SEMIGROUPS OF ANALYTIC FUNCTIONS AND COMPOSITION OPERATORS

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Let U be an open set in the complex plane $\mathbb{C}$. A one-parameter semigroup $\{\phi_t\}$ of holomorphic mappings of U into itself is a homomorphism $t \mapsto \phi_t$ of the additive semigroup of nonnegative real numbers $\mathbb{R}^+$ into the semigroup (under composition) of all analytic mappings of U into U such that ϕ_0 is the identity map of U and $\phi_t(z)$ is continuous in (t, z) on $\mathbb{R}^+ \times U$. We also write $\phi(t, z)$ for $\phi_t(z)$, and denote $\frac{\partial\phi(t, z)}{\partial t}$ by $\phi_1(t, z)$. In this paper we study the collection $\mathscr{S}(U)$ of all such one-parameter semigroups on U for U the right half-plane or the open unit disc Δ, and then apply the results to a treatment of strongly continuous one-parameter semigroups of composition operators on $H^p(\Delta)$, $1 \le p < \infty$.

In Section 1, we show for an arbitrary open set U that if $\{\phi_t\} \in \mathscr{S}(U)$, then there is a unique analytic function G on U (called the infinitesimal generator of $\{\phi_t\}$) such that $\phi_1(t, z) = G(\phi(t, z))$ on $\mathbb{R}^+ \times U$. In Section 2 we characterize and concretely describe the class of all infinitesimal generators for the case where U is the right half-plane. This involves proving the existence of a global solution to the initial value problem $\phi_1(t, z) = G(\phi(t, z))$, $\phi(0, z) = z$ for appropriate analytic functions G (see Theorems (2.6) and (2.13) below). In Section 3, after rephrasing these results so as to characterize the generators for the case where U is Δ, we study the strongly continuous one-parameter semigroups of composition operators on $H^p(\Delta)$, $1 \le p < \infty$, and characterize their infinitesimal generators in Theorem (3.7). For $1 \le p < \infty$, every $\{\phi_t\} \in \mathscr{S}(\Delta)$ gives rise to a strongly continuous semigroup $\{T_t\}$ of composition operators on $H^p(\Delta)$. The point spectrum of the infinitesimal generator of $\{T_t\}$, in certain cases, is taken up in Section 4, where an interplay with logarithmic potentials develops.

Throughout what follows, we denote composition of mappings by $\circ$ and differentiation with respect to z by $'$.

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1. THE INFINITESIMAL GENERATOR OF A SEMI-GROUP OF HOLOMORPHIC MAPPINGS

(1.1) THEOREM. *Let* U *be an open set in* $\mathbb{C}$, *and let* $\{\phi_t\}$, $t \in \mathbb{R}^+$, *be a one-parameter semigroup of holomorphic mappings of* U *into* U. *Then there is*

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a holomorphic mapping G: U → $\mathbb{C}$ *such that*

$$\frac{\partial \phi(t, z)}{\partial t} = G(\phi(t, z)) \qquad \text{for } t \in \mathbb{R}^+, z \in U. \tag{1.2}$$

Proof. Let K be a compact convex subset of U. For a suitable $\alpha \in (0, 1)$, the compact set $\bigcup \{\phi_t(K): 0 \le t \le \alpha\}$ has its convex hull (which is compact) contained in U. Therefore, there is an $\eta \in (0, \alpha]$ with the property

$$|\phi_{2t}(z) - 2\phi_t(z) + z| \le (1/10)|\phi_t(z) - z|, \qquad 0 \le t \le \eta, z \in K. \tag{1.3}$$

Indeed, the minorant in (1.3) is the absolute value of the integral of $\frac{d}{d\zeta}[\phi_t(\zeta) - \zeta]$ along the line segment from z to $\phi_t(z)$, and (by virtue of Cauchy's integral formula) this integrand has modulus less than 1/10 for sufficiently small t. Using (1.3), we find that $|\phi_t(z) - z| \le (10/19)|\phi_{2t}(z) - z|$ for $z \in K$, $0 \le t \le \eta$. For convenience, replace 10/19 by $2^{-2/3}$ in this last inequality. A straightforward argument then shows that there is a constant M (depending on K) such that

$$|\phi_t(z) - z| \le Mt^{2/3} \qquad \text{for } z \in K, 0 \le t \le 1.$$

Cover the convex hull $\tilde{K}$ of $\bigcup \{\phi_t(K): 0 \le t \le \alpha\}$ by a finite collection of closed discs, each of which is contained in a closed disc contained in U, having the same center and a strictly larger radius. Apply the last inequality to each of the larger discs (with a suitable constant M in each case), and use Cauchy's integral formula for $\frac{d}{d\zeta}[\phi_t(\zeta) - \zeta]$ to get a constant $\tilde{M}$ such that for $0 \le t \le 1$, the modulus of this derivative does not exceed $\tilde{M}t^{2/3}$ on $\tilde{K}$. By an argument like that employed in establishing (1.3), we see that for $0 \le t \le \alpha$, and $z \in K$,

$$|\phi_{2t}(z) - 2\phi_t(z) + z| \le \tilde{M}t^{2/3}|\phi_t(z) - z| \le \tilde{M}Mt^{4/3}.$$

Thus

$$|[\phi_{2t}(z) - z](2t)^{-1} - [\phi_t(z) - z]t^{-1}| \le \frac{\tilde{M}M}{2} t^{1/3} \qquad \text{for } z \in K, 0 < t \le \alpha.$$

From this we find that $\lim_n 2^n(\phi(2^{-n}, z) - z) = G(z)$ exists uniformly on compact subsets of U. In particular, G is analytic on U. For $z_0 \in U$ and $t > 0$,

$$\{\phi_s(z_0): 0 \le s \le t\}$$

is a compact subset of U. Thus $2^n[\phi(s + 2^{-n}, z_0) - \phi(s, z_0)]$ tends uniformly to

$G(\phi(s, z_0))$ for $s \in [0, t]$. By calculus we see that

$$\phi_t(z) = z + \int_0^t G(\phi_s(z))ds \qquad \text{for } z \in U, t \in \mathbb{R}^+.$$

Definition. For U and $\{\phi_t\}$ as in Theorem (1.1), the function $G: U \to \mathbb{C}$ satisfying (1.2) is uniquely determined as $\phi_1(0, \cdot)$ and is called the *infinitesimal generator* of $\{\phi_t\}$.

