OMITTED VALUES OF SINGULAR INNER FUNCTIONS

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In this paper we investigate certain properties of measures on the unit circle T associated with singular inner functions which omit values in the unit disc U . Our results are used to resolve some open questions concerning inner functions; in particular, we disprove a conjecture of Herrero concerning the structure of the inner functions under the uniform topology of H^∞, the space of bounded analytic functions on U . We assume that the reader is familiar with the basic theory of H^∞, the notion of logarithmic capacity for plane sets, and the elementary properties of universal covering surfaces for plane regions. Appropriate references would be Duren [5], Tsuji [10], and Ahlfors [1], respectively.

We briefly describe our main results below. The preliminary material is discussed in more detail and notations are established in Section 2.

1. MAIN RESULTS

If A is a (relatively) closed subset of U with (logarithmic) capacity zero, then the universal covering surface of $U \setminus A$ is conformally equivalent to U. If ϕ_A is an uniformizer of $U \setminus A$ (see 2.3), then ϕ_A is an inner function whose range is precisely $U \setminus A$. For our main result we assume $0 \in A$, so that ϕ_A is a singular inner function.

THEOREM I. *Let* A *be a closed subset of* U *of (logarithmic) capacity zero,* $0 \in A$, *and let* μ *be the singular measure on* T *associated with the conformal mapping* ϕ_A *of* U *onto* $U \setminus A$.

(a) *If* 0 *is an isolated point of* A, *then* μ *is discrete; i.e., it consists entirely of point masses.*

(b) *If* 0 *is a limit point of* A, *then* μ *is continuous; i.e., it has no point masses.*

The proofs of parts (a) and (b) require entirely different techniques and are given in Sections 3 and 4, respectively. Part (b) is actually a corollary to a stronger result, Theorem 4.2. The main ingredient is a mapping theorem which may be of some independent interest:

THEOREM II. *Let* F *be an analytic function from* U *into the left half-plane with the property that* $\liminf_{r \to 1} (1 - r)|F(r)| > 0$. *Then, for each* $M > 0$, *the disc* $\{w \in \mathbb{C} : |w - F(r)| < M\}$ *lies in the range of* F *for all* r *sufficiently close to* 1, $0 < r < 1$.

This study was originally motivated by certain conjectures of Herrero [8] concerning inner functions under the uniform norm of H^∞. In Section 5 we use

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Theorem I to arrive at the following result, which proves false one of those conjectures.

THEOREM III. *The collection of discrete singular inner functions is not closed in* H^∞.

In Section 5, we also answer some other open questions, and we discuss our results in relation to another conjecture of Herrero. We close with comments and open questions in Section 6.

2. PRELIMINARIES

2.1. *Inner Functions.* A function $f \in H^\infty$ is an *inner function* if it has unimodular radial limits at almost all (Lebesgue measure) points of T. Every inner function factors uniquely as a product $f(z) = B(z) \cdot S(z)$, $z \in U$, where B is the inner function, called a *Blaschke product,* formed by the zeros of f, and S is a *singular inner function;* that is, one which does not vanish in U.

All *singular measures* we discuss are positive, finite, Borel measures on T which are singular with respect to Lebesgue measure. Each singular measure μ gives rise to a singular inner function S_μ defined by

$$S_\mu(z) = \exp\left\{\int_T \frac{z+\xi}{z-\xi}\, d\mu(\xi)\right\}, \qquad z \in U;$$

and conversely, every singular inner function is a unimodular constant times S_μ for some unique singular μ. In the case that μ is a unit mass concentrated at 1, S_μ is denoted S_1. Specifically, $S_1(z) = \exp\{(z+1)/(z-1)\}$. If f is inner and if it factors as $f(z) = e^{i\theta} B(z) S_\mu(z)$, then S_ν is said to be a singular factor of f if $\mu - \nu \geq 0$, so that $S_\mu = S_{\mu-\nu} S_\nu$.

Two particular types of singular inner functions will be of interest to us. Let $f = e^{i\theta} S_\mu$. If μ is discrete (consists entirely of point masses), then f is termed a *discrete singular inner function;* while if μ has no discrete part, then f is termed a *continuous singular inner function.* Following Herrero, we will denote the collections of inner, singular inner, discrete singular inner, and continuous singular inner functions by $\mathscr{F}$, $\mathscr{F}_s$, $\mathscr{F}_a$, and $\mathscr{F}_c$, respectively.

