BOUNDARY VALUE ESTIMATION OF THE RANGE OF AN ANALYTIC FUNCTION

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Let S be a set of complex numbers and G a function analytic in $D = \{|z| < 1\}$. Denote the nontangential limit of G at $e^{i\theta}$, if it exists, by $G(e^{i\theta})$ and write $G(e^{i\theta}) \in S$ a.e. to mean that, for almost all θ, the limit does exist and belongs to S.

We seek conditions under which

$$G(e^{i\theta}) \in S \text{ a.e.} \Rightarrow G(D) \subset S. \tag{1}$$

Neuwirth and Newman [12] showed that if S is the positive real axis, then (1) holds for all G in the Hardy class $H^{1/2}$. Hansen [8] gave a more general approach to such results and showed, among other things, that if S is the complement of the sector $\{re^{i\theta} : r \geq 0, 0 \leq \theta \leq \pi/p\}$, where $p > 1/2$, then (1) holds for all $G \in H^p$.

Now suppose that F is also analytic in D and $p > 0$. If $F(D) \cap G(D)$ is nonempty and $G(e^{i\theta}) \notin F(D)$ a.e., then

$$G \in H^p \Rightarrow F \in H^p \tag{2}$$

[1, Theorem 4.4]. Hansen's method, somewhat strengthened, can be described as follows: If $\mathscr{F}$ is a family of functions analytic in D such that $\bigcup_{F \in \mathscr{F}} F(D)$ is a dense subset of $\mathbb{C} - S$ and $F \notin H^p$ for all $F \in \mathscr{F}$, then (1) holds for all $G \in H^p$. This assertion follows at once from the statement containing (2) and easily yields the above examples and related results.

It is classical that if S is the imaginary axis, then (1) holds for all $G \in H^1$. Tepper and Neuwirth [13], who also study (1), ask the question whether H^1 can be replaced by a larger class of functions.

We show here that in all of these examples H^p can be replaced by the larger class M^p defined below and give related results and extensions. For example, if S is compact and $\mathbb{C} - S$ is connected, then (1) holds for all $G \in M^{\log}$ (see Corollary 3). Also, if u is harmonic in the half-space $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$ and has a vanishing nontangential limit at almost every $x \in \mathbb{R}^n$, then, under a similar mild condition, u vanishes everywhere on $\mathbb{R}^{n+1}_+$ (see Theorem 3).

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1. BASIC TOOL

If $0 < \alpha < 1$, let $\Omega_\alpha(\theta)$ be the interior of the smallest convex set containing the circle $|z| = \alpha$ and the point $e^{i\theta}$. The nontangential maximal function of G is defined by

$$N_\alpha(G)(\theta) = \sup_{z \in \Omega_\alpha(\theta)} |G(z)| .$$

The following theorem, in which m denotes Lebesgue measure on $[0, 2\pi)$, provides a tool for the study of the range of G.

THEOREM 1. *Let* F *and* G *be analytic in* D *with* $F(D) \cap G(D)$ *nonempty and* $G(e^{i\theta}) \notin F(D)$ a.e. *Then*

$$m(N_\alpha(F) > \lambda) \leq c_\alpha \frac{1+|a|}{1-|a|} \cdot \frac{1+|b|}{1-|b|} m(N_\alpha(G) > \lambda), \qquad \lambda > 0, \tag{3}$$

for all $a, b \in D$ *such that* $F(a) = G(b)$. *The choice of the positive real number* c_α *depends only on* α.

The case $F(0) = G(0)$ is proved in [2]. Our proof there rests on the conformal invariance of Brownian motion, the relation between the nontangential maximal function and the Brownian maximal function, and the relation between nontangential convergence and convergence along a Brownian path to the boundary. To make the result accessible to a wider audience, we give a nonprobabilistic (but parallel) proof of Theorem 1 here.

2. SOME CONDITIONS UNDER WHICH (1) HOLDS

Let Φ be a function that is defined and positive for all large positive λ. For convenience, denote by M^Φ the class of all functions G analytic in D such that

$$\liminf_{\lambda \to \infty} \Phi(\lambda) m(N_\alpha(G) > \lambda) = 0 .$$

This class is independent of α (see (18) below, for example). If $\Phi(\lambda) = \lambda^p$, write M^p for M^Φ.

THEOREM 2. *Suppose* $\mathscr{F}$ *is a family of functions analytic in* D *such that* (i) $\bigcup_{F \in \mathscr{F}} F(D)$ *is a dense subset of* $\mathbb{C} - S$ *and* (ii) $F \notin M^\Phi$ *for all* $F \in \mathscr{F}$. *Then* (1) *holds for all* $G \in M^\Phi$.

Proof. Suppose that $G \in M^\Phi$ and $G(e^{i\theta}) \in S$ a.e. but $G(D) \not\subset S$. Then G is nonconstant so G(D) is an open set containing a point of $\mathbb{C} - S$. By (i), there is an $F \in \mathscr{F}$ such that $F(D) \cap G(D)$ is nonempty and $G(e^{i\theta}) \notin F(D)$ a.e. Now using Theorem 1 and the assumption $G \in M^\Phi$, we see that $F \in M^\Phi$. But this contradicts (ii). Therefore, $G(D) \subset S$.

Note that if $G(D) \subset S$, then $G(D) \subset S^{\circ}$, the interior of S, or G is constant.

COROLLARY 1. *If* $G \in M^1$ *and* $\operatorname{Re} G(e^{i\theta}) = 0$ a.e., *then* G *is constant.*

Proof. Let $F(z) = (1+z)/(1-z)$. Then $N_\alpha(F)(\theta) \geq N_0(\operatorname{Re} F)(\theta) \sim 1/\theta$ as $\theta \downarrow 0$, where N_0 denotes the radial maximal operator, so

$$\liminf_{\lambda\to\infty} \lambda m(N_\alpha(F) > \lambda) > 0 .$$

Since $F(D) = \{\operatorname{Re} z > 0\}$, the family $\mathscr{F} = \{F, -F\}$ satisfies the conditions of Theorem 2 relative to M^1 and $S = \{\operatorname{Re} z = 0\}$.