2. INFINITESIMAL GENERATORS IN THE CASE OF THE RIGHT HALF-PLANE

In this section we describe the infinitesimal generators of semigroups of holomorphic mappings of the right half-plane into itself. Let $\mathscr{H} = \{z \in \mathbb{C}: \operatorname{Re} z > 0\}$. We make some preliminary observations. If $G: \mathscr{H} \to \mathbb{C}$ is analytic and if the initial value problem:

$$\frac{\partial\phi(t, z)}{\partial t} = G(\phi(t, z)), \qquad \phi(0, z) = z \tag{2.1}$$

has a solution on $\mathbb{R}^+ \times \mathscr{H}$, then it follows by elementary use of the analyticity of G and the convexity of $\mathscr{H}$, that for every $z_0 \in \mathscr{H}$ and every $\tau > 0$, the initial value problem $\frac{dw}{dt} = G(w)$, $w(0) = z_0$ has a unique solution on the interval $0 \leq t \leq \tau$. In particular, (2.1) has a unique solution ϕ on $\mathbb{R}^+ \times \mathscr{H}$, and (since the initial value problem is autonomous) $\phi_{t+s} = \phi_t \circ \phi_s$ on $\mathscr{H}$ for $t, s \in \mathbb{R}^+$. It is elementary that for $z_0 \in \mathscr{H}$ and $s > 0$, the problem $\frac{dw}{dt} = G(w)$, $w(s) = z_0$ has at most one solution on the interval $0 \leq t \leq s$. It follows that for each $t \in \mathbb{R}^+$, ϕ_t is one-to-one. Standard techniques (such as the method of proof of [2, Theorem 4.1] with suitable modifications) show that $\phi(\cdot,\cdot)$ is continuous on $\mathbb{R}^+ \times \mathscr{H}$. The method of proof of [7, Theorem 9, pp. 13, 14], with obvious modifications, now shows that for $t \geq 0$, $\phi_t(\cdot)$ is analytic on $\mathscr{H}$, and its derivative with respect to the complex variable is given by: $\phi_t'(z) = \exp\left[\int_0^t G'(\phi(u, z))\, du\right]$ for $z \in \mathscr{H}$. In order to set the stage, we summarize the foregoing remarks in the following proposition.

(2.2) PROPOSITION. *Let $\mathscr{G}(\mathscr{H})$ denote the set of all infinitesimal generators of one-parameter semigroups of holomorphic mappings of $\mathscr{H}$ into itself. Let $\mathscr{S}(\mathscr{H})$ denote the set of all such one-parameter semigroups. Then $\mathscr{G}(\mathscr{H})$ consists of all analytic functions* G *on $\mathscr{H}$ such that the initial value problem* (2.1) *has a global solution ϕ on $\mathbb{R}^+ \times \mathscr{H}$. The correspondence which assigns to each member of $\mathscr{S}(\mathscr{H})$ its infinitesimal generator is one-to-one, and, for each $G \in \mathscr{G}(\mathscr{H})$, the corresponding semigroup is the unique solution on $\mathbb{R}^+ \times \mathscr{H}$ of the initial-value problem* (2.1). *If $\{\phi_t\} \in \mathscr{S}(\mathscr{H})$, then ϕ_t is univalent for all* $t \in \mathbb{R}^+$.

In order to achieve the goal of this section, we shall need the following result [6, pp. 115–116].

(2.3) DENJOY-WOLFF THEOREM. *Let f be a holomorphic map of the open unit disc $\triangle$ into itself which has no fixed point. Then there exists a unique unimodular α such that*

$$\mathrm{Re}\,[(\alpha + f(z))(\alpha - f(z))^{-1}] \geq \mathrm{Re}\,[(\alpha + z)(\alpha - z)^{-1}] \qquad \text{for } z \in \triangle\,.$$

Moreover, $\lim_n f^{[n]}(z) = \alpha$ *for all* $z \in \triangle$, *where* $f^{[n]}$ *denotes the n-fold iterate of* f.

Notation. Let $\mathscr{G}_1(\mathscr{H})$ be the class of all analytic functions on $\mathscr{H}$, not identically zero, which map $\mathscr{H}$ into its closure in $\mathbb{C}$. Let $\mathscr{G}_2(\mathscr{H})$ be the class of all analytic functions G on $\mathscr{H}$ of the form

$$G(z) = -F(z)(z - ib)^2 \qquad \text{for } z \in \mathscr{H}, \tag{2.4}$$

where $F \in \mathscr{G}_1(\mathscr{H})$, and b is a real constant. Let $\mathscr{G}_3(\mathscr{H})$ be the class of all analytic functions G on $\mathscr{H}$ of the form

$$G(z) = F(z)(\bar{\zeta} + z)(\zeta - z) \qquad \text{for } z \in \mathscr{H}, \tag{2.5}$$

where $F \in \mathscr{G}_1(\mathscr{H})$, and ζ is a constant belonging to $\mathscr{H}$.

We now take up the concrete characterization of the nontrivial generators.

(2.6) THEOREM. $\mathscr{G}(\mathscr{H}) \setminus \{0\}$ *is the disjoint union of* $\mathscr{G}_1(\mathscr{H})$, $\mathscr{G}_2(\mathscr{H})$, *and* $\mathscr{G}_3(\mathscr{H})$. *If* G *belongs to* $\mathscr{G}_1(\mathscr{H})$ *(resp.,* G *is of the form* (2.4)), *and if* $\{\phi_t\}$ *is the semi-group corresponding to* G, *then for each* $z \in \mathscr{H}$, $\phi_t(z) \to \infty$ *(resp.,* $\phi_t(z) \to ib$) *as* $t \to +\infty$. *If* G *is of the form* (2.5) *with corresponding semigroup* $\{\phi_t\}$, *then:* (i) $\phi_t(z) \to \zeta$ *as* $t \to +\infty$ *for each* $z \in \mathscr{H}$ *if and only if* $F(\mathscr{H}) \subseteq \mathscr{H}$, *and* (ii) $\phi_t(\zeta) = \zeta$ *for* $t \in \mathbb{R}^+$.

Proof. Suppose G has the form (2.5). It is easy to see that in order for (2.1) to have a solution on $\mathbb{R}^+ \times \mathscr{H}$, it suffices to show that there is a solution on $\mathbb{R}^+ \times \triangle$ of the initial-value problem,

$$\frac{\partial\psi(t, z)}{\partial t} = -\psi(t, z)A(\psi(t, z)), \qquad \psi(0, z) = z\,, \tag{2.7}$$

where $A(z) = 2\,[\mathrm{Re}\,\zeta]\,F((\bar{\zeta}z + \zeta)(1 - z)^{-1})$ for $z \in \triangle$. For any $z \in \triangle$, and any $\tau > 0$, the initial value problem $\frac{dw}{dt} = -wA(w)$, $w(0) = z$ has at most one solution on $[0, \tau)$, and the modulus of such a solution must be a decreasing function of t on $[0, \tau)$. Standard arguments with the method of successive approximations now show that (2.7) has a solution on $\mathbb{R}^+ \times \triangle$, and hence $G \in \mathscr{G}(\mathscr{H})$. Moreover, the unique solution ϕ of (2.1) is related to the unique solution ψ of (2.7) by the equation $\phi_t = L^{-1} \circ \psi_t \circ L$ for $t \in \mathbb{R}^+$, where $L(z) = (z - \zeta)(z + \bar{\zeta})^{-1}$ for $z \in \mathscr{H}$. It is obvious from (2.7) that $\psi_t(0) = 0$ for $t \in \mathbb{R}^+$. If Re F > 0 on $\mathscr{H}$, then Re A > 0 on $\triangle$, and it is an easy consequence that $\psi_t(z) \to 0$ as $t \to +\infty$ for all $z \in \triangle$. Otherwise, there is a nonzero real number α such that $A \equiv i\alpha$. But then (2.7) has the explicit solution $\psi(t, z) = e^{-i\alpha t}z$, $t \in \mathbb{R}^+$, $z \in \Delta$. The corresponding statements for $\{\phi_t\}$ establish the last sentence in the statement of (2.6).

