INVARIANT SIMPLE CLOSED CURVES ON THE TORUS

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The behavior of the orbits of a continuous flow on the torus is potentially considerably more complicated than that of a continuous flow on a portion of the plane. Specifically, nonperiodic recursion can occur on the torus. However, if there exists a simple closed invariant but not necessarily periodic curve which is not null-homotopic, then this difficulty does not arise, because by cutting along this curve one obtains a flow on a closed annulus in the plane. Our goal is to obtain sufficient conditions for the existence of such a curve.

The main theorems (Theorems 1, 2, and 7) state that such a curve exists if there exists a positive orbit on the torus satisfying the following conditions: (a) its lift to the plane does not deviate too much from a ray with rational slope, and (b) its ω-limit set is locally connected or the fixed points in its ω-limit set are totally disconnected.

An earlier version of Theorem 2 with (b) replaced by the hypothesis that there are only finitely many fixed points in the ω-limit set appears in William O'Toole's master's thesis [4], which the author directed. I wish to thank him for his assistance in organizing and clarifying some of the arguments which are needed for both results.

Let X be a metric space and let $\mathbb{R}$ denote the real numbers. A *continuous flow* on X is a continuous mapping $\pi: X \times \mathbb{R} \to X$ such that $\pi(x,0) = x$ and

$$\pi(\pi(x,s),t) = \pi(x,s+t), \qquad \text{for all } x \in X \text{ and } s, t, \in \mathbb{R}.$$

We will suppress the π and simply write xt for $\pi(x,t)$. The *set of fixed points* of a continuous flow is defined by $F = \{x: xt = x \text{ for all } t \in \mathbb{R}\}$, and the *orbit* and *positive semi-orbit* of a point x in X are defined by

$$\mathcal{O}(x) = \{xt: t \in \mathbb{R}\} \quad \text{and} \quad \mathcal{O}^+(x) = \{xt: 5t \geq 0\}$$

respectively. The ω-*limit* set which is defined by

$$\omega(x) = \bigcap_{t \geq 0} \overline{\mathcal{O}^+(xt)} = \bigcap_{t \geq 0} \{xs: s \geq t\}^-$$

will play an important role in the sequel.

Let $(\tilde{X}, p)$ be a covering space of X. Given a continuous flow π on X, there exists a unique continuous flow $\tilde{\pi}$ on $\tilde{X}$ such that

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$$p(xt) = p(x)t, \qquad \text{for all } x \in \tilde{X} \text{ and } t \in \mathbb{R}.$$

Moreover, x is a fixed point of $\tilde{\pi}$ if and only if p(x) is a fixed point of π, and for any covering transformation T we have $T(xt) = T(x)t$ for all $x \in \tilde{X}$ and $t \in \mathbb{R}$. Given a point in X, say w, we will only use the notation $\tilde{w}$, $\tilde{w}_1$, and $\tilde{w}_2$ to denote points in $p^{-1}(w)$.

Suppose π is a continuous flow on a two-manifold X. A *local cross section* of π at $x \in X$ is a subset S of X containing x which is homeomorphic to [0,1] and for which there exists an $\varepsilon > 0$, called the length of the local cross section, such that the map $(x,t) \to xt$ is a homeomorphism of $S \times [-\varepsilon, \varepsilon]$ onto the closure of an open neighborhood of x. In fact we can and will assume that

$$S_0(-\varepsilon, \varepsilon) = \{xt\colon x \in S_0 \text{ and } t \in (-\varepsilon, \varepsilon)\}$$

is open, where S_0 denotes S with its end points removed. If $x \notin F$, then there exists a local cross section of π at x [5].

For simplicity we will use the following notation for segments of curves and orbits. If S is a simple curve in some space X with a and b lying on S, then $(a,b)_S$ and $[a,b]_S$ will denote the open and closed segments of S between a and b. If π is a continuous flow on X, $x \in X$, and $s, t \in \mathbb{R}$ with $s < t$, then $(xs, xt)_\pi$ and $[xs, xt]_\pi$ will denote $\{x\tau\colon \tau \in (s,t)\}$ and $\{x\tau\colon \tau \in [s,t]\}$, respectively.