COROLLARY 2. *If* S *is the complement of the sector* $\{re^{i\theta}: r > 0, |\theta| < \pi/2p\}$, *where* $p \geq 1/2$, *then* (1) *holds for all* $G \in M^p$.

Proof. Here let $\mathscr{F} = \{f\}$ where f is analytic in D and satisfies $f^p = F$, the function of the above proof. Now use $[N_\alpha(f)]^p = N_\alpha(F)$ to complete the proof.

To see that $H^p \subset M^p$, let $G \in H^p$. Then, by a result of Hardy and Littlewood [9, Theorem 27], $N_\alpha(G) \in L^p(0, 2\pi)$. Therefore, by the Lebesgue dominated convergence theorem,

$$\lambda^p m(N_\alpha(G) > \lambda) \leq \int_{\{N_\alpha(G) > \lambda\}} |N_\alpha(G)|^p \, dm \to 0$$

as $\lambda \to \infty$, so $G \in M^p$. By Remark 4(d) below, $H^p \neq M^p$.

If S is replaced by any rotation or translation of S, Corollary 2 remains true. So Corollary 2 contains the above mentioned results of Neuwirth and Newman and of Hansen. Hansen's other results in [8] remain true and are strengthened if H^p is replaced by M^p. This includes his improvement of the Gehring-Lohwater theorem on asymptotic values.

COROLLARY 3. *Let* S *be a bounded set such that* $\mathbb{C} - \bar{S}$ *is a connected dense subset of* $\mathbb{C} - S$. *Then* (1) *holds for all* $G \in M^{\log}$.

Here $\bar{S}$ denotes the closure of S.

Proof. We can assume that $\bar{S} \subset D$. For each $a \in D - \bar{S}$, there is a simple closed analytic curve in D such that S_a, the union of the curve and the bounded component of its complement relative to $\mathbb{C}$, satisfies $\bar{S} \subset S_a \subset D$ but $a \notin S_a$. This follows from the connectedness of $\mathbb{C} - \bar{S}$. The domain $\mathbb{C} - S_a$ is doubly connected and is bounded by the above simple closed analytic curve and the point at infinity. Accordingly, there is an analytic function F_a mapping D onto $\mathbb{C} - S_a$ such that $F_a(e^{i\theta}) \in S_a$ a.e. (for example, see [7, Chapter 6]). Since $\bigcup_{a \in D - \bar{S}} F_a(D) = \mathbb{C} - \bar{S}$ is dense in $\mathbb{C} - S$, the family $\mathscr{F} = \{F_a : a \in D - \bar{S}\}$ satisfies condition (i) of Theorem 2. Relative to $M^{\log}$, condition (ii) is also satisfied as we now show. Let $f = \exp F$ where F is the function defined in the proof of Corollary 1. Then

$$N_\alpha(f) = \exp N_\alpha(\operatorname{Re} F) \geq \exp N_0(\operatorname{Re} F)$$

so

$$\liminf_{\lambda\to\infty} (\log \lambda)\, m(N_\alpha(f) > \lambda) > 0. \tag{4}$$

But $F_a(e^{i\theta}) \notin f(D) = \{|z| > 1\}$ a.e. and $F_a(D) \cap f(D)$ is nonempty. Therefore, by Theorem 1, F_a must also satisfy (4). So $\mathscr{F}$ satisfies the conditions of Theorem 2 and the proof is complete.

3. PROOF OF THEOREM 1

We shall divide the proof into several steps, each with some independent interest.

Let U be an open subset of $\mathbb{C}$ and u a function nonnegative and superharmonic in U. If $E \subset \mathbb{C}$, let $\Phi^E_{u:U}$ denote the family of all nonnegative superharmonic functions v on U such that $u \le v$ on $E \cap U$. The réduite (or reduced function) of u relative to E in U is defined by

$$R^E_{u:U}(z) = \inf\{v(z) \colon v \in \Phi^E_{u:U}\}, \qquad z \in U,$$

and this classical notion is central to what follows.

The letter c will always denote a positive real number but not necessarily the same number from one use to the next. If its choice depends on a parameter, this will be indicated by a subscript.

LEMMA 1. *If* F *is a function analytic in* D, *then*

$$m(N_\alpha(F) > \lambda) \le c_\alpha R^{\{|F(z)|>\lambda\}}_{1:D}(0) \tag{5}$$

and, for all $a \in D$,

$$R^{\{|F(z)|>\lambda\}}_{1:D}(0) \le c\,\frac{1+|a|}{1-|a|}\,R^{\{|F(z)|>\lambda\}}_{1:D}(a). \tag{6}$$

LEMMA 2. *If* F *is analytic in* D *and* $F(D) \subset W$, *an open subset of* $\mathbb{C}$, *then*

$$R^{\{|F(z)|>\lambda\}}_{1:D}(a) \le R^{\{|w|>\lambda\}}_{1:W}(F(a)), \qquad a \in D. \tag{7}$$

LEMMA 3. *If* G *is analytic in* D *and, for some* $b \in D$, $G(b) \in W$, *an open subset of* $\mathbb{C}$, *but* $G(e^{i\theta}) \notin W$ a.e., *then*

$$R^{\{|w|>\lambda\}}_{1:W}(G(b)) \le R^{\{|G(z)|>\lambda\}}_{1:D}(b). \tag{8}$$

LEMMA 4. *If* G *is analytic in* D, *then, for all* $b \in D$,

$$R^{\{|G(z)|>\lambda\}}_{1:D}(b) \le \frac{1}{4\alpha}\,\frac{1+|b|}{1-|b|}\,m(N_\alpha(G) > \lambda). \tag{9}$$

To prove Theorem 1, suppose that F and G satisfy the assumptions of the theorem and $F(a) = G(b)$. Then

$$R_{1:D}^{\{|F(z)|>\lambda\}}(a) \leq R_{1:D}^{\{|G(z)|>\lambda\}}(b).$$

For F nonconstant, this follows from Lemmas 2 and 3 with $W = F(D)$, and, for F constant, the inequality is trivial: If the left-hand side is not equal to zero, then $|G(b)| = |F(a)| > \lambda$ and both sides are equal to one.