Next, suppose G has the form (2.4). Let $\{\zeta_n\}_{n=1}^{\infty}$ be a sequence in $\mathscr{H}$ with limit ib. Put $G_n(z) = F(z)(\bar{\zeta}_n + z)(\zeta_n - z)$ for $z \in \mathscr{H}$, $n = 1, 2, \ldots$. Let $\phi^{(n)}(\cdot,\cdot)$ be the semigroup corresponding to G_n. There is a constant $M > 0$ such that $|G_n(z)| \le M$ for $|z - 1| \le 1/2$, $n = 1, 2, \ldots$. From this we see that

$$(2.8) \qquad |\phi^{(n)}(t, 1) - 1| \le 1/2 \qquad \text{for } t \in [0, (2M)^{-1}], n \ge 1 .$$

The analytic functions $\phi^{(n)}(t,\cdot)$ for $0 \le t \le (2M)^{-1}$, $n \ge 1$, leave $\mathscr{H}$ invariant, and so constitute a normal family. In view of (2.8) and Hurwitz's theorem, we conclude that every limit point of this family in the usual topology leaves $\mathscr{H}$ invariant. Straightforward reasoning shows that for each compact subset K of $\mathscr{H}$, $\{\phi^{(n)}(\cdot, \cdot)\}_{n=1}^{\infty}$ is equicontinuous on $[0, (2M)^{-1}] \times K$. By taking a subsequence $\{\phi^{(n_k)}(\cdot, \cdot)\}$ uniformly convergent on this compact product set, we see that for each $z \in K$ the initial value problem $\dfrac{dw}{dt} = G(w)$, $w(0) = z$ has a solution for $0 \le t \le (2M)^{-1}$. Thus (2.1) has a solution $f(\cdot, \cdot)$ on $[0, (2M)^{-1}] \times \mathscr{H}$. Extend f to a function $g(\cdot, \cdot)$ on $[0, M^{-1}] \times \mathscr{H}$ by setting g(t, z) equal to

$$f(t - (2M)^{-1}, f((2M)^{-1}, z)) \qquad \text{for } (2M)^{-1} \le t \le M^{-1}, z \in \mathscr{H}.$$

Clearly g is a solution of (2.1) on $[0, M^{-1}] \times \mathscr{H}$. Since this extension process can be repeated indefinitely, we see that (2.1) has a solution ϕ on $\mathbb{R}^+ \times \mathscr{H}$. We show next that $\phi_t(z) \to ib$ as $t \to +\infty$ for every $z \in \mathscr{H}$. Applying (2.1) to the function G in the present case, we see that for fixed $z_0 \in \mathscr{H}$ $\operatorname{Re}[(\phi_t(z_0) - ib)^{-1}]$ has a nonnegative derivative with respect to t, and so we have:

$$(2.9) \qquad \operatorname{Re}[(\phi_t(z_0) - ib)^{-1}] \ge \operatorname{Re}[(z_0 - ib)^{-1}] > 0 \qquad \text{for } t \in \mathbb{R}^+.$$

Moreover, since $|F| > 0$ on $\mathscr{H}$, $(1/G)$ has a primitive $\tilde{G}$ on $\mathscr{H}$. From (2.1) we find that

$$(2.10) \qquad \tilde{G}(\phi_t(z_0)) = t + \tilde{G}(z_0) \qquad \text{for } t \in \mathbb{R}^+ .$$

From (2.9) we have $\operatorname{Re}\phi_t(z_0) \ge |\phi_t(z_0) - ib|^2 \operatorname{Re}[(z_0 - ib)^{-1}]$, $t \in \mathbb{R}^+$. Thus it suffices to show that $\operatorname{Re}\phi_t(z_0) \to 0$. Also, (2.9) shows that $\phi_t(z_0)$ is a bounded function of t on $\mathbb{R}^+$. If $\operatorname{Re}\phi_t(z_0)$ does not tend to zero as $t \to +\infty$, then there are an $\varepsilon > 0$ and a strictly increasing sequence $\{t_n\} \subseteq \mathbb{R}^+$, with $t_n \to +\infty$ such that

$$\operatorname{Re}\phi_{t_n}(z_0) \ge \varepsilon \qquad \text{for all n.}$$

Thus $\{\phi_{t_n}(z_0)\}_{n=1}^{\infty}$ is contained in a compact subset K of $\mathscr{H}$. Since $\tilde{G}$ is bounded on K, (2.10) gives a contradiction.

Suppose next that $G \in \mathscr{G}_1(\mathscr{H})$. Then the function $-G(z^{-1})z^2$ belongs to $\mathscr{G}_2(\mathscr{H})$ (with $b = 0$). By what has already been shown, there is a function $\psi(\cdot, \cdot)$ on $\mathbb{R}^+ \times \mathscr{H}$

such that for each $z \in \mathscr{H}$, $\psi(0, z) = z$, $\psi(t, z) \to 0$ as $t \to +\infty$ and

$$\frac{d\psi(t, z)}{dt} = -G(\{\psi(t, z)\}^{-1})[\psi(t, z)]^2 \qquad \text{for } t \in \mathbb{R}^+.$$

It follows that the function $\phi(\cdot, \cdot)$ given by $\phi(t, z) = [\psi(t, z^{-1})]^{-1}$ is a solution to (2.1) on $\mathbb{R}^+ \times \mathscr{H}$ for the present particular G. Clearly, for each $z \in \mathscr{H}$, $\phi(t, z) \to \infty$ as $t \to +\infty$.

It is elementary that $\mathscr{G}_3(\mathscr{H})$ is disjoint from each of $\mathscr{G}_1(\mathscr{H})$ and $\mathscr{G}_2(\mathscr{H})$. The disjointness of $\mathscr{G}_1(\mathscr{H})$ and $\mathscr{G}_2(\mathscr{H})$ follows from the foregoing facts about limiting behavior, as $t \to +\infty$, of semi-groups.