Let $\mathscr{T}$ denote the torus and let $(\mathbb{R}^2, p)$ be the usual representation of its universal covering space; i.e., the covering transformations are of the form

$$T(x,y) = (x + m, y + n)$$

where m and n are arbitrary integers. This group of covering transformations will be denoted by $\mathfrak{G}$. When we are working with a continuous flow on $\mathscr{T}$, notation like xt, $\mathscr{O}(x)$, etc., for $x \in \mathbb{R}^2$ will always refer to the lifted flow.

One source of continuous flows on $\mathscr{T}$ is a system of differential equations

$$\text{(E)} \qquad \frac{dx}{dt} = \Phi_1(x,y); \quad \frac{dy}{dt} = \Phi_2(x,y),$$

where Φ_1 and Φ_2 are continuously differentiable and $\Phi_i(x + n, y + m) = \Phi_i(x,y)$ for any pair of integers n and m. In this case the orbits of the lifted flow are the solutions of (E) in $\mathbb{R}^2$ and our theorems can be applied by examining these solutions. When $\Phi_1(x,y) \equiv 1$ we have the classical case considered by Poincaré, and Theorem 7 generalizes his result that a periodic orbit occurs on the torus if the rotation number is rational.

The idea that a curve does not deviate too much from a ray will have two meanings. The first, which is defined below, is geometrical and the second, which appears in Theorem 7, is analytical. Let $\beta\colon [a,\infty) \to \mathbb{R}^2$ be a simple curve. We say that β is of the *type of a ray* if $\lim_{t\to\infty} |\beta(t)| = \infty$, where $|\ |$ is the usual norm on $\mathbb{R}^2$, and if there exists a pair of parallel lines L_1 and L_2 such that $\beta(t)$ lies between L_1 and L_2 for all t. When in addition to these conditions the slope of L_1 is rational, we say β is of the *type of a rational ray.* Given a continuous flow on $\mathcal{T}$, we will say that $\mathcal{O}^+(w)$ is of the type of a ray if the curve $t \to \tilde{w}t$ for $t \geq 0$ is of the type of a ray. It is easy to check that if $\mathcal{O}^+(w)$ is of the type of a rational ray, then for every $x \in \mathbb{R}^2$, $\mathcal{O}(x)$ lies between two parallel lines.

THEOREM 1. *Let π be a continuous flow on $\mathcal{T}$ and let $w \in \mathcal{T}$. If $\mathcal{O}^+(w)$ is of the type of a rational ray and $\omega(w)$ is locally connected, then $\omega(w)$ contains a simple closed invariant curve which is not null-homotopic.*

Proof. Since the ω-limit sets of a continuous flow on a compact Hausdorff space are compact and connected, we know that $\omega(w)$ is arcwise connected. Let $\tilde{\omega}(w)$ be an arc component of $p^{-1}(\omega(w))$, and note that $p(\tilde{\omega}(w)) = \omega(w)$.

Next, arguing by contradiction, we will show that p is not one-to-one on $\tilde{\omega}(w)$. By the Hahn-Mazurkiewicz theorem there exists α: $[0,1] \to \omega(w)$ which is onto. Let $\tilde{\alpha}$ be a lift of α lying in $\tilde{\omega}(w)$, and observe that $\tilde{\alpha}$ must map $[0,1]$ onto $\tilde{\omega}(w)$ if p is one-to-one on $\tilde{\omega}(w)$. Thus $\tilde{\omega}(w)$ is compact. Now it is easy to see that there exists $\varepsilon > 0$ such that

$$\{x \in \mathbb{R}^2 : d(x,\tilde{\omega}(w)) \leq \varepsilon\} \cap p^{-1}(\omega(w)) = \tilde{\omega}(w).$$