Therefore, in view of Lemmas 1 and 4, Theorem 1 follows.

We shall prove the above lemmas in the following order: 2, 3, 4, 1.

Proof of Lemma 2. Let $u \in \Phi_{1:W}^{\{|w|>\lambda\}}$. Then $u(F) \in \Phi_{1:D}^{\{|F(z)|>\lambda\}}$ so that

$$R_{1:D}^{\{|F(z)|>\lambda\}} \leq u(F),$$

which implies (7).

Proof of Lemma 3. First, under the conditions of the lemma,

$$R_{1:W}^{\{|w|\geq\lambda\}}(G(b)) \leq R_{1:D}^{\{|G(z)|\geq\lambda\}}(b). \tag{8'}$$

If $|G(b)| \geq \lambda$, each side is one, so, to prove this, we can assume that $|G(b)| < \lambda$.

Let $W_1 \subset W_2 \subset \ldots$ be a sequence of bounded open sets such that $G(b) \in W_j \uparrow W$, $\bar{W}_j \subset W$, and each point of ∂W_j is a regular boundary point for the solution of the Dirichlet problem in W_j. Such a sequence exists [11, Corollary 8.28]. Let $u_j = R_{1:W_j}^{\{|w|\geq\lambda\}}$. From the definition of the réduite, it follows that $u_j \leq u_{j+1}$ on W_j so we may define u_∞ on W by setting $u_\infty = \lim_{k\to\infty} u_{j+k}$ on W_j, $j \geq 1$. Then, as we shall show,

$$R_{1:W}^{\{|w|\geq\lambda\}} \leq u_\infty \tag{10}$$

and

$$u_j(G(b)) \leq R_{1:D}^{\{|G(z)|\geq\lambda\}}(b), \tag{11}$$

and (8′) will follow.

To prove (10) and (11), we shall use the following properties of u_j: u_j is nonnegative and superharmonic on W_j, $u_j = 1$ on $\{|w| \geq \lambda\} \cap W_j$, and u_j is harmonic on $\{|w| < \lambda\} \cap W_j$ with

$$\lim_{\substack{w\to w_0 \\ w\in W_j}} u_j(w) = 0 \qquad \text{if } |w_0| < \lambda,\ w_0 \in \partial W_j. \tag{12}$$

These follow from the fact that each point of $\{|w| = \lambda\} \cap W_j$ is regular for $\{|w| < \lambda\} \cap W_j$ [11, Theorem 8.26] so that here u_j is already lower semi-continuous, hence superharmonic (see Theorem 9.25 and Lemma 7.11 of [11]).

Therefore, u_∞ is nonnegative and superharmonic on W and $u_\infty = 1$ on $\{|w| \geq \lambda\} \cap W$. That is, $u_\infty \in \Phi_{1:U}^{\{|w|\geq\lambda\}}$, which gives (10).

To prove (11), let $v \in \Phi_{1:D}^{\{|G(z)|\geq\lambda\}}$ and

$$f(z_o) = \liminf_{\substack{z \to z_o \\ z \in B}} [v(z) - u_j(G(z))], \qquad z_o \in \partial B, \tag{13}$$

where B is the component of $\{|G(z)| < \lambda\} \cap G^{-1}(W_j)$ containing b. Note that v is superharmonic in $B \subset D$ and $u_j(G)$ is harmonic in B with $u_j(G) \leq 1$ there. If $z_o \in D \cap \partial B$ and $|G(z_o)| \geq \lambda$, then $v(z_o) \geq 1$ so, by the lower semi-continuity of v in D, $f(z_o) \geq v(z_o) - 1 \geq 0$. If $z_o \in D \cap \partial B$ and $|G(z_o)| < \lambda$, then $G(z_o) \in \partial W_j$ so, by (12), $f(z_o) \geq v(z_o) - 0 \geq 0$. Therefore,

$$f(z_o) \geq 0, \qquad z_o \in D \cap \partial B. \tag{14}$$

The other part of the boundary of B is small:

$$\mu_b^B(\partial D \cap \partial B) = 0, \tag{15}$$

where μ_z^B is harmonic measure relative to B at the point z. To see this, let $\partial_o D$ denote the set of all $e^{i\theta}$ such that either the nontangential limit of G at $e^{i\theta}$ does not exist or $G(e^{i\theta}) \in W$. Let $\partial_o B = (\partial D \cap \partial B) - \partial_o D$. By the assumption of the lemma, $\mu_o^D(\partial_o D) = 0$ and, as we shall see below, $\partial_o B$ is countable. Therefore, $\mu_o^D(\partial D \cap \partial B) = 0$ and we can conclude that $\mu_b^B(\partial D \cap \partial B) \leq \mu_b^D(\partial D \cap \partial B) = 0$.

