. Denote the arc joining $r_1 e^{i\theta_4}$ and $r_2 e^{i\theta_2}$ by γ_4. The Jordan domain bounded by the arc of $|z| = r_1$ joining the endpoints of γ_3 and γ_4, γ_3, γ_4 and the arc of $|z| = r_2$ with $\theta_1 \le \theta \le \theta_2$ we denote by H_2 (see Fig. 2). Finally, we call the Jordan domain $H = H_1 \cup H_2$ an Ω_λ neighborhood of z_0 (see Fig. 3). For our purposes the essential feature of the Ω_λ neighborhood is that if γ: $z = z(t)$ is any curve that satisfies $|\arg[z'(t)/z(t)]| < \lambda$ and passes through a point z_1 of the quadrilateral (3.3), then γ can intersect the boundary of the Ω_λ neighborhood H only at a point of $|z| = r_1$ or of $|z| = r_2$. Indeed γ must lie "between" the λ-spiral and the $(-\lambda)$-spiral through z_1 and these two spirals do not intersect any of the γ_i (see Fig. 4). We are now in a position to prove the theorem of this section.

THEOREM. *Let* $f \in \bar{A}$. *Then* $\mathbb{C} - f(U)$ *is a Jordan arc* γ: $z = z(t)$ $(t_0 \le t < \infty)$, $|z(t)|$ *is strictly increasing and for each* $t_1 \ge t_0$,

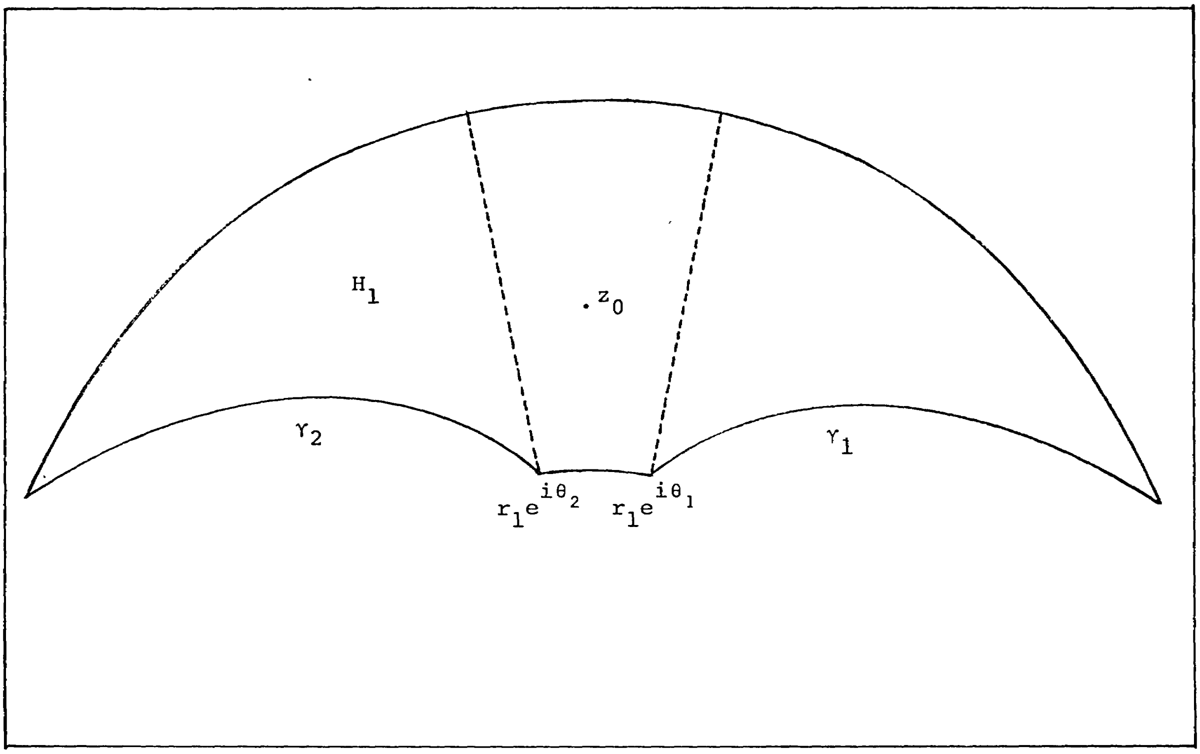
H_1
$\cdot z_0$
γ_2
γ_1
$r_1e^{i\theta_2}$
$r_1e^{i\theta_1}$

