LACUNARITY AND LIPSCHITZ PROPERTIES IN TOTALLY DISCONNECTED ABELIAN GROUPS

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In this paper we are concerned with investigating two ways in which Lipschitz properties of a function interact with lacunarity properties of its Fourier series in the setting of a totally disconnected compact abelian group. The first part of this paper deals with the lacunarity condition which forces a local Lipschitz condition to become global. This has been investigated on the circle by Izumi, Izumi and Kahane [2] where it is found that the Hadamard lacunarity condition is precisely the right condition. In our setting a complete characterization is obtained only for a restricted class of totally disconnected groups. However, this class is large enough to contain the Cantor groups. The second part of this paper deals with the lacunarity condition which together with a Lipschitz condition forces absolute convergence of the Fourier series. This has been investigated on the n-dimensional torus by Benke [1] where the lacunarity condition is almost characterized. In the present setting analogous results are obtained.

Let G be a compact totally disconnected abelian group whose dual group Γ is countable. Then there is a sequence of open subgroups $G = G_0 \supset G_1 \supset \dots \supset \{0\}$ which forms a base of neighborhoods at the identity. Let $\{0\} = \Gamma_0 \subset \Gamma_1 \subset \dots \subset \Gamma$ be the sequence of annihilators. Since G is totally disconnected Γ is a torsion group and we may assume without loss of generality that the Γ_j are finite and that Γ_{n+1}/Γ_n is cyclic. The cyclicity condition is only needed in Theorem 4.

Definition. Let m_j be the cardinality of Γ_j. Define ρ on $G \times G$ as follows. If $x = y$ then $\rho(x,y) = 0$. If $x \neq y$ then

$$\rho(x,y) = m_{k(x,y)}^{-1}$$

where $k(x,y)$ satisfies $x - y \in G_{k(x,y)} \setminus G_{k(x,y)+1}$.

Proposition. *ρ is a translation invariant metric on* G.

Proof. If $x - y \in G_j \setminus G_{j+1}$ then $(x+z) - (y+z) \in G_j \setminus G_{j+1}$ and hence ρ is translation invariant.

Next, since $\{G_j\}_{j=0}^{\infty}$ is a neighborhood base $\rho(x,y) \neq 0$ for all $x \neq y$. Furthermore, since $x - y \in G_j \setminus G_{j+1}$ if and only if $y - x \in G_j \setminus G_{j+1}$ it follows that $\rho(x,y) = \rho(y,x)$.

To show the triangle inequality, by translation invariance, it suffices to show $\rho(x,0) \leq \rho(x,z) + \rho(z,0)$. Suppose $x \in G_j \setminus G_{j+1}$ then $\rho(x,0) = m_j^{-1}$. If $z \in G_{j+1}$ then $x - z \in G_j \setminus G_{j+1}$ and hence $\rho(x,z) = m_j^{-1}$ which gives the inequality. If $z \notin G_{j+1}$ then $\rho(z,0) \geq m_j^{-1}$ which also gives the inequality.

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That ρ generates the original topology is immediate from the observation that the ε-balls of ρ are simply the subgroups G_i.

Definition. A complex function f satisfies a Lipschitz condition of order α $(0 < \alpha < \infty)$ at $x \in G$ if there is a constant $M(x, \alpha)$ so that

$$|f(x) - f(y)| \leq M\rho(x, y)^{\alpha}.$$

If $M(x, \alpha)$ is a bounded function of x, then f is said to belong to the class $\mathrm{Lip}_\alpha(G)$.

The definition analogous to this one for functions on the real line becomes trivial for $\alpha > 1$. In this setting however, for any $\alpha > 1$ there exist non-trivial functions which belong to $\mathrm{Lip}_\alpha(G)$. An example of such a function is

$$f(x) = \sum_{n=1}^{\infty} m_{n-1}^{-\alpha} \langle x, \gamma_n \rangle$$

where $\gamma_n \in \Gamma_n \setminus \Gamma_{n-1}$. This may be seen as follows. Let $x \in G_k, y \in G$, then

$$|f(x + y) - f(y)| \leq \sum_{n=1}^{\infty} m_{n-1}^{-\alpha} |\langle x + y, \gamma_n \rangle - \langle y, \gamma_n \rangle|.$$

However, $\langle x + y, \gamma_n \rangle = \langle y, \gamma_n \rangle$ for $\gamma_n \in \Gamma_k$, so that in the above sum, the terms for which $n \leq k$ vanish. Hence

$$|f(x + y) - f(y)| \leq 2 \sum_{n=k+1}^{\infty} m_{n-1}^{-\alpha} = m_k^{-\alpha} \left(2 \sum_{n=k+1}^{\infty} (m_{n-1}/m_k)^{-\alpha} \right)$$

Since $m_{n+1}/m_n \geq 2$ it follows that f belongs to $\mathrm{Lip}_\alpha(G)$.