Because $\lim_{t\to\infty} |\tilde{w}t| = \infty$ for every $\tilde{w} \in p^{-1}(w)$, there exist sequences

$$\{\tilde{w}_n\}_{n=1}^{\infty} \subset p^{-1}(w) \quad \text{and} \quad \{t_n\}_{n=1}^{\infty} \subset \mathbb{R}$$

such that $d(\tilde{w}_n t_n, \tilde{\omega}(w)) = \varepsilon$, $t_n \to \infty$, and $w_n t_n \to \tilde{z}$ as $n \to \infty$. By our choice of ε, $p(\tilde{z}) \notin \omega(w)$; and because $wt_n \to p(\tilde{z})$ and $t_n \to \infty$ as $n \to \infty$, we must have $p(\tilde{z}) \in \omega(w)$. This contradiction establishes our claim that p is not one-to-one on $\omega(\tilde{w})$. We can push this a little further by noting that $T(\tilde{\omega}(w)) \cap \tilde{\omega}(w) \neq \emptyset$ implies $T(\tilde{\omega}(w)) = \tilde{\omega}(w)$ and hence $p^{-1}(z) \cap \tilde{\omega}(w)$ contains more than one point for all $z \in \omega(w)$.

Since $\omega(x)$ is a compact invariant set, it must contain a minimal set. By Theorem 3.3 in [3] this minimal set can not contain an almost periodic point which is not periodic, because $\mathcal{O}^+(w)$ is of the type of a rational ray. Therefore, $\omega(w)$ contains a fixed point or a periodic orbit. Using local cross sections, it is easy to check that a periodic orbit in $\omega(x)$ can not be null-homotopic, because $\lim_{t\to\infty} \tilde{w}t = \infty$. From now on we will assume that $\omega(w) \cap F \neq \emptyset$.

Let $a \in F \cap \omega(w)$, $\tilde{a}_1, \tilde{a}_2 \in \tilde{\omega}(w)$, and let β be an arc in $\tilde{\omega}(w)$ joining $\tilde{a}_1$ and $\tilde{a}_2$. We will show that the range of β is an invariant set; that is, if x is in the range of β, then so is $\mathcal{O}(x)$. We may as well assume x is not a fixed point. Let S be a local cross section of length ε at x. Consider $\tilde{\omega}(w) \cap S$ as a subset of S. If it has interior, then w must be positively recurrent, which is impossible by Theorem 3.3 in [3]. Hence there exists $\delta > 0$ such that $|t - t_0| < \delta$ implies $\beta(t) \in (x(-\varepsilon), x\varepsilon)_\pi$, where $\beta(t_0) = x$. Writing $\beta(t) = x\rho(t)$ with $|\rho(t)| < \varepsilon$ for $t \in (t_0 - \delta, t_0 + \delta)$ defines a continuous one-to-one function from $(t_0 - \delta, t_0 + \delta)$ into $(-\varepsilon, \varepsilon)$. Since $\rho(t_0) = 0$, the range of ρ must be an open interval containing 0, and we have established that $\{xt : xt \in \beta([0,1])\}$ is an open subset of $\mathcal{O}(x)$.

This set is clearly a closed subset of $\mathcal{O}(x)$, which implies $\mathcal{O}(x) \subset \beta([0,1])$ by the connectedness of $\mathcal{O}(x)$.

If $p \circ \beta$ is a simple curve on $\mathcal{T}$, then we are finished. If not, we must extract such a curve from it. To do this, let

$$A = \{\sigma \in [0,1]: \text{there exists } \sigma' \in (\sigma, 1] \text{ and } T \in \mathfrak{G} \text{ such that } T(\beta(\sigma)) = \beta(\sigma')\}.$$

It is straightforward to show that A is a nonempty closed subset of [0, 1] not containing 1. Let $\sigma_0 = \sup A \in A$ and let $\gamma = \beta | [\sigma_0, \sigma_0']$ where $T(\beta(\sigma_0)) = \beta(\sigma_0')$ for some $T \in \mathfrak{G}$ and $\sigma_0' \in (\sigma_0, 1]$. Clearly $p \circ \gamma$ is a simple closed curve on $\mathcal{T}$ which is not null-homotopic. To prove that $p \circ \gamma$ is invariant it suffices to show that $\beta(\sigma_0)$ is a fixed point, and this is a consequence of the relations

$$\mathcal{O}(\beta(\sigma_0)) \subset \beta([0,1]), \ \mathcal{O}(\beta(\sigma_0')) \subset \beta([0,1]), \text{ and } T(\mathcal{O}(\beta(\sigma_0))) = \mathcal{O}(\beta(\sigma_0')),$$

which we already know.