Fig. 1

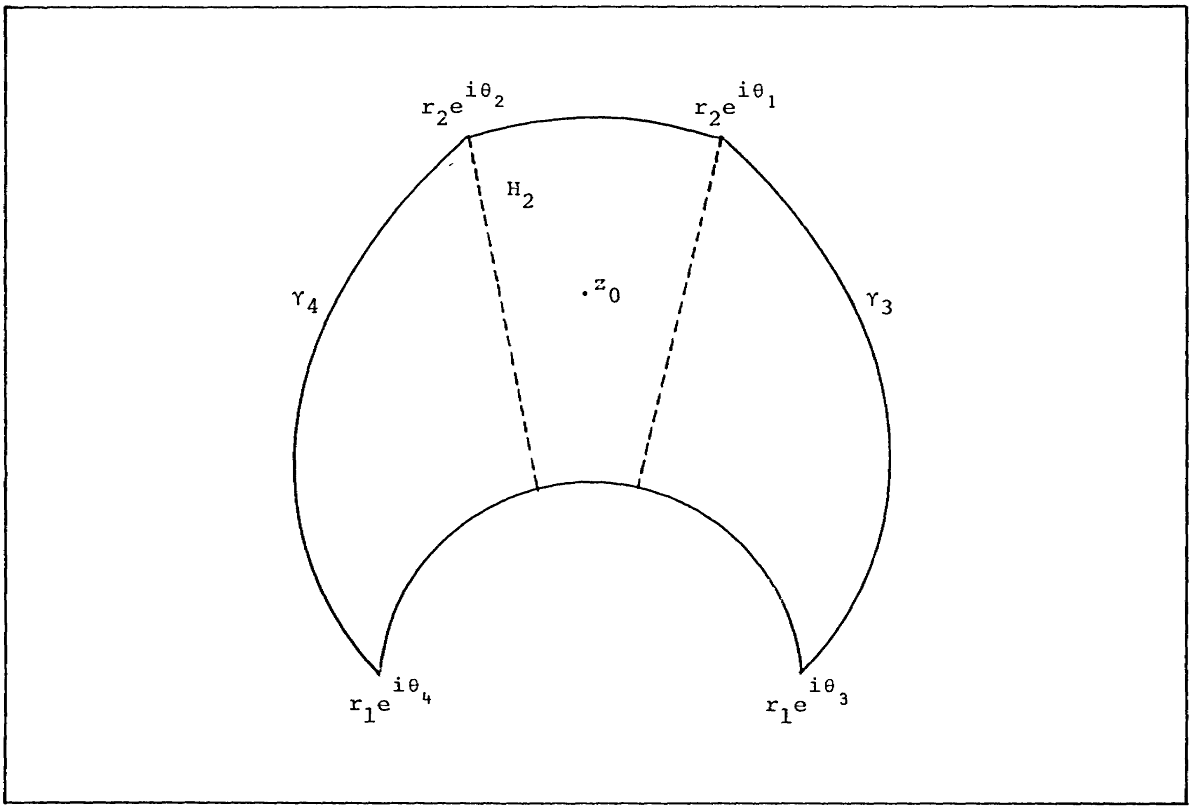
$r_2e^{i\theta_2}$
$r_2e^{i\theta_1}$
H_2
γ_4
$\cdot z_0$
γ_3
$r_1e^{i\theta_4}$
$r_1e^{i\theta_3}$