To show that $\partial_o B$ is countable, we follow a line of argument similar to that of Hansen [8, page 191]. Let $\Omega_\alpha^n(\theta) = \{z \in \Omega_\alpha(\theta): |z| > 1 - 1/n\}$. For each $e^{i\theta} \in \partial_o B$, $G(z) \to G(e^{i\theta}) \notin \bar{W}_j$ as $z \to e^{i\theta}$, $z \in \Omega_\alpha(\theta)$, so there is a positive integer $n = n(\theta)$ such that $\Omega_\alpha^n(\theta)$ is disjoint from $G^{-1}(W_j)$, hence disjoint from B. Suppose that $\partial_o B$ is uncountable. Then there are numbers $\theta_1 < \theta_2 < \theta_3$ in $[0, 2\pi)$ with $e^{i\theta_k} \in \partial_o B$ and $n(\theta_k) = n$, say, such that no two of the $\Omega_\alpha^n(\theta_k)$ are disjoint. The connected set B is a subset of one of the components of $D - \bigcup_{k=1}^{3} \Omega_\alpha^n(\theta_k)$. Therefore, at least one of the $e^{i\theta_k}$ cannot be a boundary point of B, which is a contradiction. Therefore, $\partial_o B$ is countable.

The function f, defined by (13), is lower semi-continuous on ∂B, hence Borel measurable, and is bounded from below. Therefore, the upper solution for the Dirichlet problem in B corresponding to the boundary function f is $\int_{\partial B} f\, d\mu_z^B$ [11, Theorem 8.13]. Since, by (13), $v - u_j(G)$ is in the upper class of f,

$$\int_{\partial B} f\, d\mu_b^B \leq v(b) - u_j(G(b)).$$

By (14) and (15), the integral on the left is nonnegative. Therefore, $u_j(G(b)) \leq v(b)$, which implies (11), and completes the proof of (8′).

To obtain (8), replace λ by $\lambda + 1/n$ in (8′) and use the fact [5, page 111] that if U is an open set of $\mathbb{C}$ and $E_1 \subset E_2 \subset \ldots$ are subsets of $\mathbb{C}$ with union E, then

$$R^{E_n}_{1:U} \uparrow R^{E}_{1:U} \qquad \text{as } n \uparrow \infty . \tag{16}$$

Proof of Lemma 4. Let $E = \{\theta \in [0,2\pi): N_\alpha(G)(\theta) > \lambda\}$ and v be the Poisson integral of the characteristic function of E:

$$v(z) = \frac{1}{2\pi}\int_E \frac{1-r^2}{1-2r\cos(\theta - t) + r^2}\,dt, \qquad z = re^{i\theta} .$$

Let $\beta = \pi/2\alpha$. Then v is harmonic in D and, as we shall show,

$$\beta v \geq 1 \qquad \text{on } \{|G(z)| > \lambda\} . \tag{17}$$

Therefore, $\beta v \in \Phi^{\{|G(z)|>\lambda\}}_{1:D}$ and

$$R^{\{|G(z)|>\lambda\}}_{1:D}(z) \leq \beta v(z) \leq \frac{\beta}{2\pi}\,\frac{1+r}{1-r}\int_E dt ,$$

which implies (9).

To show (17), fix z satisfying $|G(z)| > \lambda$. If $|z| \leq \alpha$, then $E = [0, 2\pi)$ so $v(z) = 1$ and $\beta v(z) \geq 1$ holds trivially. If $|z| > \alpha$, let $E_z = \{\theta \in [0,2\pi): z \in \Omega_\alpha(\theta)\}$ and A_z be the corresponding arc: $A_z = \{e^{i\theta}: \theta \in E_z\}$. Since $E_z \subset E$,

$$2\pi v(z) \geq \int_{E_z} \frac{1-r^2}{1-2r\cos(\theta - t) + r^2}\,dt .$$

Note that each of the two lines containing z and an endpoint of the arc A_z is tangent to the circle with radius α and center at the origin. Consider the arc on the unit circle between these two lines that is opposite to A_z. It is a classical fact [14, Vol. I, page 99] that the integral on the right-hand side of the last inequality is the length of this opposite arc. A simple argument using similar triangles shows that this length is not less than 4α. Therefore, $2\pi v(z) \geq 4\alpha$, so $\beta v(z) \geq 1$.

Proof of Lemma 1. To prove (5), we assume that F is continuous in $|z| \leq 1$, for otherwise replace F by F_r, $0 < r < 1$, where $F_r(z) = F(rz)$.

Suppose that $0 < \alpha \leq \beta < 1$, $0 < h < 1 - \beta$, and $\Omega_{\alpha,h}(\theta)$ is the set of all $z \in \Omega_\alpha(\theta)$ with $|z| > 1 - h$. Let $N_{\alpha,h}(F)(\theta) = \sup |F(z)|$ for $z \in \Omega_{\alpha,h}(\theta)$. Then

$$m(N_\beta(F) > \lambda) \leq c_{\beta,h}\, m(N_{\beta,h}(F) > \lambda) \leq c_{\alpha,\beta,h}\, m(N_{\alpha,h}(F) > \lambda) . \tag{18}$$

For if $|F(z)| \leq \lambda$ for all z satisfying $|z| \leq 1 - h/2$, then $N_{\beta,h}(F) = N_\beta(F)$ on the set $\{N_\beta(F) > \lambda\}$. So, for this case, the left-hand inequality of (18) does hold. If $|F(z)| > \lambda$ for some z satisfying $|z| \leq 1 - h/2$, then, by the maximum principle,

$|F(z)| > \lambda$ for some z satisfying $|z| = 1 - h/2$. Let $E_z = \{\theta: z \in \Omega_{\beta,h}(\theta)\}$. Then $E_z \subset \{N_{\beta,h}(F) > \lambda\}$ and the left-hand inequality of (18) must hold here with $c_{\beta,h} = 2\pi/m(E_z)$, which depends on β and h but not on z. For a proof of the right-hand side of (18), see [3, page 531]. In the present one-dimensional setting, a somewhat more geometrical proof can also be constructed.

So to prove (5), it is enough, in view of (16) and (18), to show that

$$m(N_{\alpha,h}(F) > \lambda) \leq c_{\alpha,h} R_{1:D}^{\{|F(z)| \geq \lambda\}}(0) \tag{19}$$

for α and h small.