It should also be noted that since the metric depends on the choice of the neighborhood system $\{G_n\}$ so does the space $\mathrm{Lip}_\alpha(G)$. If $\{G_{n_k}\}$ is a subsequence of $\{G_n\}$, then the associated space $\mathrm{Lip}_\alpha(G)'$ contains $\mathrm{Lip}_\alpha(G)$ and may in fact be strictly larger. For example, let $\{G_n\}_{n=0}^{\infty}$ be such that the sequence $\{m_{n+1}/m_n\}$ is unbounded, and consider $\{G_{2n}\}_{n=0}^{\infty}$. Fix any $\alpha > 0$. Let

$$f(x) = \sum_{k=0}^{\infty} m_{2k-2}^{-\alpha} \langle x, \gamma_k \rangle$$

where $\gamma_k \in \Gamma_{2k}$. For $h \in G_{2n}$ it is shown exactly as in the previous example, that

$$|f(x + h) - f(x)| \leq C\, m_{2n}^{-\alpha}$$

So that $f \in \mathrm{Lip}_\alpha(G)'$. Now for each n, $\{\langle h, \gamma_{n+1} \rangle : h \in G_{2n+1}\}$ is a nontrivial subgroup of the circle. Hence there exists $h_{2n+1} \in G_{2n+1}$ so that

$$|\langle h_{2n+1}, \gamma_{n+1} \rangle - 1| > \sqrt{3}\,.$$

Then

$$|f(h_{2n+1}) - f(0)| = \left| \sum_{k=n+1}^{\infty} m_{2k-2}^{-\alpha} (\langle h_{2n+1}, \gamma_k \rangle - 1) \right|$$
$$> |m_{2n}^{-\alpha} (\langle h_{2n+1}, \gamma_{n+1} \rangle - 1| - \sum_{k=n+2}^{\infty} m_{2k-2}^{-\alpha} 2$$
$$> m_{2n}^{-\alpha} \sqrt{3} - m_{2n+2}^{-\alpha} 4 > m_{2n}^{-\alpha}$$

for all n sufficiently large. This shows that $f \notin \mathrm{Lip}_\alpha(G)$.

The choice of a neighborhood base $\{G_n\}$ which is most natural depends upon the application. For example in the next proposition, the conclusion concerning the behavior of $\hat{f}$ is more detailed when the neighborhood base is finer. In theorem 4a taking $E = \Gamma$, the thinner the neighborhood base is, the stronger the theorem.

Proposition. If f *belongs to* $\mathrm{Lip}_\alpha(G)$ *with Lipschitz constant* M, *then for* $\gamma \in \Gamma_j \setminus \Gamma_{j-1}, |\hat{f}(\gamma)| \leq Mm_{j-1}^{-\alpha}$.

Proof. For $x \in G$,

$$2\hat{f}(\gamma) = \int_G f(y)\langle -y, \gamma\rangle dy - \int_G f(y-x)\langle -y,\gamma\rangle dy$$
$$+ (\langle x, \gamma\rangle + 1) \int_G f(y-x)\langle -y, \gamma\rangle dy.$$

If $\gamma \in \Gamma_j \setminus \Gamma_{j-1}$ then $\{\langle z, \gamma\rangle : z \in G_{j-1}\}$ is a non-trivial subgroup of the circle, so there exists an $x \in G_{j-1}$ such that $|\langle x, \gamma\rangle - (-1)| \leq 1$. Using this x in the above inequality, we have

$$|\hat{f}(\gamma)| \leq \frac{1}{2} \sup_y |f(y) - f(y-x)| + \frac{1}{2} |\hat{f}(\gamma)|$$

and so $|\hat{f}(\gamma)| \leq M\rho(x,0)^\alpha \leq Mm_{j-1}^{-\alpha}$.