To complete the proof of the theorem, suppose $G \in \mathscr{G}(\mathscr{H}) \setminus \{0\}$, with corresponding semigroup $\{\phi_t\}$, and let t_0 be a fixed positive real number such that ϕ_{t_0} is not the identity map of $\mathscr{H}$. Let S be the linear fractional transformation given by $S(z) = (z - 1)(z + 1)^{-1}$, and let $\{\psi_t\}$ be the semigroup of analytic mappings of Δ into Δ specified by $\psi_t = S \circ \phi_t \circ S^{-1}$ for $t \in \mathbb{R}^+$. Either ψ_{t_0} has a (necessarily unique) fixed point $z_0 \in \Delta$ or the Denjoy-Wolff theorem, (2.3), applies to ψ_{t_0}. In the former case, we have for $t \in \mathbb{R}^+$, $\psi_{t_0}(\psi_t(z_0)) = \psi_t(\psi_{t_0}(z_0)) = \psi_t(z_0)$. Hence z_0 is a common fixed point of the semigroup $\{\psi_t\}$. Thus $S^{-1}z_0$ (which we denote by ζ) is a common fixed point of $\{\phi_t\}$. In particular, $G(\zeta)$ must vanish. Let L be the linear fractional transformation given by $L(z) = (z - \zeta)(z + \bar{\zeta})^{-1}$, and let $\{\tilde{\psi}_t\}$ be the semigroup on Δ defined by $\tilde{\psi}_t = L \circ \phi_t \circ L^{-1}$. Clearly, $\tilde{\psi}_t(0) = 0$ for $t \in \mathbb{R}^+$. By Schwarz's lemma, $|\tilde{\psi}_t(z)| \le |z|$ for $t \in \mathbb{R}^+$, $z \in \Delta$. It follows that for each $z \in \Delta$, $|\tilde{\psi}_t(z)|$ is a decreasing function of t, and hence for $z \in \mathscr{H}$,

$$|\{\phi_t(z) - \zeta\}\{\phi_t(z) + \bar{\zeta}\}^{-1}|^2$$

is a decreasing function of t. Since the derivative at $t = 0$ of this last expression does not exceed zero, simple calculations show that

$$0 \ge \operatorname{Re}[G(z)(\overline{z - \zeta})(z + \bar{\zeta})^{-1}] \qquad \text{for } z \in \mathscr{H}. \tag{2.11}$$

Since $G(\zeta) = 0$, there is an analytic function F on $\mathscr{H}$ such that

$$G(z) = F(z)(\bar{\zeta} + z)(\zeta - z) \qquad \text{for } z \in \mathscr{H}.$$

Substitution of this expression for G in (2.11) easily gives the conclusion that $F \in \mathscr{G}_1(\mathscr{H})$.

There remains the case in which ψ_{t_0} does not have a fixed point. Let α correspond to ψ_{t_0} as in Theorem (2.3). It follows from Cauchy's integral formula and the Lebesgue bounded convergence theorem that $\psi_{nt_0} \to \alpha$ uniformly on compact subsets of Δ. For each $z \in \Delta$, $\{\psi_t(z)\colon 0 \le t \le t_0\}$ is a compact subset of Δ, and so we find easily that $\psi_t(z) \to \alpha$ as $t \to +\infty$ for $z \in \Delta$. In particular, for each $t > 0$, ψ_t does not have a fixed point in Δ, and Theorem (2.3) allows us to deduce that

$$\operatorname{Re}[(\alpha + \psi_t(z))(\alpha - \psi_t(z))^{-1}] \ge \operatorname{Re}[(\alpha + z)(\alpha - z)^{-1}] \qquad \text{for } t > 0, z \in \Delta.$$

It follows that for each $z \in \Delta$,

$$\text{(2.12)} \qquad \operatorname{Re}[(\alpha + \psi_t(z))(\alpha - \psi_t(z))^{-1}] \text{ is an increasing function of } t.$$

At this juncture there are two possibilities to consider: $\alpha \neq 1$ or $\alpha = 1$. In the former event, there is a real number b such that $\alpha = S(ib)$, and by obvious calculations, (2.12) can be rephrased as follows: for each $z \in \mathscr{H}$,

$$\operatorname{Re}[(ib\phi_t(z) - 1)(ib - \phi_t(z))^{-1}]$$

is an increasing function of t. If we take the derivative with respect to t of this last expression and set $t = 0$, we find that for $z \in \mathscr{H}$, $\operatorname{Re}[G(z)(z - ib)^{-2}] \leq 0$. Hence $G \in \mathscr{G}_2(\mathscr{H})$, if $\alpha \neq 1$. On the other hand, if $\alpha = 1$, then (2.12) can be rephrased to say that for each $z \in \mathscr{H}$, $\operatorname{Re}\phi_t(z)$ is an increasing function of t. Hence $G \in \mathscr{G}_1(\mathscr{H})$, if $\alpha = 1$. The proof of the theorem is complete.

Remark. It is evident from the limiting behavior of semigroups (as $t \to +\infty$) that if G has a representation in either of the forms (2.4), (2.5), then such a representation is unique.

The concrete description of $\mathscr{G}(\mathscr{H})$ in Theorem (2.6), which owes its formulation to the Denjoy-Wolff theorem, does not provide us with a unifying criterion that characterizes the class $\mathscr{G}(\mathscr{H})$. As will be seen in the proof of the next theorem, the Schwarz-Pick inequality leads to the formulation of such a criterion.

(2.13) THEOREM. *Let* $G: \mathscr{H} \to \mathbb{C}$ *be analytic, and writing* $z = x + iy$, *let* $\operatorname{Re} G(z) = u(x, y)$. *Then* $G \in \mathscr{G}(\mathscr{H})$ *if and only if*

$$\text{(2.14)} \qquad x\frac{\partial u}{\partial x} \leq u \quad \text{on } \mathscr{H}.$$

Proof. Suppose $G \in \mathscr{G}(\mathscr{H})$ with corresponding semigroup $\{\phi_t\}$. For $z_0 \in \mathscr{H}$, $\phi(\cdot, \cdot)$ is analytic in (t, z) in a neighborhood of $(0, z_0)$ (as may be seen, *e.g.*, by treating t as a complex variable and using successive approximations). This gives a $\delta > 0$ such that for $0 \leq t < \delta$,

$$\text{(2.15)} \qquad \phi(t, z_0) = z_0 + G(z_0)t + O(t^2);$$

$$\text{(2.16)} \qquad \phi_t'(z_0) = 1 + G'(z_0)t + O(t^2).$$

We know from the Schwarz-Pick inequality that for $t \in \mathbb{R}^+$, $z \in \mathscr{H}$,

$$|\phi_t'(z)| \leq z^{-1}\operatorname{Re}\phi_t(z).$$

In particular,

$$\text{(2.17)} \qquad \operatorname{Re}\phi_t'(z_0) \leq [\operatorname{Re}\phi_t(z_0)]/\operatorname{Re} z_0.$$

Substitute (2.15) and (2.16) into (2.17) to get for $0 < t < \delta$

$$(2.18) \qquad t \operatorname{Re} G'(z_0) \le t[\operatorname{Re} G(z_0)](\operatorname{Re} z_0)^{-1} + O(t^2).$$

Divide (2.18) by t, and let $t \to O^+$ to obtain (2.14).