Assume that $|F(0)| < \lambda$, the other case being trivial. Let

$$p_t(z) = \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}, \qquad z = re^{i\theta}.$$

Then, by a classical disintegration theorem for the balayage (for example, see [11, Theorem 12.21 and Lemma 7.11]),

$$R_{1:D}^{\{|F(z)| \geq \lambda\}}(0) = \frac{1}{2\pi}\int_0^{2\pi} R_{p_t:D}^{\{|F(z)| \geq \lambda\}}(0)\, dt$$

and (19) follows from the fact that, for all small α and h,

$$R_{p_t:D}^{\{|F(z)| \geq \lambda\}}(0) \geq c_{\alpha,h} \qquad \text{if } N_{\alpha,h}(F)(t) > \lambda. \tag{20}$$

To show (20), suppose without loss of generality that $t = 0$ and $N_{\alpha,h}(F)(0) > \lambda$. Then there is a positive number $s < 1$ such that $B = \{|z - s| \leq \alpha(1 - s)\}$ contains a point z in $\Omega_{\alpha,h}(0)$ satisfying $|F(z)| > \lambda$. Choose α and h small enough so that $s > 1 - 2h$ and $0 \notin B$. Let $A = \{|F(z)| \geq \lambda\}$ and $C = \{z: \bar{z} \in A\}$. Then $v \in \Phi_{p_0:D}^A$ if and only if the map $z \to v(\bar{z})$ belongs to $\Phi_{p_0:D}^C$. Accordingly,

$$R_{p_0:D}^A(0) = R_{p_0:D}^C(0). \tag{21}$$

Let U be the component of $D - (A \cup B \cup C)$ containing the origin. Then

$$1 \leq R_{p_0:D}^{A \cup B \cup C}(0) \leq 2R_{p_0:D}^A(0) + R_{p_0:D}^B(0) \tag{22}$$

where the right-hand side follows from (21) and the subadditivity property of the réduite.

To show the left-hand side, first note that $1 \notin \partial U$. For suppose, on the contrary, that $1 \in \partial U$. Then, by the definition of U and the initial assumption that F is continuous on $|z| \leq 1$, a path in $U \cup \{1\}$ from 0 to 1 exists such that $|F(z)| \leq \lambda$ along this path. Since U is symmetrical with respect to the real axis, the "conjugate" path also has this property. The two paths do not touch B so $B \subset V$ where V is the open set between the two paths. By the definition of B, $|F(z)| > \lambda$ for some $z \in B \subset V$ but $|F(z)| \leq \lambda$ for all $z \in \partial V$. This contradicts the maximum principle; therefore, $1 \notin \partial U$.

Now let $v \in \Phi_{p_o:D}^{A \cup B \cup C}$. Then, for $z_o \in D \cap \partial U \subset A \cup B \cup C$,

$$f(z_o) = \liminf_{\substack{z \to z_0 \\ z \in U}} [v(z) - p_o(z)] \geq v(z_o) - p_o(z_o) \geq 0,$$

and, for $z_o \in \partial D \cap \partial U$, $f(z_o) \geq -p_o(z_o) = 0$. Therefore, $f(z_o) \geq 0$ for all $z_o \in \partial U$ and, by the minimum principle (for example, see [11, Corollary 4.3]), the superharmonic function $v - p_o$ is nonnegative in U. In particular, $1 = p_o(0) \leq v(0)$, which implies the left-hand side of (22).

We now show that, for all small α and h,

$$R_{p_o:D}^{B}(0) \leq 1 - c_{\alpha,h}, \tag{23}$$

which, in view of (22), gives (20) for $t = 0$. Let $v(z) = \gamma \log |z| / \log \alpha$ where $\gamma = 2/(1-\alpha)(1-s)$. Then v is nonnegative and superharmonic in D and satisfies $v(z) \geq \gamma$ in $|z| \leq \alpha$. Let $\varphi(z) = (s - z)/(1 - sz)$. If $z \in B$, then $|\varphi(z)| \leq \alpha$ and $v(\varphi(z)) \geq \gamma \geq p_o(z)$. Therefore, $v(\varphi) \in \Phi_{p_o:D}^{B}$, which implies that

$$R_{p_o:D}^{B}(0) \leq v(\varphi(0)) = v(s) = \frac{\gamma \log \frac{1}{s}}{\log \frac{1}{\alpha}} \leq \frac{2}{s(1-\alpha)\log \frac{1}{\alpha}}.$$

Since $s \geq 1 - 2h$, the last expression converges to 0 as $\alpha, h \to 0$. This implies (23) and completes the proof of (5), the first part of Lemma 1.

To prove the second part, let $\varphi(z) = (a - z)/(1 - \bar{a}z)$. Then, by the univalence of φ and the definition of the réduite,

$$R_{1:D}^{\{|F(\varphi(z))| > \lambda\}} = R_{1:D}^{\{|F(z)| > \lambda\}}(\varphi). \tag{24}$$

By Lemma 4 applied to F, and by (5) and (24),

$$\begin{aligned} R_{1:D}^{\{|F(z)|>\lambda\}}(0) &= R_{1:D}^{\{|F(\varphi(z))|>\lambda\}}(a) \\ &\leq \frac{1}{4\alpha} \cdot \frac{1+|a|}{1-|a|}\, m(N_\alpha(F(\varphi)) > \lambda) \\ &\leq c_\alpha \cdot \frac{1+|a|}{1-|a|}\, R_{1:D}^{\{|F(\varphi(z))|>\lambda\}}(0) \\ &= c_\alpha \cdot \frac{1+|a|}{1-|a|}\, R_{1:D}^{\{|F(z)|>\lambda\}}(a), \end{aligned}$$

which implies (6) with a constant independent of α. Therefore, the proof of Lemma 1, and hence of Theorem 1, is complete.