For each $\alpha \in U$, ψ_α will denote the Möbius transformation of U onto itself given by $\psi_\alpha(z) = (z-\alpha)/(1-\bar{\alpha}z)$, $z \in U$. Of course, these are inner functions.

Finally, for later reference we note the following properties of inner functions:

(a) $\mathscr{F}$ and $\mathscr{F}_s$ are closed in H^∞.

(b) $\mathscr{F}_a \cup \mathscr{F}_c \neq \mathscr{F}_s$ since, in general, a singular measure μ may have both discrete and continuous parts. $\mathscr{F}_a \cap \mathscr{F}_c$ consists of trivial inner functions only (*i.e.,* unimodular constants).

(c) If $f \in \mathscr{F}$ and $g \in H^\infty$, $\|g\|_\infty \leq 1$, then a necessary and sufficient condition for $f \circ g \in \mathscr{F}$ is that $g \in \mathscr{F}$. The sufficiency is well known (see [9, Appendix]), and the necessity is clear by considering radial limits.

(d) For each $f \in \mathscr{F}$, $\|\psi_\alpha \circ f - \psi_\beta \circ f\|_\infty \to 0$ as $\alpha \to \beta$ in U .

2.2. *Omitted and Exceptional Values.* $\mathscr{A}$ will denote the collection of (relatively) closed subsets of U with (logarithmic) capacity zero.

(a) Every closed countable subset of U is in $\mathscr{A}$. However, $\mathscr{A}$ also contains uncountable subsets of U . Any closed subset of a set in $\mathscr{A}$ is in $\mathscr{A}$.

(b) If $A \in \mathscr{A}$, then $U \setminus A$ is connected.

(c) $A \in \mathscr{A}$ if and only if $\psi_\alpha(A) \in \mathscr{A}$ for all $\alpha \in U$.

For each $f \in \mathscr{F}$ we define the *exceptional set* E(f) and the *omitted set* O(f) as follows:

$$E(f) = \{\alpha \in U: \psi_\alpha \circ f \quad \text{has a nontrivial singular factor}\};$$
$$O(f) = \{\alpha \in U: \psi_\alpha \circ f \text{ is singular}\}.$$

Clearly, $O(f) \subseteq E(f)$.

2.3. *Frostman's Construction.* A famous theorem of O. Frostman [7] states: *If* f *is inner, then* E(f) *is a set of capacity* 0. Less well known is his converse: *For each* $A \in \mathscr{A}$, *there is an inner function* f *such that* $O(f) = E(f) = A$. His construction goes as follows (see [4, Ch. 2, Section 8]).

For $A \in \mathscr{A}$ let R_A be the *universal covering surface* of $U \setminus A$ with covering projection $\pi: R_A \to U \setminus A$. By the uniformization theorem, R_A is conformally equivalent to U. If $\tilde{\phi}_A: U \to R_A$ is a mapping which realizes this equivalence, then $\phi_A: U \to U$ defined by $\phi_A = \pi \circ \tilde{\phi}_A$ is an inner function, and $O(\phi_A) = E(\phi_A) = A$. ϕ_A is called an *uniformizer* of $U \setminus A$. Note that $\tilde{\phi}_A$, and hence ϕ_A, is determined only up to composition with a conformal mapping of U. However, we will see that our results are independent of the choice of ϕ_A , so all statements referring to ϕ_A assume that some particular choice has been made.

The uniformizer of $U \setminus A$ may be characterized as follows, the proof is standard.

2.4. PROPOSITION. *Suppose* $A \in \mathscr{A}$ *and* $\phi \in \mathscr{F}$. *Then* $\phi = \phi_A$ *if and only if, for each* $f \in H^\infty$ *with* $f(U) \subseteq U \setminus A$, *there exists* $g \in H^\infty$, $\|g\|_\infty \leq 1$, *such that* $f = \phi \circ g$.

2.5. COROLLARIES. (a) *If* $A \in \mathscr{A}$, $\alpha \in U$, *and* $B = \psi_\alpha(A)$, *then* $\phi_B = \psi_\alpha \circ \phi_A$.