Conversely, suppose (2.14) holds. We recall some basic facts about the hyperbolic metric ρ on $\mathscr{H}$ (see, for example, [1]). For z_1, z_2 in $\mathscr{H}$, $\rho(z_1, z_2) = \min \int_\gamma (\operatorname{Re} z)^{-1}|dz|$, where the minimum extends over all paths γ in $\mathscr{H}$ from z_1 to z_2. The metric ρ induces the usual Euclidean topology on $\mathscr{H}$, and the ρ-compact sets are the subsets of $\mathscr{H}$ which are closed and bounded with respect to ρ. Let $z_0 \in \mathscr{H}$, and suppose that for some $b > 0$, the initial value problem $\frac{dw}{dt} = G(w)$, $w(0) = z_0$, has a (necessarily unique) solution $\phi(t, z_0)$ for $0 \le t < b$. If we square the expression

$$(2.19) \qquad \left|\frac{d\phi(t, z_0)}{dt}\right| / \operatorname{Re}(\phi(t, z_0))$$

and differentiate with respect to t, we get $2|G(\phi(t, z_0))|^2[\operatorname{Re}\phi(t, z_0)]^{-3}$ multiplied by $[\operatorname{Re}(\phi(t, z_0))] \operatorname{Re} G'(\phi(t, z_0)) - \operatorname{Re} G(\phi(t, z_0))$. Thus by (2.14), the expression (2.19) is a decreasing function of t. Thus $\rho(z_0, \phi(t, z_0))$ does not exceed

$$b|G(z_0)| / \operatorname{Re} z_0 \qquad \text{for } 0 \le t < b.$$

Thus $\{\phi(t, z_0): 0 \le t < b\}$ is contained in a compact subset of $\mathscr{H}$. Standard arguments now show that (2.1) has a solution on $\mathbb{R}^+ \times \mathscr{H}$.

(2.20) COROLLARY. *$\mathscr{G}(\mathscr{H})$ is a cone with vertex at* 0 *in the linear space of all analytic functions on $\mathscr{H}$.*

Suppose next that G (resp., $\tilde{G}$) belongs to $\mathscr{G}(\mathscr{H})$ with corresponding semigroup $\{\phi_t\}$ (resp., $\{\tilde{\phi}_t\}$), and $\lambda > 0$. Theorem (2.13) and Corollary (2.20) do not provide a method for constructing the semigroups corresponding to λG and $(G + \tilde{G})$ from $\{\phi_t\}$ and $\{\tilde{\phi}_t\}$. The semigroup corresponding to λG is obviously given by $\phi(\lambda t, z)$. We sketch a method for obtaining the semigroup $\{\psi_t\}$ corresponding to $(G + \tilde{G})/2$. For each positive integer n, we construct an "approximate semigroup" $\psi_n(\cdot, \cdot)$ on $\mathbb{R}^+ \times \mathscr{H}$ satisfying the requirements: for each $z \in \mathscr{H}$,

$$(2.21) \qquad \frac{d\psi_n(t,z)}{dt} = \begin{cases} G(\psi_n(t,z)) & \text{for } 2kn^{-1} \le t \le (2k+1)n^{-1},\ k = 0, 1, 2, \dots, \\ \tilde{G}(\psi_n(t,z)) & \text{for } (2k+1)n^{-1} \le t \le 2(k+1)n^{-1}, \\ & k = 0, 1, 2, \dots; \end{cases}$$

$$\psi_n(0, z) = z,$$

where the differential equations indicated in (2.21) involve only one-sided derivatives at the end-points of the indicated intervals. The construction of ψ_n is accomplished by repeated alternate use of the global solutions ϕ and $\tilde{\phi}$ in appropriate initial value problems. It is easy to see that for each $t \in \mathbb{R}^+$, $\psi_n(t,\cdot)$ is an analytic map of $\mathscr{H}$ into $\mathscr{H}$. Using (2.21), we find that there is a positive t_0 such that

$$|\psi_n(t, 1) - 1| < 2^{-1} \quad \text{for } 0 \leq t \leq t_0,\ n = 1, 2, \ldots . \tag{2.22}$$

Let $\mathscr{F}$ be the family of functions on $\mathscr{H}$ given by $\{\psi_n(t, \cdot): t \in [0, t_0], n = 1, 2, \ldots\}$. As in the proof that $\mathscr{G}_2(\mathscr{H}) \subseteq \mathscr{G}(\mathscr{H})$ (Theorem (2.6)), we see that $\mathscr{F}$ is a normal family such that every limit point of $\mathscr{F}$ in the usual topology leaves $\mathscr{H}$ invariant. With the aid of this, we find that $\{\psi_n(\cdot, \cdot)\}$ is equicontinuous on compact subsets of $[0, t_0] \times \mathscr{H}$. Thus we arrive at a subsequence $\{\psi_{n_k}(\cdot, \cdot)\}$ which converges uniformly on compact subsets of $[0, t_0] \times \mathscr{H}$. It follows from (2.21) that the limit ψ satisfies, on $[0, t_0] \times \mathscr{H}$,

$$\frac{\partial \psi(t, z)}{\partial t} = 2^{-1}[G(\psi(t, z)) + \tilde{G}(\psi(t, z))], \qquad \psi(0, z) = z. \tag{2.23}$$

As before, we can extend ψ so as to satisfy (2.23) on $\mathbb{R}^+ \times \mathscr{H}$.

3. SEMI-GROUPS OF COMPOSITION OPERATORS ON HARDY SPACES OF THE DISC

We begin this section with the observation that Proposition (2.2), together with the remarks in Section 2 leading up to it, remains true word for word if $\mathscr{H}$ is replaced by Δ throughout.

Notation. Let $\mathscr{P}$ be the class of all analytic functions F on Δ such that Re $F \geq 0$, and F is not the zero function. Let $\mathscr{A}$ be the class of all functions G on Δ of the form

$$G(z) = \bar{\alpha} F(z)(z - \alpha)^2, \tag{3.1}$$

where $|\alpha| = 1$, $F \in \mathscr{P}$. Let $\mathscr{B}$ be the class of all functions G on Δ of the form

$$G(z) = F(z)(\bar{\beta} z - 1)(z - \beta), \tag{3.2}$$

where $\beta \in \Delta$, $F \in \mathscr{P}$.

In the preceding notation, Theorem (2.6) and the remark immediately following it can be rephrased as follows.

(3.3) THEOREM. *$\mathscr{G}(\Delta) \setminus \{0\}$ is the disjoint union of $\mathscr{A}$ and $\mathscr{B}$. If* G *has the form* (3.1), *then its corresponding semigroup* $\{\phi_t\}$ *satisfies:* $\phi_t(z) \to \alpha$ *as* $t \to +\infty$ *for each* $z \in \Delta$. *If* G *has the form* (3.2), *then for its corresponding semigroup* $\{\phi_t\}$ *we have:* (i) $\phi_t(z) \to \beta$ *as* $t \to +\infty$ *for each* $z \in \Delta$ *if and only if* Re $F > 0$ *on* Δ; *and* (ii) $\phi_t(\beta) = \beta$ *for all* $t \in \mathbb{R}^+$. *A representation in either of the forms* (3.1), (3.2) *is unique.*

We remark in passing that, as would be expected, the description of $\mathscr{G}(\mathbb{C})$ contrasts with that of $\mathscr{G}(\Delta)$. Since a univalent entire function is a first degree polynomial, it is easy to see that $\mathscr{G}(\mathbb{C})$ consists of the constant functions and the first degree polynomials.