Definitions. (1) Let $E \subset \Gamma$ denote $E \cap (\Gamma_j \setminus \Gamma_{j-1})$ by E_j and let

$$q(j,E) = \max\{k : (\gamma + \Gamma_k) \cap E = \{\gamma\} \text{ for all } \gamma \in E_j\}$$

When E_j is non-empty, and let $q(j,E) = j-1$ when E_j is empty. E satisfies the *lacunarity condition* L if $\sup\{m_{j-1} m_{q(j,E)}^{-1} : j = 0,1,\ldots\} < \infty$.

(2) $E \subset \Gamma$ satisfies the *local-global Lipschitz property for* α if for every complex function f on G with supp $\hat{f} \subset E$, f does not satisfy a Lipschitz condition of order α at any point unless f belongs to the class $\mathrm{Lip}_\alpha(G)$.

THEOREM 1. *If* $E \subset \Gamma$ *satisfies the lacunarity condition* L *and if card* E_j *is bounded with respect to* j, *then* E *satisfies the local-global Lipschitz property for all* α.

Proof. Suppose E satisfies the lacunarity condition L, and suppose that f satisfies a Lipschitz condition of order α at x_0 and the spectrum of f is contained in E.

Put $D_k(x) = \sum_{\gamma \in \Gamma_k} \langle x, \gamma \rangle$. Then since Γ_k is a subgroup whose dual is G/G_k, $D_k = (\text{card}\, G/G_k)\delta_k \circ \eta_k$ where η_k is the natural homomorphism from G onto G/G_k and δ_k is the delta function at $0 + G_k$ in G/G_k. Since

$$\text{card}\, G/G_k = \text{card}\, \Gamma_k = m_k,$$

$D_k = m_k 1_{G_k}$ where 1_{G_k} denotes the indicator function of G_k.

Now take $\gamma \in E$, then for some j, $\gamma \in \Gamma_j \setminus \Gamma_{j-1}$. Also putting

$$g(x) = f(x + x_o) - f(x_o),$$

we have

$$|\hat{g}(\gamma)| = |\hat{f}(\gamma)| \qquad \text{for all } \gamma \neq 0,$$

g satisfies a Lipschitz condition of order α at 0 (with the same Lipschitz constant as f), and the spectrum of f and g are essentially the same. Then

$$\hat{g} * \hat{D}_{q(j,E)}(\gamma) = \sum_{\xi \in \Gamma_{q(j,E)} + \gamma} \hat{g}(\xi) = \hat{g}(\gamma),$$

since by the definition of $q(j, E)$ the coset $\Gamma_{q(j,E)} + \gamma$ intersects E only at γ. Hence

$$\hat{g}(\gamma) = \int_G g(x) D_{q(j,E)}(x) \langle -x, \gamma \rangle \, dx = m_{q(j,E)} \int_{G_{q(j,E)}} g(x) \langle -x, \gamma \rangle \, dx.$$

So

$$\begin{aligned} |\hat{g}(\gamma)| &\leq m_{q(j,E)} \sup \{ |g(x)| \rho(x,0)^{-\alpha} : x \in G_{q(j,E)} \} \int_{G_{q(j,E)}} \rho(x,0)^{\alpha} dx \\ &\leq m_{q(j,E)}\, C \sum_{k=q(j,E)}^{\infty} \int_{G_k \setminus G_{k+1}} \rho(x,0)^{\alpha} dx \\ &\leq m_{q(j,E)}\, C \sum_{k=q(j,E)}^{\infty} (\text{measure } G_k)\, m_k^{-\alpha}. \end{aligned}$$

Now measure $G_k = m_k^{-1}$ since G is the disjoint union of m_k cosets of G_k. Hence

$$|\hat{f}(\gamma)| \leq m_{q(j,E)}^{-\alpha}\, C \sum_{k=q(j,E)}^{\infty} (m_{q(j,E)}/m_k)^{1+\alpha}.$$

But $m_k = \ell_1 \dots \ell_k$ where $\ell_i = \text{card}\, \Gamma_i/\Gamma_{i-1}$, so this last sum is

$$1 + \sum_{r=1}^{\infty} (\ell_{q+1} \cdots \ell_{q+r})^{-1-\alpha} \leq (1 - 2^{-1-\alpha})^{-1}.$$

Hence by the lacunarity hypothesis L,

$$|\hat{f}(\gamma)| \le C' m_{j-1}^{-\alpha}. \tag{1}$$

where C' depends only on E, f and α.