THEOREM 2. *Let π be a continuous flow on $\mathcal{T}$ and let $w \in \mathcal{T}$. If $\mathcal{O}^+(w)$ is of the type of a rational ray and $\omega(w) \cap F$ is totally disconnected, then $\omega(w)$ contains a simple closed invariant curve which is not null-homotopic.*

Proof. Suppose $\omega(w) \subset F$. Since $\omega(w)$ is connected and $\omega(w) \cap F$ is totally disconnected, $\omega(w) = \{z_0\}$ and $\{wt: t \geq 0\} \cup \{z_0\}$ is an arc in $\mathcal{T}$ which has a bounded lift to $\mathbb{R}^2$. This is impossible because we have assumed that $\lim_{t\to\infty} |\tilde{w}t| = \infty$; so there exists $z \in \omega(w) \setminus F$. Let S be a local cross section at z and consider some $\tilde{w}$. There exist parallel rational lines L_1 and L_2 such that $\mathcal{O}^+(\tilde{w})$ lies between L_1 and L_2. We will say that two lifts $\tilde{S}_1$ and $\tilde{S}_2$ of S are *equivalent* if $T(\tilde{S}_1) = \tilde{S}_2$ implies $T(L_1) = L_1$, $T \in \mathfrak{G}$. Clearly the lifts of S lying between L_1 and L_2 belong to only a finite number of equivalence classes. Because $z \in \omega(w)$ and $\lim_{t\to\infty} |\tilde{w}t| = \infty$, we know that $\mathcal{O}^+(\tilde{w})$ must cross infinitely many lifts of S lying between L_1 and L_2, and hence two equivalent ones. In particular, we can choose $0 < t_1 < t_2$ and equivalent lifts $\tilde{S}_1$ and $\tilde{S}_2$ of S such that $\tilde{w}t_i \in \tilde{S}_i$ and $\{\tilde{w}t: t_1 < t < t_2\}$ does not cross any lifts of S which are equivalent to $\tilde{S}_1$. Let

$$\tilde{\gamma}_u = \bigcup_{n=-\infty}^{\infty} T^n\{[\tilde{w}t_1, \tilde{w}t_2]_\pi \cup (T(\tilde{w}t_1), \tilde{w}t_2)_{\tilde{S}_2}\},$$

where T is the covering transformation carrying $\tilde{S}_1$ onto $\tilde{S}_2$. Then as in [3, Lemma 3.2], $\tilde{\gamma}_u$ divides the plane into two regions—one positively invariant and the other negatively. Moreover, $\tilde{\gamma}_u$ is "parallel" to L_1. (This curve $\tilde{\gamma}_u$ is not quite what I previously called a control curve in [3] because we do not know whether or not its projection on the torus is simple. For the present argument the direction of $\tilde{\gamma}_u$ is the crucial thing.) Since $\mathcal{O}^+(\tilde{w})$ lies between L_1 and L_2, it can intersect only a finite number of distinct translates of $\tilde{\gamma}_u$ by covering transformations, and it can intersect such a curve only once. Therefore, $\omega(w) \cap (wt_1, wt_2)_S = \emptyset$, and from this we readily have the following:

There exist real numbers $0 < s_1 < s_2$ such that $ws_1, ws_2 \in S$;

$$J = [ws_1, ws_2]_\pi \cup (ws_1, ws_2)_S$$

is a simple closed curve which is not null-homotopic; and $\omega(w) \cap (ws_1, ws_2)_S = \emptyset$.

Using the technique in [1], we can construct a new flow with the new orbits contained in the old ones and with $F \cup J$ as the set of fixed points. For the moment it is necessary to let π and π' denote the old and new flows, respectively, and to use notation like $\omega(x, \pi)$, $\mathcal{O}^+(x, \pi')$, etc. Choose τ such that $\mathcal{O}^+(w\tau, \pi) \cap J = \emptyset$ and let $w' = w\tau$. Clearly $\omega(w, \pi) = \omega(w', \pi')$. If we cut $\mathcal{T}$ along J we obtain a flow on a closed annulus which can easily be extended to the whole plane because the boundary consists of fixed points. In view of Theorem 1 it now suffices to prove the following theorem:

THEOREM 3. *Let π be a continuous flow on an open subset* W *of* $\mathbb{R}^2$ *and let* $w \in W$. *If* $\mathcal{O}^+(w)$ *is bounded and* $\omega(w) \cap F$ *is totally disconnected, then* $\omega(w)$ *is locally connected.*

Proof. If $x \in \omega(x)$, then from the Poincaré-Bendixson theorem for flows in the plane we know that x is periodic or fixed, and in either case the theorem is trivially true. Similarly, if $\omega(x) \subset F$, then $\omega(x)$ consists of a single fixed point and there is nothing to prove. We will now assume that $y \in \omega(x)$, $x \notin \omega(x)$, and $\omega(x) \neq \{y\}$. It suffices to show that $\omega(x)$ is connected im kleinen at y [2, Theorem 3-11, page 114]. Let $D = \{z: |z - y| \leq r\}$ and let $D' = \{z: |z - y| \leq r/2\}$ where r is any positive real number less than half the diameter of $\omega(x)$. We will show that only a finite number of the components of $\omega(x) \cap D$ meet D' from which it follows that $\omega(x)$ is connected im kleinen at y because these components are compact.

Given $y' \in \omega(x) \cap D'$, we will construct a compact connected subset C of $\omega(x)$ which contains y', meets ∂D, and can be nicely approximated by arcs in $\mathcal{O}^+(x) \cap D$. There exists a sequence $\{t_n\}$ in $\mathbb{R}$ such that $t_n \to \infty$ and $xt_n \to y'$. For each t_n, pick $\tau_n < t_n < \sigma_n$ such that $x\tau_n, x\sigma_n \in \partial D$ and $|y - xt| < r$ for all $t \in (\tau_n, \sigma_n)$. We can assume that $(\tau_n, \sigma_n) \cap (\tau_m, \sigma_m) = \emptyset$ for $n \neq m$. By taking subsequences we can also assume that $[x\tau_n, x\sigma_n]_\pi \to C$ in the Hausdorff metric. It follows that C is a compact connected subset contained in $\omega(x) \cap D$ which meets both ∂D and D'. Suppose $y'' \in C$ and $[y''\alpha, y''\beta]_\pi \subset D^0 = \{z: |z - y| < r\}$, where $\alpha < 0 < \beta$. Using the continuity of π, it can readily be shown that

$$[y''\alpha, y''\beta]_\pi \subset C.$$

Now suppose infinitely many components of $\omega(x) \cap D$ meet D'. We can then use the preceding construction to obtain a sequence of disjoint compact connected subsets $\{C_n\}$ in $\omega(x) \in D$ which meet D' and ∂D and which have the above invariance property. By taking a subsequence we can assume that $C_n \to C'$ in the Hausdorff metric. Clearly C' is not a point. Since C' is also a compact connected subset of $\omega(x)$, $C' \not\subset F$. Let $y'' \in C' \setminus F$ and let S be a local cross section of length ε at y'' such that $S_0[-\varepsilon, \varepsilon] \cap \omega(x) = [y''(-\varepsilon), y''\varepsilon]_\pi \subset D^0$. For large n we must have $C_n \cap S_0(-\varepsilon, \varepsilon) \neq \emptyset$ and hence $[y''(-\varepsilon), y''\varepsilon]_\pi \subset C_n$, which contradicts the disjointness of the C_n's and completes the proof.

For the next two results we will need the following notation: If a and b are distinct points in $\mathbb{R}^2$, then $\mathscr{L}(a, b)$ and $\mathscr{LS}(a, b)$ will denote the line and line segment, respectively, determined by a and b.