If $0 < p \leq \infty$ and ψ is an analytic map of Δ into Δ, then, as is well known, ψ induces a continuous linear transformation C_ψ: $H^p(\Delta) \to H^p(\Delta)$ defined by the formula $C_\psi f = f \circ \psi$ [4, p. 29]; C_ψ is called *the composition operator on* $H^p(\Delta)$ *induced by* ψ. For fixed p, the correspondence between analytic maps of Δ into itself and the composition operators they induce is one-to-one, since $\psi = C_\psi \chi_1$ (from now on, χ_n denotes the function given by $\chi_n(z) \equiv z^n$, $n = 0, 1, 2, \ldots$).

(3.4) THEOREM. *Suppose* $1 \leq p < \infty$. *There is a one-to-one correspondence between* $\mathscr{S}(\Delta)$ *and the strongly continuous one-parameter semigroups of composition operators on* $H^p(\Delta)$. *For* $\{\phi_t\} \in \mathscr{S}(\Delta)$, *the corresponding semigroup of composition operators is given by:*

$$T_t f = f(\phi_t) \qquad \text{for } t \in \mathbb{R}^+, f \in H^p(\Delta). \tag{3.5}$$

Proof. Suppose $\{\phi_t\} \in \mathscr{S}(\Delta)$. We show that the semigroup $\{T_t\}$ defined by (3.5) is strongly continuous on $H^p(\Delta)$. For each $t \in \mathbb{R}^+$, let $\sum_{n=0}^{\infty} a_{n,t} z^n$ be the Maclaurin series expansion for ϕ_t. Since each ϕ_t is univalent (Proposition (2.2) applied to Δ), we have by the area theorem [5, Lemma 1.1]

$$\sum_{n=1}^{\infty} n\,|a_{n,t}|^2 \leq 1 \qquad \text{for } t \in \mathbb{R}^+. \tag{3.6}$$

It follows from (3.6) that $\{\phi_t\colon t \in \mathbb{R}^+\}$ is a totally bounded subset of $H^2(\Delta)$. Suppose $\{t_n\} \subseteq \mathbb{R}^+$ and $t_n \to s$. Then there are a subsequence $\{t_{n_k}\}$ and an $f \in H^2(\Delta)$ such that $\|\phi_{t_{n(k)}} - f\|_2 \to 0$. In particular, for each $z \in \Delta$, $\phi_{t_{n(k)}}(z) \to f(z)$. But $\phi_t(z)$ is a continuous function of t, and so $\|\phi_{t_{n(k)}} - \phi_s\|_2 \to 0$. Consequently, the map $t \mapsto \phi_t$ is continuous from $\mathbb{R}^+$ into $H^2(\Delta)$. If we go out to the boundary $|z| = 1$, and use the Lebesgue bounded convergence theorem, we see that $t \mapsto \phi_t$ is continuous from $\mathbb{R}^+$ into $H^p(\Delta)$, and in fact for each polynomial P, $t \mapsto T_t P$ is continuous from $\mathbb{R}^+$ into $H^p(\Delta)$. Since for each $t \in \mathbb{R}^+$

$$\|T_t\| \leq [(1 + |\phi_t(0)|)(1 - |\phi_t(0)|)^{-1}]^{1/p},$$

$\{T_t\}$ is uniformly bounded on compact subsets. Since the polynomials are dense in $H^p(\Delta)$, it follows from the foregoing that $\{T_t\}$ is strongly continuous.

Conversely, suppose $\{T_t\}$ is a strongly continuous semigroup of composition operators on $H^p(\Delta)$, and let $\{\phi_t\}$ be the family of holomorphic functions satisfying (3.5). By strong continuity (applied to χ_1), $t \mapsto \phi_t$ is continuous from $\mathbb{R}^+$ into $H^p(\Delta)$. The continuity of $\phi(\cdot, \cdot)$ on $\mathbb{R}^+ \times \Delta$ follows easily by Cauchy's integral formula.

(3.7) THEOREM. *Suppose* $1 \leq p < \infty$, *and* $\{\phi_t\} \in \mathscr{S}(\Delta)$. *Let* G *be the infinitesimal generator of* $\{\phi_t\}$, $\{T_t\}$ *the semigroup of composition operators in* (3.5), *and* Γ *the infinitesimal generator of* $\{T_t\}$. *Then the domain of* Γ, $\mathscr{D}(\Gamma)$, *consists of all* $f \in H^p(\Delta)$ *such that* $Gf' \in H^p(\Delta)$, *and* $\Gamma(f) = Gf'$ *for* $f \in \mathscr{D}(\Gamma)$.

Proof. If $\phi_t = \chi_1$ for $t \in \mathbb{R}^+$, the assertion is trivial. Thus we assume that $G \in \mathscr{G}(\Delta)\setminus\{0\}$. Since $T_t\chi_0 = \chi_0$, $\|T_t\| \geq 1$ for $t \in \mathbb{R}^+$. By [3, Lemma VIII.1.4 and Theorem VIII.1.11], $\omega_0 = \lim_{t\to+\infty} t^{-1}\log\|T_t\|$ exists, $0 \leq \omega_0 < +\infty$, and if $\operatorname{Re}\lambda > \omega_0$, then λ is in the resolvent set of Γ. The description of $\mathscr{D}(\Gamma)$ in the statement of the present theorem certainly defines a linear manifold M in $H^p(\Delta)$. The linear transformation $\mathscr{E}$: $M \to H^p(\Delta)$ given by $\mathscr{E}f = Gf'$ clearly extends Γ.

Suppose first that G has the form (3.1). In particular, $|G| > 0$ on Δ. Let h be a primitive on Δ of $(1/G)$. Then for $t \in \mathbb{R}^+$, $z \in \Delta$,

$$h(\phi(t, z)) = t + h(z). \tag{3.8}$$

Pick $r > \max\{\omega_0, \log[(1 + |\phi(1, 0)|)(1 - |\phi(1, 0)|)^{-1}]\}$. We show that $(\mathscr{E} - r)$ is one-to-one. Suppose $f \in H^p(\Delta)$, and $Gf' = rf$ on Δ. There is a complex constant K such that $f = K\exp(rh)$. From this and (3.8) we see that for $t \in \mathbb{R}^+$, $f(\phi_t) = e^{rt}f$. Thus $e^{rt}\|f\|_1 \leq (1 + |\phi_t(0)|)(1 - |\phi_t(0)|)^{-1}\|f\|_1$. If f is not the zero function, then by taking $t = 1$ in this last inequality, we get a contradiction to the choice of r. Since $(\mathscr{E} - r)$ is one-to-one and extends $(\Gamma - r)$, while the range of the latter is $H^p(\Delta)$, it follows that $\mathscr{E} = \Gamma$.