4. EXTENSIONS AND REMARKS

(a) Theorem 1 remains true if, in (3), F and G are replaced by Re F and Re G or by $(\mathrm{Re}\, F)^+$ and $(\mathrm{Re}\, G)^+$. In fact, let ψ be any nonnegative subharmonic function on $\mathbb{C}$. Then, under the conditions of Theorem 1,

$$m(N_\alpha(\psi(F)) > \lambda) \le c_\alpha \frac{1+|a|}{1-|a|} \frac{1+|b|}{1-|b|} m(N_\alpha(\psi(G)) > \lambda) \tag{25}$$

for all $a, b \in D$ such that $F(a) = G(b)$.

Apart from notational changes, the proof is almost identical to the proof of Theorem 1. For example, the only nontrivial change needed in the proof of Lemma 1 occurs in the argument that $1 \notin \partial U$. If $1 \in \partial U$, then, by the subharmonicity of $\psi(F)$,

$$\limsup_{\substack{z \to z_0 \\ z \in V}} \psi(F(z)) \le \psi(F(z_0)) \le \lambda$$

for all $z_0 \in \partial V - \{1\}$. But $\mu_z^V(\{1\}) \le \mu_z^D(\{1\}) = 0$, $z \in V$, and, since the subharmonic function $\psi(F)$ is bounded on the compact set $\bar{D}$ and hence on V,

$$\psi(F(z)) \le \int_{\partial V} \lambda \, d\mu_z^V = \lambda, \qquad z \in V,$$

which gives a contradiction.

Let M_ψ^Φ denote the class of all functions G analytic in D such that

$$\liminf_{\lambda \to \infty} \Phi(\lambda) m(N_\alpha(\psi(G)) > \lambda) = 0 .$$

Then, by (25), Theorem 2 remains true if M^Φ is replaced by M_ψ^Φ.

So, for example, Corollary 1 can be somewhat strengthened: If $G \in M_{|\mathrm{Re}|}^1$ and $\mathrm{Re}\, G(e^{i\theta}) = 0$ a.e., then G is constant.

To gain some insight into the growth condition here placed on the real part of G, note that, for $\varepsilon > 0$, it cannot be replaced by the slightly weaker condition

$$\liminf_{\lambda \to \infty} \lambda m(N_\alpha(\mathrm{Re}\, G) > \lambda) \le \varepsilon . \tag{26}$$

To see this, consider $G = \delta F$, where $F(z) = (1+z)/(1-z)$, and notice that

$$R_{1:F(D)}^{\{|\mathrm{Re}\, w| > \lambda\}}(x+iy) \le x/\lambda, \qquad x > 0,$$

so, by the ψ-analogues of Lemma 1 and 2, $m(N_\alpha(\mathrm{Re}\, F) > \lambda) \le c_\alpha/\lambda$. Therefore, for small δ, (26) holds and $\mathrm{Re}\, G(e^{i\theta}) = 0$ a.e., but G is not constant.

Corollary 2 remains true if M^p is replaced by $M_{\mathrm{Re}^+}^p$ where Re^+ denotes the mapping $w \to (\mathrm{Re}\, w) \vee 0$. Finally, Corollary 3 remains true if, for example, $M^{\log}$ is replaced by $M^{\log}_{\mathrm{Re}+}$.

(b) If F and G are analytic in D and F is subordinate to G (that is, $F = G(\varphi)$ for some analytic function $\varphi: D \to D$ with $\varphi(0) = 0$), then (3) and (25) hold with $a = b = 0$. Observe that $R_{1:D}^{\{\psi(F)>\lambda\}} \leq R_{1:D}^{\{\psi(G)>\lambda\}}(\varphi)$ and apply the ψ-analogues of (5) and (9).

(c) The constant in (3) satisfies $c_\alpha > 1$ and, in fact, is large if α is small: let $0 < \varepsilon < 1$. By Runge's theorem, there is an entire function G such that $|G(1/2) - 2| < \varepsilon$ but $|G(z)| < \varepsilon$ for all z in D not in the ε-neighborhood of $[1/2, 1]$. Let $F(z) = G(1/2)$, $z \in D$. Then $m(N_\alpha(F) > 1) = 2\pi$ but $m(N_\alpha(G) > 1) \to 0$, implying that $c_\alpha \to \infty$, as $\varepsilon, \alpha \to 0$. Moreover, by the geometry of the problem,

$$\liminf_{\alpha \to 0} \alpha c_\alpha > 0;$$

the other direction, $\limsup_{\alpha \to 0} \alpha c_\alpha < \infty$ follows from Lemmas 1–4, since, in (5), $c_\alpha \leq c_{1/2}$ for $\alpha \leq 1/2$. Note that here $G(e^{i\theta}) \notin F(D) = \{G(1/2)\}$ a.e. because otherwise, by Privalov's Theorem [14, Vol. II, page 203], G would be constant.

It is perhaps a little more surprising that the absolute constant in (6) satisfies $c > 1$: Use Runge's theorem again together with the well known fact [10, page 109] that

$$R_{1:D}^{[-1,0]}(r) = 1 - \frac{4}{\pi} \arctan r^{1/2}, \qquad 0 \leq r < 1.$$

Note, for example, that

$$\frac{1+r}{1-r} R_{1:D}^{[-1,0]}(r) = 2/3 \not\geq 1 \qquad \text{if } r = 1/3.$$

(d) Let f be a nonnegative function in $L^1(0, 2\pi)$, u its Poisson integral, and v any harmonic conjugate of u in D. Then $F = u + iv$ belongs to M^1 but does not belong to H^1 unless a further restriction is placed on f: $f \in L \log L$. To show this, we can assume that $v(0) = 0$. Then

$$\lambda m(N_\alpha(F) > \lambda) \leq c_\alpha ||f||_1, \qquad \lambda > 0. \tag{27}$$