LEMMA 4. *Let* α: $[0,\infty) \to \mathbb{R}^2$ *be a simple curve such that* $|\alpha(t)| \to \infty$ *as* $t \to \infty$. *If there exists a constant* $D > 0$ *such that for any* $[a,b] \subset \mathbb{R}$,

$$d(\alpha(t), \mathscr{L}(\alpha(a), \alpha(b)) < D, \qquad \textit{for all } t \in [a,b],$$

then α *is of the type of a ray.*

Proof. Let $L_n = \mathscr{L}(\alpha(0), \alpha(n))$, let θ_n be the angle L_n makes with the positive x-axis. Choose a convergent subsequence $\{\theta_{n_k}\}$ with its limit denoted by θ. Let L be the line through $\alpha(0)$ with slope equal to $\tan\theta$, and suppose

$$d(\alpha(t_0), L) > 3D.$$

Let ϕ_k be the acute angle between L and L_{n_k}. Clearly, $\phi_k \to 0$ as $k \to \infty$. Let $L^\perp$ be the line through $\alpha(t_0)$ perpendicular to L, and let $x_k = L^\perp \cap L_{n_k}$. Clearly,

$$x_k \to x = L \cap L^\perp, \qquad \text{as } k \to \infty.$$

Choose k so that $n_k > |t_0|$, $|x_k - \alpha(t_0)| > 2D$, and $\cos\phi_k > 1/2$. Then

$$d(\alpha(t_0), L_{n_k}) = \cos\phi_k\, d(x_k, \alpha(t_0)) > D.$$

This is a contradiction because $|t_0| < n_k$ implies that $d(\alpha(t_0), L_{n_k}) < D$. Therefore, $\alpha(t)$ lies between the two lines parallel to L and at a distance of 3D from L.

THEOREM 5. *Let* π *be a continuous flow on* $\mathscr{T}$ *and let* $w \in \mathscr{T}$. *If* $\omega(w) \not\subset F$ *and* $|\tilde{w}t| \to \infty$ *as* $t \to \infty$, *then* $\mathscr{O}^+(w)$ *is of the type of a ray.*

Proof. The conclusion certainly holds if w is periodic, and we will assume throughout the rest of the proof that w is not periodic. Let $z \in \omega(w) \setminus F$ and let S be a local cross section at z. Because $|\tilde{w}t| \to \infty$ as $t \to \infty$ we can find $0 < t_1 < t_2$ such that $\gamma = [wt_1, wt_2]_\pi \cup (wt_1, wt_2)_S$ is a simple closed curve which is not null-homotopic. Let $\{\tilde{\gamma}_n\}_{n=-\infty}^{\infty}$ be the universal lifts of γ; that is, each $\tilde{\gamma}_n$ is arc component of $p^{-1}(\gamma)$ homeomorphic to $\mathbb{R}$ which divides $\mathbb{R}^2$ into two open connected sets. Moreover, $\mathscr{O}^+(x)$ can cross a given $\tilde{\gamma}_n$ at most once [3, Lemma 3.2].

We will now assume that $\mathscr{O}^+(\tilde{w})$ is not of the type of a ray. Then $\mathscr{O}^+(wt_2)$ can not lie between any two $\tilde{\gamma}_n$'s, and we can find $t_4 > t_3 > t_2$ such that $\tilde{w}t_3$ and $\tilde{w}t_4$ lie on adjacent $\tilde{\gamma}_n$'s. Now $\{\tilde{\gamma}_n\}_{n=-\infty}^{\infty}$ and $\{T([\tilde{w}t_3, \tilde{w}t_4]_{\tilde{\pi}}): T \in \mathfrak{G}\}$ divide the plane into bounded congruent regions which an orbit can leave only by crossing some $\tilde{\gamma}_n$. In particular, if r is the common diameter of these regions and $|x - xt| > qr$, then $(x, xt)_\pi$ crosses q different $\tilde{\gamma}_n$'s.

By assuming that $\mathscr{O}^+(w)$ is not of type of a ray, we know from Lemma 4 that we can find sequences $\{a_i\}_{i=1}^{\infty} \subset \mathscr{O}^+(\tilde{w})$ and $\{t_i\}_{i=1}^{\infty} \subset \mathbb{R}$ such that $t_i > 0$ and $\sup\{d(a_i t, \mathscr{L}(a_i, a_i t_i)): 0 \le t \le t_i\} > i$. Moreover, these sequences can be chosen so that $\mathscr{L}(a_i, a_i t_i) \cap (a_i, a_i t_i)_{\tilde{\pi}} = \emptyset$. It follows that

$$J_i = \mathscr{B}(a_i, a_i t_i) \cup (a_i, a_i t_i)_{\tilde{\pi}}$$

is a simple closed curve and contains no points z in its interior such that $p(z) = p(a_i)$ or $p(z) = p(a_i t_i)$ [3, Lemma 2.2].