(b) *If* $A \in \mathscr{A}$ *and* $f \in \mathscr{F}$, *then* $O(f) \supseteq A$ *if and only if there is a* $g \in \mathscr{F}$ *with* $f = \phi_A \circ g$.

(c) *If* $A = \{0\}$, *then* $\phi_A = S_1$. To see this, let $e^{i\theta} S_\mu$ be any singular inner fι nction and write

$$g(z) = (\log e^{i\theta} S_\mu(z) + 1)/(\log e^{i\theta} S_\mu(z) - 1)$$

using any determination of the logarithm. Clearly, $e^{i\theta} S_\mu = S_1 \circ g$.

2.6. *Angular Derivatives.* Througout this paper, r denotes a real number, $0 < r < 1$. Let $g \in H^\infty$, $\|g\|_\infty \leq 1$.

If g has radial limit 1 at the point $1 \in T$, it can be shown that

$$\lim_{r\to 1}\frac{1-g(r)}{1-r}=\lim_{r\to 1}\frac{|1-g(r)|}{1-r}=c,$$

where the common limit c is either real or infinite. Moreover, if $c<\infty$, then also $c=\lim_{r\to 1} g'(r)$, and we say g has the *finite angular derivative* c at 1. Note that by Schwarz's lemma, $c>0$. The properties of angular derivatives are developed by Carathéodory [2, Sections 298-299].

Putting all of this together, we see that g has radial limit 1 and finite angular derivative c at 1 if and only if

(1) $$\limsup_{r\to 1}\frac{|1-g(r)|}{1-r}=c<\infty .$$

This allows a very concise statement of a result due to Fisher which is central to the proof of Theorem I(b).

2.7. THEOREM. (Fisher [6]). *Let* $g\in H^{\infty}$, $\|g\|_{\infty}\le 1$, *and let* S_{μ} *be the singular inner factor of* $S_1\circ g$. *Then*

$$\mu\{1\}=\liminf_{r\to 1}\frac{1-r}{|1-g(r)|}.$$

3. PROOF OF THEOREM I(a)

First we should point out why the conclusions of Theorem I do not depend on the choice of the uniformizer ϕ_A. In [9, Section 5] it was shown that if $f\in\mathscr{F}_c$ and $g\in\mathscr{F}$, then $(f\circ g)\in\mathscr{F}_c$. Now, if $0\in A$ and if ϕ_1 and ϕ_2 are any two uniformizers of $U\setminus A$, then there is a conformal mapping ψ of U such that

$$\phi_1=\phi_2\circ\psi \quad\text{and}\quad \phi_2=\phi_1\circ\psi^{-1}.$$

We see then that $\phi_1\in\mathscr{F}_c$ if and only if $\phi_2\in\mathscr{F}_c$. Likewise, ϕ_1 has a factor in $\mathscr{F}_c$ if and only if ϕ_2 also has a factor in $\mathscr{F}_c$; that is, $\phi_1\in\mathscr{F}_a$ if and only if $\phi_2\in\mathscr{F}_a$.

3.1. THEOREM I(a). *If* $A\in\mathscr{A}$ *and* 0 *is an isolated point of* A, *then* ϕ_A *is a discrete singular inner function.*

Proof. For convenience choose ϕ_A so that $\phi_A(0)>0$. Then $\phi_A=S_{\mu}$ for some singular measure μ. Let G consist of those points $\xi\in T$ for which $\lim_{r\to 1} S_{\mu}(r\xi)=0$. It is known that G has full μ-measure (see [5, Theorem 1.2]); therefore, it will suffice to prove that G is at most countable.

Choose $\delta>0$ so that if $B=\{z:|z|<\delta\}$, then $A\cap\bar{B}=\{0\}$. Let Ω be the open set $\pi^{-1}(B)\subseteq R_A$ with open components Ω_n, $n=1,2,\ldots$. The countability of this collection follows from the fact that R_A is separable. Observe that for each n, $R_A\setminus\Omega_n$ is connected. For suppose that a and b are arbitrary points of $R_A\setminus\Omega_n$.