We will now show that f belongs to $\mathrm{Lip}_\alpha(G)$. Note first that since $m_j \ge 2^j$, and the cardinality of E_j is bounded in j, f has an absolutely convergent Fourier series. Hence we write

$$\begin{aligned} |f(x+y) - f(y)| &= \left| \sum_{\gamma \in \Gamma} \hat{f}(\gamma)\langle y,\gamma\rangle [\langle x,\gamma\rangle - 1] \right| \\ &\le \sum_{j=1}^{\infty} \sum_{\gamma \in \Gamma_j \setminus \Gamma_{j-1}} |\hat{f}(\gamma)| \, |\langle x,\gamma\rangle - 1| \\ &\le \sum_{j=1}^{\infty} C' m_{j-1}^{-\alpha} \sum_{\gamma \in E_j} |\langle x,\gamma\rangle - 1|. \end{aligned}$$

Now $x \in G_N \setminus G_{N+1}$ for some N, so that $\rho(x,0) = m_N^{-1}$, and $\langle x,\gamma\rangle = 1$ for all $\gamma \in \Gamma_N$. Hence in this last sum the index j needs only to run through $N+1, N+2, \ldots$. This together with the hypothesis that the cardinality of E_j is bounded in j gives that the last sum is less than or equal to

$$C'' \sum_{j=N+1}^{\infty} m_{j-1}^{-\alpha} \le C'' m_N^{-\alpha} \sum_{j=N+1}^{\infty} (m_{j-1}/m_N)^{-\alpha},$$

which is less than or equal to $C''' \rho(x,0)^\alpha$. Hence f belongs to $\mathrm{Lip}_\alpha(G)$.

We have a partial converse to this theorem.

THEOREM 2. If $E \subset \Gamma$ satisfies the local-global Lipschitz property for some $\alpha > 0$, then E satisfies the lacunarity condition L.

Proof. Assume E does not satisfy condition L. Given $\alpha > 0$ we will construct a function with spectrum in E which fails to have the local-global Lipschitz property for α.

Since $\liminf_{j\to\infty} m_{q(j)}/m_{j-1} = 0$ we can select a sequence of indices $\{j_k\}_{k=1}^{\infty}$ so that $m_{q(j_k)}/m_{j_k-1} = o(k^{-2/\alpha})$. From E_{j_k} select γ_k and γ'_k so that

$$\gamma_k - \gamma'_k \in \Gamma_{q(j_k)+1} \setminus \Gamma_{q(j_k)}.$$

Now define

$$f(x) = \sum_{k=1}^{\infty} k^{-2} m_{q(j_k)}^{-\alpha} (\langle x,\gamma_k\rangle - \langle x,\gamma'_k\rangle).$$

Then

$$|f(x) - f(0)| \le \sum_{k=1}^{\infty} k^{-2} m_{q(j_k)}^{-\alpha} \, |\langle x, \gamma_k - \gamma_k' \rangle - 1|. \tag{1}$$

Now suppose $\rho(x,0) = m_{K+1}^{-1}$ then $x \in G_{K+1} 1 \setminus G_{K+2}$ and therefore $\langle x, \gamma_k - \gamma_k' \rangle = 1$ for all k such that $q(j_k) \le K$. Hence the sum in (1) only runs through those k which satisfy $q(j_k) > K$ and therefore is less than or equal to

$$2m_{K+1}^{-\alpha} \sum_{q(j_k)>K} k^{-2} (m_{K+1} / m_{q(j_k)})^{\alpha},$$

which is less than or equal to

$$2m_{K+1}^{-\alpha} \sum_{s=0}^{\infty} 2^{-\alpha s} = C\rho(x,0)^{\alpha}.$$

So f satisfies a Lipschitz condition of order α at 0. Since $k^{2/\alpha} m_{q(j_k)} / m_{j_k - 1} \to 0$, $\hat{f}(\gamma_k) \ne O(m_{j_k-1}^{-\alpha})$ and therefore by the proposition f is not in $\mathrm{Lip}_\alpha(G)$.

In certain groups we do get a complete characterization.