Suppose, finally, that G has the form (3.2). If λ is a complex number, and f is an analytic function on Δ such that f is not the zero function and $Gf' = \lambda f$ on Δ, then pick r such that $|\beta| < r < 1$, and f has no zeros on $|z| = r$. We have:

$$(2\pi i)^{-1}\int_{|z|=r} [f'(z)/f(z)]\,dz = (2\pi i)^{-1}\int_{|z|=r} [\lambda/G(z)]\,dz = \lambda[F(\beta)(|\beta|^2 - 1)]^{-1}. \tag{3.9}$$

By virtue of the argument principle, we infer that the set of eigenvalues of $\mathscr{E}$ is countable in this case. In particular, we can choose a real number $\lambda > \omega_0$ such that $(\mathscr{E} - \lambda)$ is one-to-one. As in the previous case, $\mathscr{E} = \Gamma$, and the proof is complete.

(3.10) COROLLARY. *If* $1 \leq p < \infty$, *and* $\{T_t\}$ *is a strongly continuous one-parameter semigroup of composition operators on* $H^p(\Delta)$, *then* $\{T_t\}$ *has a bounded infinitesimal generator if and only if* T_t *is the identity operator for every* $t \in \mathbb{R}^+$.

Proof. Suppose the infinitesimal generator Γ of $\{T_t\}$ is a bounded operator on $H^p(\Delta)$. Then, in the notation of Theorem (3.7), $\Gamma(f) = Gf'$ for every $f \in H^p(\Delta)$. Taking f to be χ_1, we see that $G \in H^p(\Delta)$. By taking f to be χ_n, $n = 1, 2, \ldots$, we see that $n\|G\|_p \leq \|\Gamma\|$. Hence $G \equiv 0$.

4. AN EIGENVALUE PROBLEM.

In this section we examine the point spectrum of the infinitesimal generator for a certain type of semigroup of composition operators. An interplay will arise between the foregoing circle of ideas and potential theory. Let F be an analytic function on Δ such that $\operatorname{Re} F > 0$ and $F(0) > 0$. The Herglotz representation gives a Borel measure μ on the unit circle K, with $\mu \geq 0$, $\mu(K) > 0$ such that

$$(4.1) \qquad 1/F(z) = \int_K (w+z)(w-z)^{-1} d\mu(w) .$$

Clearly any such measure μ can arise in (4.1) by suitable initial choice of F. We rewrite (4.1) in the form

$$(4.2) \qquad 1/F(z) = [1/F(0)] \int_K (w+z)(w-z)^{-1} d\sigma(w) ,$$

where $\sigma \geq 0$, $\sigma(K) = 1$. Let $G(z) = -zF(z)$. By Theorem (3.3), G is the infinitesimal generator of a semigroup of holomorphic mappings of $\triangle$ into $\triangle$. For $1 \leq p < \infty$, let Γ_p be the infinitesimal generator of the corresponding semigroup of composition operators on $H^p(\triangle)$. It is apparent from (3.9) that the point spectrum of Γ_p is a subset of $\{-F(0)k\colon k = 0, 1, 2, \ldots\}$. The null space of Γ_p is the set of all constant functions. We have:

(4.3) THEOREM. *Suppose* $1 \leq p < \infty$. *For each positive integer* k, $-F(0)k$ *belongs to the point spectrum of* Γ_p *if and only if* $H^p(\triangle)$ *contains the function* E_k *given by*

$$(4.4) \qquad E_k(z) = \exp\left[-k \int_0^z \zeta^{-1}(F(\zeta) - F(0))(1/F(\zeta))\, d\zeta\right] .$$

In this case, the eigenmanifold corresponding to $-F(0)k$ *is the span of the function* $z^k E_k(z)$.

Proof. If $-F(0)k$ is in the point spectrum, and f is a non-zero vector in $\mathscr{D}(\Gamma_p)$ such that $Gf' = -F(0)kf$, then it follows from (3.9) that there is an analytic function g on $\triangle$ such that $f(z) = z^k g(z)$ and $|g| > 0$ on $\triangle$. Obvious simplification of the equation $zF(z)f' - kF(0)f = 0$ gives $g' + k[zF(z)]^{-1}(F(z) - F(0))g = 0$. The theorem follows easily.

If we use (4.2) in the integral in (4.4) and interchange the order of integration, we see that this integral is equal to $-2 \int_K \log[1/(1-\bar{w}z)]\, d\sigma(w)$, where "log" denotes the principal branch of the logarithm. In view of this, the next proposition is a corollary of Theorem (4.3).

(4.5) PROPOSITION. *The point spectrum of* Γ_p *is*

$$\{-F(0)k\colon k = 0, 1, 2, \ldots\} \qquad \textit{for all } p \in [1, \infty)$$

if and only if the function e_σ *given by*

$$(4.6) \qquad e_\sigma(z) = \exp\left[\int_K \log\{1/(1-\bar{w}z)\}\, d\sigma(w)\right] \qquad \textit{for } z \in \triangle$$

belongs to $H^p(\triangle)$ *for all finite* p.

In view of Proposition (4.5), we shall concentrate throughout the remainder of this section on functions of the form (4.6).

(4.7) THEOREM. *Let ν be a finite positive measure on the Borel sets of* K, *and suppose* ε *is a positive number such that* K *can be covered by a finite collection of closed arcs* $J_1, \ldots, J_n$ *with* $\nu(J_k) < \varepsilon$ *for* $k = 1, 2, \ldots, n$. *Then* $e_\nu \in H^p(\Delta)$ *for* $p < \varepsilon^{-1}$.

Proof. There is no loss of generality in making the further assumption that each J_k is a proper subset of K. For each k, let S_k be the sector $\{rw: r \in [0, 1], w \in J_k\}$. If η is any positive number, we strictly extend each J_k at both end-points so as to obtain a larger closed arc $\tilde{J}_k$, still a proper subset of K, such that $\nu(\tilde{J}_k) < \varepsilon + \eta$. The distance between S_k and $K \setminus \tilde{J}_k$ is a positive number a_k. Let m denote normalized Lebesgue measure on K, and let p be any positive real number. For $z \in J_k$,

$$(4.8)\qquad |e_\nu(rz)|^p \leq a_k^{-p\nu(K\setminus \tilde{J}_k)} \exp\left[\int_{\tilde{J}_k} \log\{1/|w - rz|^p\}\, d\nu(w)\right].$$

Application of the arithmetic-geometric mean inequality to the integral in (4.8) yields

$$(4.9)\qquad |e_\nu(rz)|^p \leq a_k^{-p\nu(K\setminus \tilde{J}_k)}[\nu(\tilde{J}_k)]^{-1} \int_{\tilde{J}_k} |w - rz|^{-p\nu(\tilde{J}_k)}\, d\nu(w)$$

(if $\nu(\tilde{J}_k) = 0$, the last two factors are deleted from the majorant in (4.9)). Integration of (4.9) with respect to m and interchange of the order of integration give

$$\int_{J_k} |e_\nu(rz)|^p dm(z) \leq a_k^{-p\nu(K\setminus \tilde{J}_k)} \int_K |1 - rz|^{-p\nu(\tilde{J}_k)}\, dm(z)\,.$$

We see then that $e_\nu \in H^p(\Delta)$ if $p\nu(\tilde{J}_k) < 1$ for $k = 1, 2, \ldots, n$. Thus $e_\nu \in H^p(\Delta)$ for $p < (\varepsilon + \eta)^{-1}$, where η is an arbitrary positive number. The desired conclusion follows.