Since R_A is connected, we can choose a path γ: $[0, 1] \to R_A$ with $\gamma(0) = a$, $\gamma(1) = b$, and consider $\Gamma = \pi \circ \gamma$. Γ is a path from $\pi(a)$ to $\pi(b)$ in $U \setminus A$ and, because of the conditions on B, is clearly homotopic in $U \setminus A$ to a path $\tilde{\Gamma}$ from $\pi(a)$ to $\pi(b)$ which misses B. If $\tilde{\gamma}$ is the lifting of $\tilde{\Gamma}$ to R_A (*i.e.,* $\pi \circ \tilde{\gamma} = \tilde{\Gamma}$) with $\tilde{\gamma}(0) = a$, then $\tilde{\gamma}(1) = b$. Thus the path $\tilde{\gamma}$ lies in $R_A \setminus \Omega_n$ and connects a to b.

Fix a point $\xi \in G$. If L is the image on R_A of the radial segment to ξ under the mapping ϕ_A, then from some point on, the curve L will lie entirely within one of the components of Ω; call it $\Omega_{n(\xi)}$. The countability of G follows if we show that the map $\xi \to n(\xi)$ is one-to-one. Defining $V \subseteq U$ by $V = \tilde{\phi}_A^{-1}(\Omega_{n(\xi)})$, it is enough to prove that $\bar{V} \cap T = \{\xi\}$.

Clearly $\xi \in \bar{V} \cap T$. Suppose there exists $\xi_1 \neq \xi$ with $\xi_1 \in \bar{V} \cap T$. Because ϕ_A has radial limits of modulus 1 almost everywhere on T, we can choose $\rho \in T$ so that ρ and $-\rho$ lie in different arcs of $T \setminus \{\xi_1, \xi\}$ and so that

$$\lim_{r\to 1} |\phi_A(r\rho)| = 1 = \lim_{r\to 1} |\phi_A(-r\rho)| . \tag{2}$$

Since V is connected, the choice of ρ ensures that the line segment $[\rho, -\rho]$ intersects V, while (2) ensures that the segment leaves V near both ends. However, this implies that $U \setminus V$ is not connected, contradicting the fact that

$$\tilde{\phi}_A(U \setminus V) = R_A \setminus \Omega_{n(\xi)}$$

is connected. Therefore, $\bar{V} \cap T = \{\xi\}$, and the proof is complete.

3.2. *Remarks.* When A is not a singleton, the measure μ above is not a "typical" discrete singular measure. Results of Seidel and Lohwater (see [4, Theorems 5.13 and 5.14]) imply, for example, that the closed support of μ is a perfect subset of T with positive (logarithmic) capacity. Moreover, our proof shows that μ has a *countable number* of points of density, which is strictly stronger than the conclusion that μ is discrete.

4. PROOF OF THEOREM I(b)

4.1. THEOREM II. *Let* F *be an analytic function from* U *into the left half-plane* $\{w \in \mathbb{C} : \mathrm{Re}\, w < 0\}$ *such that*

$$\liminf_{r\to 1} (1 - r)|F(r)| > 0 . \tag{3}$$

Then, for each $M > 0$, *the disc* $\{w \in \mathbb{C}: |w - F(r)| < M\}$ *lies in the range of* F *for all* r *sufficiently close to* 1.

Proof. First observe that it is sufficient to prove the conclusion in the case that $M = 1/3$. For other values of M, just apply this result to an appropriate dilation F_t of F, $F_t(z) = tF(z)$, $0 < t < \infty$. Let Ψ denote the Möbius transformation $\Psi(z) = (z + 1)/(z - 1)$, $z \neq 1$. Ψ is its own inverse and maps the left half plane conformally onto U. Define $g: U \to U$ by $g = \Psi \circ F$. Condition (3) on F implies $\liminf_{r\to 1} (1 - r)|1 - F(r)| > 0$ which converts precisely to the condition (1) on g, so we know that as $r \to 1$,

$$|1 - g(r)| \to 0, \quad |g'(r)| \to c, \quad \text{and} \tag{4}$$
$$|1 - g(r)|/(1 - r) \to c \qquad \text{for some } c,\ 0 < c < \infty.$$

Choose r sufficiently near 1 that $g'(r) \neq 0$, and define the auxiliary functions

$$g_r(z) = (g \circ \psi_r)(z)\ ; \qquad h_r(z) = \frac{g_r(z) - g(r)}{g'_r(0)}.$$

Then $|g_r'(0)| = |g'(r)|(1 - r^2) \neq 0$, so h_r is well-defined and satisfies

(1) $h_r(0) = 0$

(2) $h_r'(0) = 1$

(3) $\|h_r\|_\infty \leq 2/|g_r'(0)|$.