This well known inequality is an immediate consequence of Theorem 1: Compare F with $G(z) = u(0)(1+z)/(1-z)$ and note that $F(0) = G(0) = \int_0^{2\pi} f\, d\theta/2\pi$. (The function $\tilde{f}$ conjugate to f satisfies $|\tilde{f}(\theta)| = |v(e^{i\theta})| \leq N_\alpha(F)(\theta)$ a.e. so (27) implies Kolmogorov's weak-type (1,1) inequality for the conjugate function.) Let $f_n = f \wedge n$ and F_n be the corresponding analytic function with $v_n(0) = 0$. Then

$$\lambda m(N_\alpha(F) > 2\lambda) \leq \lambda m(N_\alpha(F - F_n) > \lambda) + \lambda m(N_\alpha(F_n) > \lambda)$$

and, since $F_n \in H^1$, the last expression approaches zero as $\lambda \to \infty$. Therefore, by (27) applied to $F - F_n$,

$$\limsup_{\lambda\to\infty} \lambda m\,(N_\alpha(F) > 2\lambda) \le c_\alpha \,||f - f_n||_1 \downarrow 0 \qquad \text{as } n \uparrow \infty,$$

which implies that $F \in M^1$. Since f is nonnegative, $\tilde{f} \in L^1$ if and only if $f \in L \log L$ [14, Vol. I, page 254], equivalently, $F \in H^1$ if and only if $f \in L \log L$.

Therefore, for all positive real p, H^p is a proper subclass of M^p: with $0 \le f \in L^1$, $f \notin L \log L$, and F as above, let G be an analytic function satisfying $G^p = F$. Then $G \in M^p$ but $G \notin H^p$.

On the other hand, the Nevanlinna class N is not a subset of $M^{\log}$: consider $\exp\{(1 + z)/(1 - z)\}$. Allen Shields, in response to the present work, has asked about the relationship between $M^{\log}$ and the uniform Nevanlinna class N^+ (denoted by N' in [13]). We can show that N^+ is a proper subclass of $M^{\log}$.

To see that $N^+ \subset M^{\log}$, let $G \in N^+$ and define the nonnegative function f in $L^1(0, 2\pi)$ by $f(\theta) = \log^+ |G(e^{i\theta})|$. Let u be the Poisson integral of f. Then

$$\log^+ |G(z)| \le u(z), \qquad z \in D,$$

so that $m(N_\alpha(\log^+|G|) > \lambda) \le m(N_\alpha(u) > \lambda)$. By the paragraph containing (27), the right-hand side of this inequality is of smaller order than λ^{-1} as $\lambda \to \infty$. This implies that $G \in M^{\log}$.

We now show that $M^{\log} \not\subset N^+$. In fact, no matter how rapidly the function Φ of Section 2 grows, $M^\Phi \not\subset N$, a larger class than N^+. To prove this, we may assume that $\Phi(\lambda)$ is positive for all $\lambda > 0$. In the following, J_n is an arc of ∂D, D_n is the union of all Stolz domains $\Omega_\alpha(\theta)$ such that $e^{i\theta} \notin J_n$, I_n is any subarc of J_n having strictly positive distance from D_n, and G_n is an entire function. We assume that the family $\{J_n\}$ is disjoint. Using Runge's theorem successively for $n = 1, 2, \ldots$, we may choose these to satisfy

$$\begin{aligned}
&|G_n(z)| < 2^{-n}, && z \in D_n, \\
&|G_n(z)| > 1 + \exp|I_n|^{-2}, && z \in I_n, \\
&|J_{n+1}| < |J_n|/2, && \\
&\Phi(\lambda_n)|J_{n+1}| < 2^{-n-1}, &&
\end{aligned}$$

where $|J_n|$ denotes the length of J_n and $\lambda_n - 1$ is the maximum modulus of $G_1 + \ldots + G_n$ in D. Note that $N_\alpha(G_n)(\theta) \le 2^{-n}$ if $e^{i\theta} \notin J_n$. Let

$$G(z) = \sum_{n=1}^{\infty} G_n(z), \qquad z \in D.$$

Then G is analytic in D and, on the set where $e^{i\theta} \notin \bigcup_{k>n} J_k$,

$$N_\alpha(G) \le N_\alpha(G_1 + \dots + G_n) + \sum_{k>n} N_\alpha(G_k) \le \lambda_n .$$

This implies that $N_\alpha(G)$ is finite a.e. so [13, Chapter 14] the nontangential limit $G(e^{i\theta})$ exists a.e. From all of this we see that $G \in M^\Phi$:

$$\Phi(\lambda_n)\, m(N_\alpha(G) > \lambda_n) \le \Phi(\lambda_n) \sum_{k>n} |J_k| \le 2\Phi(\lambda_n)|J_{n+1}| \le 2^{-n};$$

but that $G \notin N$:

$$\int_0^{2\pi} \log^+ |G(e^{i\theta})|\, d\theta \ge |I_n| \log^+ \exp |I_n|^{-2} = |I_n|^{-1} \to \infty \qquad \text{as } n \to \infty.$$

(e) If $a \in D$, then $R_{1:D}^{\{|F(z)|>\lambda\}}(a)$ is the probability that a complex Brownian motion starting at a hits $\{|F(z)| > \lambda\}$ before leaving D. Accordingly, the proofs of (5) and (9) give a nonprobabilistic approach to Theorems 3 and 3′ of [4].