COROLLARY. *Suppose* m_j / m_{j-1} *is bounded in* j. *Then* E *satisfies the local-global Lipschitz property for all* $\alpha > 0$ *if and only if* E *satisfies the lacunarity condition* L. *Moreover* E *satisfies the local-global Lipschitz property for all* $\alpha > 0$ *if and only if it satisfies the property for some* $\alpha > 0$.

Proof. By the definition of q(j), for each pair of distinct $\gamma, \gamma' \in E_j$, γ and γ' belong to distinct cosets of $\Gamma_{q(j)}$ in Γ_j. Hence

$$\mathrm{card}\, E_j \le m_j / m_{q(j)} = (m_{j-1} / m_{q(j)})(m_j / m_{j-1}).$$

So in the presence of the condition that m_j / m_{j-1} is bounded, the condition that card E_j is bounded follows from L.

The remainder of this paper concerns the relationship between Lipschitz conditions, lacunarity and absolute convergence. The next result will be needed later but is interesting in its own right. It is an analogue of a theorem of Kahane [3, p. 66].

THEOREM 3. *Let* m_{n+1} / m_n *be bounded and take* $\beta > 0$. *Consider the random series*

$$f(x) = \sum_{\gamma \in \Gamma} \pm\, q(\gamma) \langle x, \gamma \rangle.$$

If there exists a constant $C(\beta, f)$ *so that* $\|1_{\Gamma_n \setminus \Gamma_{n-1}} q\|_2 \le C m_n^{-\beta}$ *then there exists a constant* $K(\beta, f)$ *so that the random series almost surely represents a function whose modulus of continuity* ω_f *satisfies*

$$\omega_f(h) \le K (\log \rho(h,0)^{-1})^{1/2} \rho(h,0)^{\beta}.$$

In particular, f *satisfies a Lipschitz condition of order* $\beta - \varepsilon$ *for all* $0 < \varepsilon < \beta$.

Proof. If $\gamma \in \Gamma_n$, then $\langle x, \gamma \rangle = 1$ for all $x \in G_n$ which has measure m_n^{-1}. So by [3, p. 55] we have that for the random polynomial $g(x) = \sum_{\gamma \in \Gamma_n}' \pm q(\gamma)\langle x, \gamma \rangle$,

$$P(\|g\|_\infty \geq (3 \log(2m_n K))^{1/2} \|q\|_2) \leq 2K^{-1} \qquad \text{for any } K > 2. \tag{1}$$

Consider $|f(x+h) - f(x)| = \left| \left(\sum_{\Gamma_n} + \sum_{\Gamma \setminus \Gamma_n} \right) \pm q(\gamma)(\langle h, \gamma \rangle - 1)\langle x, \gamma \rangle \right|$. If $h \in G_n$, then the first sum on the right vanishes. Therefore given $h \in G_k$ we have $\rho(h, 0) \leq m_k^{-1}$, and then $\omega_f(h) \leq 2\|f_k\|_\infty$ where $\hat{f}_k = 1_{\Gamma \setminus \Gamma_k} \hat{f}$. Let $n_j = j + k$ for $j = 0, 1, \ldots$ and put $\hat{g}_j = 1_{\Gamma_{n(j)} \setminus \Gamma_{n(j-1)}} \hat{f}$. Noting that $f_k = \sum_{j=1}^{\infty} g_j$ we want to estimate $\|g_j\|_\infty$. Letting $K = 2^{n_j}$ and applying (1) to g_j we have

$$P(\|g_j\|_\infty \geq 3 (\log m_{n_j} + (k + j + 1) \log 2)^{1/2} \|g_j\|_2) \leq 2^{-k-j+1}.$$

Now by hypothesis $\|g_j\|_2 \leq C m_{n_j}^{-\beta}$. So except on a set E_k of probability not greater than 2^{-k+1} we have $\|f_k\|_\infty \leq 3 \sum_{j=1}^{\infty} (\log m_{n_j} + (k + j + 1) \log 2)^{1/2} C m_{n_j}^{-\beta}$. This, together with $m_{n_j} \leq A^{n_j}$ gives

$$\begin{aligned} \|f_k\|_\infty &\leq C_1 m_k^{-\beta} \sum_{j=1}^{\infty} ((k + j) \log A + (k + j + 1) \log 2)^{1/2} (m_{n_j} / m_k)^{-\beta} \\ &\leq C_2 m_k^{-\beta} k^{1/2} \sum_{j=1}^{\infty} j^{1/2} (2^{-\beta})^j \leq C_3 \rho(k, 0)^\beta k^{1/2}. \end{aligned}$$