(4.10) THEOREM. *If* ν *is a finite positive measure on the Borel sets of* K, *then* $e_\nu \in H^p(\Delta)$ *for all finite* p *if and only if* ν *annihilates singletons.*

Proof. The "if" part is immediate from Theorem (4.7). On the other hand, suppose $z_0 \in K$, and $\nu(\{z_0\}) > 0$. Then $|e_\nu(rz_0)| \geq 2^{-\nu(K\setminus\{z_0\})}(1 - r)^{-\nu(\{z_0\})}$. If $0 < p < \infty$ and $e_\nu \in H^p(\Delta)$, then $|e_\nu(rz_0)| = 0((1 - r)^{-1/p})$. It follows that $p \leq [\nu(\{z_0\})]^{-1}$.

We conclude this section by investigating conditions sufficient to insure that $e_\nu \in H^\infty(\Delta)$, or, equivalently, that the logarithmic potential u_ν given by

$$(4.11)\qquad u_\nu(z) = -\int_K \log|w - z|\, d\nu(w)\,, \qquad z \in \Delta,$$

is bounded. We recall that by a standard application of Fubini's theorem the right-hand side of (4.11) defines (m-almost everywhere on K) a function of z belonging to L^1 (dm) whose Poisson integral is u_ν.

(4.12) THEOREM. *Let* ν *be a finite positive measure on the Borel sets of* K, *and define* f_ν *on* $[0, 2\pi]$ *by setting* $f_\nu(t) = \nu\{e^{iu}: 0 < u \leq t\}$. *Either of the following conditions is sufficient for* e_ν *to be in* $H^\infty(\triangle)$:

(i) ν *is absolutely continuous, and the Radon-Nikodym derivative* $\dfrac{d\nu}{dm}$ *belongs to* $L \log^+ L$;

(ii) *if* ω_ν *denotes the modulus of continuity of* f_ν *then* $t^{-1}\omega_\nu(t)$ *is integrable on* $0 \leq t \leq 2\pi$.

Proof. Without loss of generality, we assume that $\nu(K) = 1$. Let F be an analytic function on $\triangle$ such that $\operatorname{Re} F > 0$, $F(0) > 0$, and (4.2) holds with σ replaced by ν. Then by virtue of the discussion just preceding Proposition (4.5),

$$e_\nu(z) = \exp\left[-2^{-1}\int_0^z \zeta^{-1}(F(\zeta) - F(0))(1/F(\zeta))\,d\zeta\right].$$

Differentiation with respect to z gives,

$$e_\nu'/e_\nu = -(2z)^{-1}[1 - (F(0)/F(z))]\,. \tag{4.13}$$

The left-hand side of (4.13) is the derivative of $\int_K \log[1/(1-\bar{w}z)]\,d\nu(w)$. The right-hand side of (4.13) belongs to $H^1(\triangle)$ if and only if $1/F$ belongs to $H^1(\triangle)$. In view of [4, Theorems 4.3 and 4.4] and (4.2), we see that this last condition is equivalent to (i). Thus, if (i) holds, then it follows from Hardy's inequality that $H^\infty(\triangle)$ contains $\int_K \log[1/(1-\bar{w}z)]\,d\nu(w)$.

Suppose next that (ii) holds. Let γ be the measure $(\nu - m)$. Although γ is not a positive measure, we define f_γ and ω_γ by analogy with the definitions of f_ν and ω_ν Clearly, $t^{-1}\omega_\gamma(t)$ is integrable on $[0, 2\pi]$ and $f_\gamma(0) = f_\gamma(2\pi) = 0$. Since, for all $z \in \triangle$,

$$\int_K \log|w - z|\,dm(w) = 0,$$

we have

$$u_\nu(z) = -\int_K \log|w - z|\,d\gamma(w) = -\int_0^{2\pi} \log|e^{it} - z|\,df_\gamma(t), \qquad z \in \triangle.$$

Integration by parts on the right-hand side of this last equation gives

$$u_\nu(z) = -2^{-1}\int_0^{2\pi} f_\gamma(t)\,\mathrm{Im}[(e^{it}+z)(e^{it}-z)^{-1}]\,dt. \tag{4.14}$$

Since $t^{-1}\omega_\gamma(t)$ is integrable on $[0, 2\pi]$, it follows by [8, Theorem III.13.30] that the conjugate function of f_γ is continuous. It follows easily that u_ν is bounded.

Remarks. (1) The discussion in [8, p. 197] shows the existence of a positive singular measure ν_0 on K having total mass 1 such that f_{ν_0} satisfies a Lipschitz condition. By Theorem (4.12)-(ii), $e_{\nu_0} \in H^\infty(\Delta)$. Thus the absolute continuity of ν is not a necessary condition for e_ν to be bounded. (2) The absolute continuity of ν is not a sufficient condition for e_ν to be bounded. For, suppose to the contrary that for every real function $f \in L^1(m)$, the convolution $g * f \in L^\infty(m)$, where g is defined by $g(z) = -\log|1-z|$ for $z \in K$. By the closed graph theorem, there is a constant $B > 0$ such that $\|f * g\|_\infty \leq B\|f\|_1$ for all real $f \in L^1(m)$. Let $\{f_n\}$ be the Fejér kernel. By Alaoglu's theorem, there are a $\psi \in L^\infty(m)$ and a subnet $f_{n_j} * g$ which converges weak-star in $L^\infty(m)$ to ψ. Since $\|f_n * g - g\|_1 \to 0$, we are led to the absurd conclusion that $g \in L^\infty(m)$.

REFERENCES

1. L. Ahlfors, *Conformal invariants: topics in geometric function theory.* McGraw-Hill, New York, 1973.
2. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations.* McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.
3. N. Dunford and J. T. Schwartz, *Linear operators. I. General Theory.* Pure & Applied Mathematics, Vol. 7. Interscience Publishers, Inc., New York, 1958.
4. P. Duren, *Theory of H^p spaces.* Pure and Applied Mathematics, Vol. 38. Academic Press, New York-London, 1970.
5. W. Hayman, *Multivalent functions.* Cambridge Tracts in Mathematics and Mathematical Physics, No. 48. Cambridge University Press, Cambridge, 1958.
6. M. H. Heins, *On the pseudo-periods of the Weierstrass zeta functions. II.* Nagoya Math. J., 30 (1967), 113–119.
7. W. Hurewicz, *Lectures on ordinary differential equations.* The Technology Press of the Massachusetts Institute of Technology, Cambridge, Mass.; John Wiley & Sons, Inc., New York, 1958.
8. A. Zygmund, *Trigonometric series.* 2nd ed., Vol. I. Cambridge University Press, New York, 1959.

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