Fig. 2

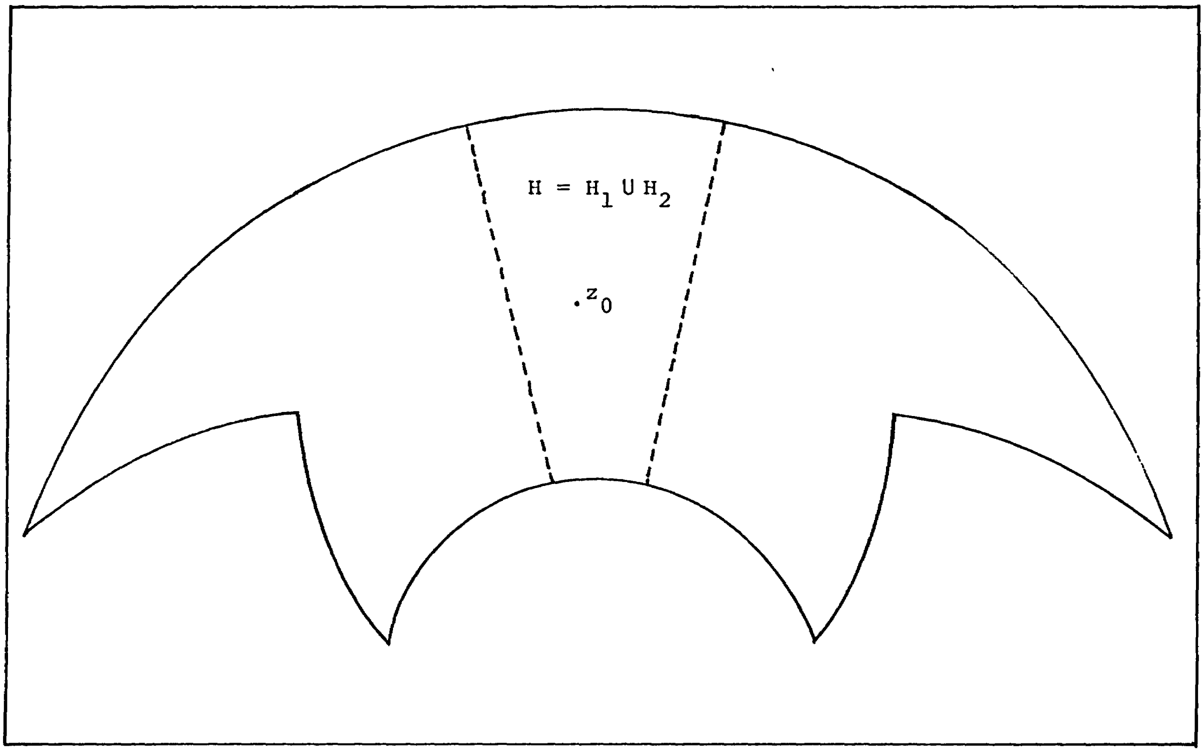

Fig. 3

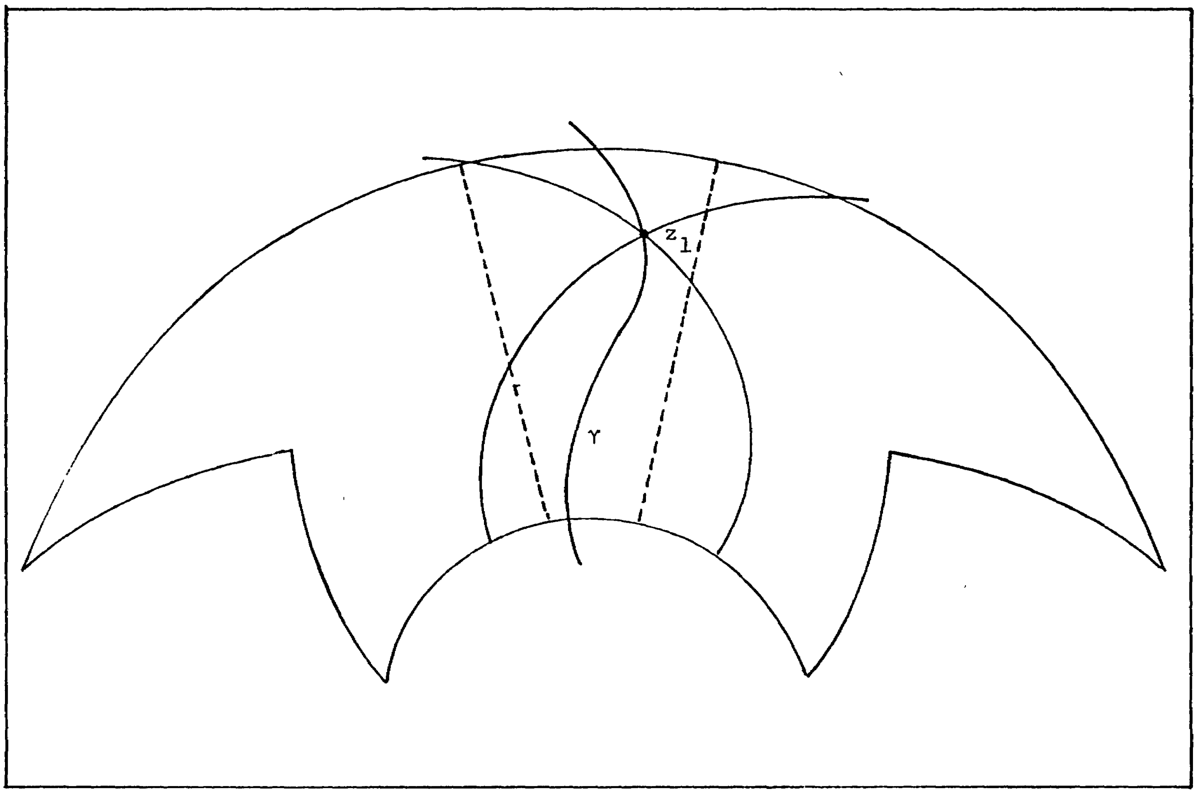

Fig. 4

$$\limsup_{t \to t_1} \left| \arg \frac{z(t) - z(t_1)}{z(t_1)} \right| \le \frac{\pi}{4}. \tag{3.4}$$

In particular, at any point where z(t) *is differentiable,*

$$\left| \arg \frac{z'(t)}{z(t)} \right| \le \frac{\pi}{4}.$$

Proof. In order to prove that $\mathbb{C} - f(U)$ is a monotone arc, it suffices, by an argument due to Brickman [1], to show that each circle $|w| = r$ meets $\mathbb{C} - f(U)$ in at most one point. Suppose on the contrary that some circle $|w| = r$ intersects $\mathbb{C} - f(U)$ in two points w_1 and w_2 which we may suppose are boundary points of $f(U)$. Let $f_n \in A$ be chosen such that $f_n \to f$ in $H(U)$ and such that there exist points w_n on the boundary of $f_n(U)$ with $w_n \to w_1$. Choose disjoint $\Omega_{\pi/4}$ neighborhoods of w_1 and w_2, say Ω' and Ω'', respectively. We may assume that the bounding circular arcs of Ω_1 lie on the same two circles as do the bounding circular arcs of Ω_2. Since $f_n \in A$, the boundary of $f_n(U)$ has the $\pi/4$-property (and in particular is monotonic). Thus if w_n is sufficiently close to w_1, then by the property of $\Omega_{\pi/4}$ neighborhoods mentioned above, the boundary curve of $f_n(U)$ does not pass through Ω''. That is, $\Omega'' \subset f_n(U)$ for n sufficiently large. However, $w_2 \in \Omega''$ is a boundary point of $f(U)$ and so Ω'' contains points of

$$f(U) = \ker f_n(U)$$

($\ker f_n(U)$ = kernel of $\{f_n(U)\}$ [9; p. 29]; the previous equality is a consequence of the Carathéodory Kernel Theorem). It follows that $\Omega'' \subset \ker f_n(U) = f(U)$. But, this contradicts the fact that w_2 belongs to the boundary of $f(U)$.