Now $2^k \leq m_k$ so $k \leq \log m_k / \log 2$ and

$$\omega_f(h) \leq C_4 (\log \rho(h, 0)^{-1})^{1/2} \rho(h, 0)^\beta \tag{2}$$

except possibly on E_k. Since $\sum P(E_k) < \infty$, by the Borel-Cantelli lemma (2) holds for all but a finite number of k's except possibly on a set of probability 0. Hence by choosing some appropriate new constant, (2) holds for all k almost surely.

Definition. Let $E \subset \Gamma$ be called a Lip α set for $\alpha > 0$ if any $f \in \mathrm{Lip}_\alpha(G)$ whose spectrum is contained in E satisfies $\sum |\hat{f}(\gamma)| < \infty$.

THEOREM 4. *Let* m_{n+1}/m_n *be bounded.*

(a) *If for some* $\alpha > 0$, $\sum_n (\mathrm{card}\, E_n)^{1/2} m_n^{-\alpha} < \infty$, *then* E *is a* Lip α *set.*

(b) *If* E *is a* Lip α *set for some* $\alpha > 0$, *then*

$$\sum_n (\text{card } E_n)^{1/2} m_n^{-\alpha-\varepsilon} < \infty \qquad \textit{for all } \varepsilon > 0.$$

Proof. (a) Suppose for some n we have $\gamma \in \Gamma_n \setminus \Gamma_{n-1}$ and $h \in G_{n-1} \setminus G_n$ then the value $\langle -h, \gamma \rangle$ depends only on the cosets of h and γ relative to G_n and Γ_{n-1} respectively. In fact, Γ_n / Γ_{n-1} is the dual group of G_{n-1}/G_n under this action. Since both of these groups are isomorphic to $\mathbb{Z}_{\ell_n}$, there exists a coset $h + G_n \neq G_n$ so that

$$\langle -h + G_n, \gamma + \Gamma_{n-1} \rangle \subset \{e^{2\Pi ik/\ell_n} : k = 1, \ldots, \ell_n - 1\rangle$$

for all cosets $\gamma + \Gamma_{n-1} \neq \Gamma_{n-1}$. Therefore

$$|1 - \langle -h, \gamma \rangle| > 2 \sin \pi \ell_n^{-1} > \delta \qquad \text{for some } \delta > 0 \text{ independent of n.}$$

Now $\hat{f}_h(\gamma) - \hat{f}(\gamma) = (\langle -h, \gamma \rangle - 1)\hat{f}(\gamma)$, and therefore $\hat{f}(\gamma) \leq \delta^{-1} |\hat{f}_h(\gamma) - \hat{f}(\gamma)|$. Using this bound and the hypothesis that f satisfies a Lipschitz condition of order α we have

$$\sum_{\gamma \in \Gamma_n \setminus \Gamma_{n-1}} |\hat{f}(\gamma)|^2 \leq \delta^{-1} \|f_h - f\|_2^2 \leq C^2 m_{n-1}^{-2\alpha}.$$

Therefore

$$\sum_\gamma |\hat{f}(\gamma)| = \sum_n \sum_{E_n} |\hat{f}(\gamma)| \leq \sum_n (\text{card } E_n)^{1/2} \Bigg(\sum_{\Gamma_n \setminus \Gamma_{n-1}} |\hat{f}(\gamma)|^2 \Bigg)^{1/2}$$

$$\leq \sum_n (\text{card } E_n)^{1/2} C m_{n-1}^{\alpha} < \infty$$

by hypothesis.

(b) Let $b_n = (\text{card } E_n)^{-1/2} m_n^{-\alpha-\varepsilon}$ and define $q(\gamma) = b_n$ if $\gamma \in E_n$ and $q(\gamma) = 0$ if $\gamma \notin E$. By Theorem 3 there is a sequence of $\pm$ signs so that f satisfies a Lipschitz condition of order α. Hence, since E is a Lip α set,

$$\sum |\hat{f}(\gamma)| = \sum_n (\text{card } E_n)^{1/2} m_n^{-\alpha-\varepsilon} < \infty.$$

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