(f) Some of our methods and results carry over to other domains and to higher dimensions. For example, consider a function u defined on the half-space $\mathbb{R}^{n+1}_+$, the set of all $z = (x, y)$ with $x \in \mathbb{R}^n$ and $y > 0$. Here let m denote Lebesgue measure on $\mathbb{R}^n$ and

$$N_\alpha(u)(x) = \sup\{|u(s,y)| : |x - s| < \alpha y\}, \qquad x \in \mathbb{R}^n .$$

THEOREM 3. *If* u *is harmonic in* $\mathbb{R}^{n+1}_+$, *or is nonnegative and subharmonic in* $\mathbb{R}^{n+1}_+$, *with a vanishing nontangential limit at almost every* $x \in \mathbb{R}^n$ *and*

$$\liminf_{\lambda\to\infty} \lambda m(N_\alpha(u) > \lambda) = 0, \tag{28}$$

then u *vanishes everywhere on* $\mathbb{R}^{n+1}_+$.

This is equivalent to the following maximum principle: If u is subharmonic in $\mathbb{R}^{n+1}_+$ with a nonpositive nontangential limit superior at almost every $x \in \mathbb{R}^n$ and the one-sided nontangential maximal function $N_\alpha(u^+)$ is controlled as in (28), then $u(z) \le 0$ for all $z \in \mathbb{R}^{n+1}_+$.

Proof. It is enough to prove the theorem for u nonnegative and subharmonic. Let $E = \{N_\alpha(u) \ge \lambda\}$ and

$$v(z) = \int_E \frac{y}{(|x - s|^2 + y^2)^{(n+1)/2}}\, ds \tag{29}$$

for $z = (x, y) \in \mathbb{R}^{n+1}_+$. Up to a multiplicative constant, v is the Poisson integral of the characteristic function of E; hence v is harmonic and bounded. As we shall show, there is a positive number β such that

$$u(z) \wedge \lambda \le \beta\lambda v(z), \qquad z \in \mathbb{R}^{n+1}_+. \tag{30}$$

From this inequality and Fubini's theorem, we obtain

$$\int_{\mathbb{R}^n} |u(x,y)| \wedge \lambda \, dx \le \beta\lambda \int_{\mathbb{R}^n} v(x,y)\, dx = c_{\alpha,n}\lambda m(N_\alpha(u) \ge \lambda) .$$

So letting $\lambda \uparrow \infty$ and using (28), we see that $\int_{\mathbb{R}^n} |u(x,y)|\, dx = 0$, which implies that u vanishes everywhere on $\mathbb{R}^{n+1}_+$.

To prove (30), we note that if $u(z) \ge \lambda$ and $|x-s| < \alpha y$, then $s \in E$. Therefore, if $u(z) \ge \lambda$ then

$$v(z) \ge \int_{\{|x-s|<\alpha y\}} \frac{y}{(|x-s|^2 + y^2)^{(n+1)/2}}\, ds$$

where the right-hand side depends on α and n but not on z. So, with β the reciprocal of the right-hand side, $\{u \ge \lambda\} \subset \{\beta v \ge 1\}$, which implies part of (30).

We shall complete the proof by showing that (30) holds also on the open set $\{u < \lambda\}$. Although we could do this using purely potential-theoretic methods, we shall instead use Doob's fundamental results connecting submartingales with subharmonic functions [6]. Let $Z = \{Z_t, t \ge 0\}$ be a Brownian motion in $\mathbb{R}^{n+1}$ starting at a point z in $\{u < \lambda\}$. Let σ be the first time Z leaves $\{u < \lambda\}$ and τ the first time Z leaves $\mathbb{R}^{n+1}_+$; necessarily $\sigma \le \tau$. Similarly, let $\sigma_1, \sigma_2, \ldots$ be stopping times such that $0 = \sigma_1 < \ldots < \sigma_j \to \sigma$ everywhere on the underlying probability space as $j \to \infty$. Such a sequence of stopping times exists. Since, with probability one, the sample paths $t \to u(Z_t)$ are continuous [6, pages 104, 120] on the interval $[0, \tau)$,

$$\lim_{j\to\infty} u(Z_{\sigma_j}) = u(Z_\sigma) = \lambda \le \beta\lambda v(Z_\sigma)$$

almost surely on $\{\sigma < \tau\}$. We now show that $\lim_{j\to\infty} u(Z_{\sigma_j}) = 0$ almost surely on $\{\sigma = \tau\}$.

Let $E_h = \{x \in \mathbb{R}^n : |x| < r + \alpha h, N_{\alpha,h}(u)(x) > \varepsilon\}$ and v_h be defined analogously to v in (29) where ε, h, r are positive numbers and

$$N_{\alpha,h}(u)(x) = \sup\{|u(s,y)| : |x-s| < \alpha y, 0 < y < h\} .$$

If $u(x,y) > \varepsilon$, $|x| < r$, $0 < y < h$, then, as above, $\beta v_h(x,y) \ge 1$. Since u vanishes almost everywhere at the boundary of the half-space, $m(E_h) \downarrow 0$ so the integral $v_h(z) \downarrow 0$ as $h \downarrow 0$. From these facts and Doob's weak-type inequality for martingales applied to the nonnegative martingale $\{v_h(Z_{\sigma_j \wedge t}), t \ge 0\}$ starting at $v_h(z)$, we obtain

$$\begin{aligned} P_z(\limsup_{j\to\infty} u(Z_{\sigma_j}) > \varepsilon, |Z_\sigma| < r, \sigma = \tau) &\le \lim_{j\to\infty} P_z(\sup_{0\le t\le\sigma_j} \beta v_h(Z_t) \ge 1) \\ &\le \beta v_h(z) \downarrow 0 \qquad \text{as } h \downarrow 0, \end{aligned}$$

which implies that $\lim_{j\to\infty} u(Z_{\sigma_j}) = 0$ almost surely on $\{\sigma = \tau\}$.

Accordingly, the bounded submartingale $\{(u - \beta\lambda v)(Z_{\sigma_j}), j \geq 1\}$ starting at $(u - \beta\lambda v)(z)$ has, almost surely, a nonpositive limit, say L. Therefore,

$$(u - \beta\lambda v)(z) \leq E_z L \leq 0 ,$$

which completes the proof of (30).

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