It remains to show condition (3.4) is satisfied by $\gamma: z = z(t)$. Again choose $f_n \to f$, $f_n \in A$. Let $z(t_1)$ be a point on γ and $w_n \to z(t_1)$ with w_n on the boundary of $f_n(U)$. Choose an $\Omega_{\pi/4}$ neighborhood of $z(t_1)$, say Ω. If n is sufficiently large, then by the $\Omega_{\pi/4}$ property of Ω and the fact that $f_n \in A$, all the curves γ_n which form the boundary of $f_n(U)$ can intersect the boundary of Ω only along the bounding circular arcs. In the quadrilateral of the form (3.3) associated with Ω, we first let θ_1 and $\theta_2 \to \arg z(t_1)$ and $r_1 \to |z(t_1)|$, holding r_2 fixed. It follows that all limit points of the γ_n which lie in the annulus $|z(t_1)| < |z| < r_2$ must lie in the Jordan domain bounded by an arc of the $(-\pi/4)$-spiral emanating from $z(t_1)$, an arc of the $\pi/4$-spiral emanating from $z(t_1)$ and an arc of $|w| = r_2$ that joins the points of these spirals on $|w| = r_2$. This establishes (3.4) for $t > t_1$. If $t < t_1$ we argue in the same way except that now we hold r_1 fixed and let $r_2 \to |z(t_1)|$.

It is not hard to show that the converse of this theorem is also true. This follows from a construction similar to that used to show that the estimate $|a_2| > \sqrt{2}$ is best possible in A.

As was pointed out earlier, $E(\overline{\text{co}}S) \subset \bar{A}$ and so we have the following

COROLLARY. *If* $f \in E(\overline{\text{co}}S)$, *then* $\mathbb{C} - f(U)$ *is a Jordan arc* $\gamma: z = z(t)$ $(t_0 \le t < \infty)$, *that satisfies the condition* (3.4).

In particular, this shows that members of $E(\overline{co}S)$ have a "generalized" $\pi/4$-property. This fact complements the information contained in Brickman's result [1].

In conclusion we note that the functions of the form (2.7) with $\pi/4 < \lambda < \pi/2$ map onto the complement of a spiral which has Brickman's monotonic property. However, the previous corollary rules out the possibility that these functions belong to $E(\overline{co}S)$.

Acknowledgement. The authors would like to express appreciation to Carl FitzGerald and Lawrence Zalcman for helpful conversations during the preparation of this paper.

REFERENCES

1. L. Brickman, *Extreme points of the set of univalent functions.* Bull. Amer. Math. Soc. 76 (1970), 372–374.
2. L. Brickman and D. R. Wilken, *Support points of the set of univalent functions.* Proc. Amer. Math. Soc. 42 (1974), 523–528.
3. ———, *Subordination and insuperable elements.* Michigan Math. J. 23 (1976), 225–233.
4. N. Dunford and J. T. Schwartz, *Linear Operators. I. General Theory.* Pure and Applied Mathematics, Vol. 7. Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London; 1958.
5. H. Grunsky, *Neue Abschätzungen zur konformen Abbildung ein-und mehrfach zusammenhängender Bereiche.* Schr. math. Semin. u. Inst. angew. Math. Univ. Berlin 1 (1932), 95–140.
6. W. Hengartner and G. Schober, *Some new properties of support points for compact families of univalent functions in the unit disc.* Michigan Math. J. 23 (1976), 207–216.
7. J. A. Hummel, *Lectures on variational methods in the theory of univalent functions.* Lecture Note #8. University of Maryland Department of Mathematics, College Park, Maryland, 1972.
8. A. Pfluger, *Lineare Extremalprobleme bei schlichten Funktionen.* Ann. Acad. Sci. Fenn. Ser. AI No. 489 (1971), 32 pp.
9. Ch. Pommerenke, *Univalent Functions.* Vandenhoeck und Ruprecht, Göttingen, 1975.

Department of Mathematics
University of Maryland
College Park, Maryland 20742