Now, by a result of Dieudonné (see Tsuji [10, Theorem VI. 10.]), $h_r(U) \supseteq \{w : |w| < |g_r'(0)|/16\}$. Unraveling our auxiliary functions, we see that $g(U)$ contains the disc $D_r = \{w : |w - g(r)| < |g'(r)|^2(1 - r^2)^2(16)^{-1}\}$ and hence that $F(U)$ contains the set $B_r = \Psi(D_r)$. Of course, $F(r) \in B_r$ and B_r is a disc because Ψ is a Möbius transformation. We will be finished if we can show that for r sufficiently near 1, the distance $d_r = \inf\{|F(r) - w| : w \in \partial B_r\}$ is larger than 1/3.

Fix r temporarily. Let C_1, C_2, and C_3 be circles centered at 1 with radii, respectively,

(i) $|1 - g(r)| + |g'(r)|^2(1 - r^2)^2(16)^{-1}$,

(ii) $|1 - g(r)|$, and

(iii) $|1 - g(r)| - |g'(r)|^2(1 - r^2)^2(16)^{-1}$.

We see that C_2 passes through $g(r)$, the center of D_r, while C_1 and C_3 are tangent to D_r. The images $\Psi(C_1)$, $\Psi(C_2)$, and $\Psi(C_3)$ are also circles centered at 1; and of course $\Psi(C_2)$ passes through $F(r)$, and $\Psi(C_1)$ and $\Psi(C_3)$ are tangent to B_r. A moment's reflection shows that the distance d_r is either the distance from $\Psi(C_2)$ to $\Psi(C_3)$ or from $\Psi(C_2)$ to $\Psi(C_1)$. Furthermore, it is clear that the latter distance is smaller. Let

$$s_1 = 1 - |1 - g(r)| - |g'(r)|^2(1 - r^2)^2(16)^{-1},$$
$$s_2 = 1 - |1 - g(r)|.$$

Then

$$d_r = |\Psi(s_1) - \Psi(s_2)|$$
$$= \left| \left[\frac{2 - |1 - g(r)| - |g'(r)|^2(1 - r^2)^2(16)^{-1}}{-|1 - g(r)| - |g'(r)|^2(1 - r^2)^2(16)^{-1}} \right] - \left[\frac{2 - |1 - g(r)|}{-|1 - g(r)|} \right] \right|$$

$$= \left| \frac{|g'(r)|^2 (1-r)^2 (1+r)^2}{8|1-g(r)|^2 + \frac{1}{2}|1-g(r)|\,|g'(r)|^2 (1-r)^2 (1+r)^2} \right|$$

$$= \left| \frac{|g'(r)|^2 (1+r)^2}{8\left|\frac{1-g(r)}{1-r}\right|^2 + \frac{|1-g(r)|\,|g'(r)|^2 (1+r)^2}{2}} \right| .$$

Finally, let $r \to 1$. The conditions (4) imply $d_r \to 1/2$. In particular, for r sufficiently close to 1, $d_r > 1/3$, proving Theorem II.

4.2. THEOREM. *Let* h *be a nonvanishing function in* H^∞, $\|h\|_\infty \le 1$, *and assume its singular inner factor has a nontrivial discrete part. Then, for each integer* $N \ge 1$, *there exists* $\delta = \delta(N) > 0$ *such that* h *assumes every value* w, $0 < |w| < \delta$, *at least* N *times.*

Proof. Let S_μ be the singular inner factor of h, and assume without loss of generality that $\mu\{1\} > 0$. By Proposition 2.4 and Theorem 2.7, $h = S_1 \circ g$, where

$$\liminf_{r\to 1} \frac{1-r}{|1-g(r)|} = \mu\{1\} . \tag{5}$$

$F = \Psi \circ g$ then satisfies the hypotheses of Theorem II.