It is entirely possible that $|a_i - a_i t_i| \to \infty$ as $i \to \infty$, but the following argument will show that this can be avoided. Fix i and let s_i be the first time in $[0, t_i]$ that $d(a_i t, \mathscr{L}(a_i, a_i t_i))$ achieves its maximum. Let L′ be the line parallel to L on the same side as $a_i s_i$ with $d(L, L') = 2$, let L_1 be perpendicular to L at a_i, let H be the half-plane determined by L_1 containing $a_i t_i$, and let

$$B_i = \{a_i t : t \in [0, s_i], d(a_i t, L) \leq 2, \text{ and } a_i t \in H\}.$$

There exists a $b_i \in B_i$ such that $d(b_i, L_1) = \max_{z \in B_i} d(z, L_1)$ and there exist lines L_2 and L_3 in H parallel to L_1 such that $d(L_1, L_2) = d(b_i, L_1)$ and

$$d(L_1, L_3) = d(b_i, L_1) + 2.$$

Now L, L', L_2 and L_3 form a square which contains a fundamental region and thus cannot be contained inside J_i. Therefore, there exists $\tau_i \in [s_i, t_i]$ such that $a_i \tau_i = b_i \sigma_i$ lies in this square region. Set $A_i = \mathscr{B}(b_i, b_i \sigma_i) \cup (b_i, b_i \sigma_i)_{\tilde{\pi}}$ and note that $|b_i - b_i \sigma_i| \leq 2$ and the diameter of A_i goes to infinity as $i \to \infty$. Let b_i' be the unique point in $[0,1) \times [0,1)$ such that $p(b_i) = p(b_i')$. By picking a subsequence we can assume that $b_i' \to c$ and $b_i' \sigma_i \to c'$.

We can now assemble the contradiction that will complete the proof. We can find $\tilde{\gamma}_{n'}$ and $\tilde{\gamma}_{n''}$ such that c and c′ lie between them. Let q be the number of $\tilde{\gamma}_n$'s which lie between $\tilde{\gamma}_{n'}$ and $\tilde{\gamma}_{n''}$, and choose I such that $|b_i - a_i s_i| > (q + 2)r$ when $i > I$. Therefore, $(b_i, a_i s_i)_{\tilde{\pi}}$ must cross at least $q + 2$ different $\tilde{\gamma}_n$'s and consequently leave the region between $\tilde{\gamma}_{n'}$ and $\tilde{\gamma}_{n''}$, never to return again. Since $a_i s_i \in (b_i, b_i \sigma_i)_{\tilde{\pi}}$, we get the contradiction that $b_i \sigma_i$ does not converge to c′.

COROLLARY 6. *Let* π *be a continuous flow on* $\mathscr{T}$ *and let* $w \in \mathscr{T}$. *If* w *is positively recurrent and not periodic, then* $\mathscr{O}^+(w)$ *is of the type of an irrational ray.*

Proof. Use Theorem 5 and [3, Theorem 3.3 and Corollary 3.5].

THEOREM 7. *Let* π *be a continuous flow on* $\mathscr{T}$, $w \in \mathscr{T}$, $\tilde{w} \in p^{-1}(w)$, *and*

$$\tilde{w} t = (x(t), y(t)).$$

Assume that $|\tilde{w} t| = \{(x(t))^2 + (y(t))^2\}^{1/2} \to \infty$ *as* $t \to \infty$.

(a) *The* $\lim_{t \to \infty} \dfrac{y(t)}{x(t)}$ *exists as an extended real number.*

(b) *If* $\lim_{t \to \infty} \dfrac{y(t)}{x(t)}$ *is rational or* $\pm\infty$ *and either* $\omega(w)$ *is locally connected or* $\omega(w) \cap F$ *is totally disconnected, then there exists a simple closed invariant curve which is not null-homotopic.*

Proof. (a). This follows from the existence of $\lim_{t\to\infty} \frac{\tilde{w}t}{|\tilde{w}t|}$ [3, Theorem 2.1].

(b). Use Theorem 5 and either Theorem 1 or 2.

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