We need one additional property of F. We know from the properties of angular derivatives that (5) implies $\lim_{r\to 1} (1-g(r))/(1-r) = c$, $0 < c < \infty$. Therefore, $\lim_{r\to 1} (\operatorname{Im} g(r))/(1-\operatorname{Re} g(r)) = 0$. But $\operatorname{Im} g(r)/(1-\operatorname{Re} g(r))$ is the tangent of the angle the line from g(r) to 1 makes with the real axis, so we see that g maps the radial segment to 1 onto an arc which is tangent to the real axis at 1. As for $F = \Psi \circ g$, this means that

$$|F(r)| \to \infty \qquad \text{as } r \to 1$$

and that, for each θ, $\pi/2 < \theta < \pi$, for r sufficiently near 1, $\theta < \arg F(r) < (2\pi - \theta)$. That is,

$$\operatorname{Re} F(r) \to -\infty \qquad \text{as } r \to 1 . \tag{6}$$

If N is any positive integer, then by the conclusion of Theorem II, there exists r_0, $0 < r_0 < 1$, such that for all r, $r_0 \le r < 1$,

$$F(U) \supseteq \{w: |w - F(r)| < 2\pi N\} . \tag{7}$$

Let $\delta = \exp\{\operatorname{Re} F(r_0)\}$. Then (6) and (7) imply that $\exp\{F\}$ assumes each w, $0 < |w| < \delta$, at least N times. But $h = \exp\{F\}$, so we are finished.

4.3. THEOREM I(b). *If* $A \in \mathscr{A}$ *and* 0 *is a limit point of* A, *then* ϕ_A *is a continuous singular inner function.*

Proof. Assume ϕ_A has a nontrivial discrete factor and apply Theorem 4.2 with $N = 1$. We conclude that ϕ_A assumes every value $w \neq 0$ in some neighborhood of 0, contradicting the fact that 0 is a limit point of omitted values of ϕ_A. Therefore, ϕ_A must be a continuous singular inner function.

5. APPLICATIONS

5.1. *Inner Functions in* H^∞. In [8], Herrero conjectured that the collections $\mathscr{F}_a$ and $\mathscr{F}_c$ are closed in H^∞. Using Theorem I, we prove this is not true for $\mathscr{F}_a$.

5.2. THEOREM III. *$\mathscr{F}_a$ is not closed in* H^∞.

Proof: Let $A = \{0, 1/2, 1/3, 1/4, \ldots\} \subset U$. Then $A \in \mathscr{A}$ and 0 is a limit point of A. For $n \geq 2$, let $\alpha_n = 1/n$ and define $f_n = \psi_{\alpha_n} \circ \phi_A$. Each f_n is a singular inner function, and $\|\phi_A - f_n\|_\infty \to 0$ as $n \to \infty$. By Theorem I(b), $\phi_A \in \mathscr{F}_c$, so it suffices to prove that $f_n \in \mathscr{F}_a$, $n \geq 2$. But this follows from Theorem I(a) since, by Corollary 2.5(a), f_n is the uniformizer of $U \setminus \psi_{\alpha_n}(A)$ and 0 is an isolated point of $\psi_{\alpha_n}(A)$.

5.3. Herrero's conjecture that $\mathscr{F}_c = \bar{\mathscr{F}}_c$ remains open. Our results seem to lend support to this conjecture in the following sense. According to Theorem 4.2,

5.4. *If* f *is a singular inner function with a nontrivial discrete factor, then there is a* $\delta > 0$ *such that* $\psi_\alpha \circ f$ *fails to be singular if* $|\alpha| < \delta$.

That is, f is isolated in H^∞ from any singular inner functions of the form $\psi_\alpha \circ f$. Can it be that f is isolated from *all* singular inner functions (except possibly those with a common factor)? For example, if $f \in \mathscr{F}_a$, must there exist an $\varepsilon > 0$ such that $\|f - g\|_\infty \geq \varepsilon$ for every $g \in \mathscr{F}_s$ which is not a unimodular constant times f?

5.5. We can also use Theorem I to answer two questions raised by Caughran and Shields. In [3] they asked whether a discrete singular inner function can omit an uncountable number of values in U. The answer is yes; ϕ_A is such a function whenever $A \in \mathscr{A}$ is an uncountable set with 0 as an isolated point.

Seidel proved (see [4, Theorem 5.13]) that if μ is a singular measure whose closed support in R has an isolated point, then S_μ omits no value in U other than 0 . In an unpublished manuscript Caughran and Shields suggest that this conclusion may hold under the weaker hypothesis that the *discrete part* of μ has a closed support with an isolated point. The following example shows this to be false.

Choose $A \in \mathscr{A}$ to be uncountable with 0 as an isolated point. Then ϕ_A is a discrete singular inner function, and without loss of generality we may assume $\phi_A = S_\mu$ with $\mu\{1\} = 1$. Construct an inner function g so that

(a) g has radial limit 1 at 1;

(b) g has a finite angular derivative at 1; and

(c) g has a finite angular derivative at no other point of T.

Such a function can be constructed, for example, by an easy modification of the method of Lemma 5.9 in [9].

Now consider $S_\nu = S_\mu \circ g$. It was shown in [9, Section 5] that ν can have a point mass only at a point where g has a finite angular derivative. Thus ν has at most one mass point. Since $\mu\{1\} = 1$, S_1 is a factor of S_μ , so by Theorem 2.7, ν does have point mass at 1. We conclude that ν has precisely one point mass, yet $S_\nu = \phi_A \circ g$ omits the uncountable set A.

5.6. In [6] Fisher proved that for $h \in H^\infty$, $\|h\|_\infty \leq 1$, the set of α in U for which $\psi_\alpha \circ h$ has a discrete singular inner factor is at most countable. Let $\{\rho_j\}_{j=1}^\infty$ be the (countable) collection of masses associated with all the discrete singular factors of $\{\psi_\alpha \circ h : \alpha \varepsilon U\}$. In Theorem 2 of [6], Fisher proves that at most a finite number of the masses ρ_j exceed each $\delta > 0$. In a private communication, Fisher asked whether, in fact, $\sum_{j=1}^{\infty} \rho_j < \infty$. We can answer this in the negative. Note first that if μ is a singular measure, then $\mu(T) = -\log|S_\mu(0)|$. Using the set A and the notation from the proof of Theorem III, let $h = \phi_A$. Since each $f_n = \psi_{\alpha_n} \circ h$ is totally discrete, the associated masses sum to $-\log|f_n(0)|$. But as $n \to \infty$, $|f_n(0)| \to |h(0)| \neq 1$, so $\sum_{n=2}^{\infty} - \log|f_n(0)| = \infty$.

6. COMMENTS AND QUESTIONS

6.1. Let μ be a singular measure with nontrivial discrete part. Define $\delta(\mu)$ to be the largest value of δ for which the conclusion of 5.4 above holds with $f = S_\mu$. What can one say about the map $\mu \to \delta(\mu)$?

For instance, what are necessary and sufficient conditions for $\delta(\mu)$ to be less than 1 ? It is necessary that the closed support of μ be a perfect set of positive capacity in T (see Remark 3.2); however, this is far from sufficient. Using the methods of Herrero [8], for example, it can be shown that if μ has a *point* mass at the endpoint of any arc where S_μ is analytic, then $\delta(\mu) = 1$.

When $\delta(\mu) < 1$, its size would seem to depend on characteristics of the discrete part of μ . To be specific, perhaps if we normalize μ in an appropriate way, $\delta(\mu)$ will necessarily be larger than some absolute constant. This would not be too surprising since the existence of $\delta(\mu)$ depends on Theorem II, which is a result of Bloch type.

6.2 We have been concerned primarily with the *omitted* values of inner functions. To what extent do our results carry over to *exceptional* values of inner functions? More generally, the exceptional set E(h) makes sense whenever $\|h\|_\infty \leq 1$. Do our results have analogues in the general case?

Consider this result, for example: If $f \in \mathscr{F}$, then the set

$$\{\alpha \in O(f) : \psi_\alpha \circ f \text{ has a factor in } \mathscr{F}_a\}$$

is at most countable. This follows trivially from our work because this set is discrete, but it also is a special case of Fisher's result quoted in Section 5.6. Does Fisher's result hold because the α's for which $\psi_\alpha \circ h$ has a discrete singular inner factor form a discrete set?

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