A CLASS OF PURE SUBNORMAL OPERATORS

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Preliminaries. If T is a bounded operator on a Hilbert space $\mathscr{H}$, its spectrum and point spectrum will be denoted by $\sigma(T)$ and $\sigma_p(T)$, respectively. An operator S on $\mathscr{H}$ is *subnormal* if there exists a normal operator N on a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ such that N agrees with S on $\mathscr{H}$. That is, N leaves $\mathscr{H}$ invariant and N restricted to $\mathscr{H}$ is S. N is called the *minimal normal extension* of S if $\mathscr{K}$ is the smallest reducing subspace for N containing $\mathscr{H}$. S is said to be *pure subnormal* if there exists no nontrivial subspace of $\mathscr{H}$ which reduces S and on which S is normal. It is easy to see that the point spectrum of a pure subnormal operator is empty. For further properties of subnormal operators, consult P. Halmos [6] and J. Bram [1].

If K is a compact set in the plane, C(K) denotes the set of (complex-valued) continuous functions on K and R(K) represents the uniform closure of the set of rational functions with poles off K. If F is a closed subset of K and there exists a function f in R(K) such that $f(z) = 1$ for z in F and $|f(z)| < 1$ for z in $K \backslash F$, then F is called a *peak set* for R(K). A peak set consisting of a single point is called a *peak point.* For a detailed exposition of these ideas, consult T. Gamelin [5].

Let S be a pure subnormal operator with minimal normal extension N. If z is any boundary point of a component of the complement of $\sigma(S)$, then $\sigma_p(N) \cap \{z\} = \emptyset$. (See [4, p. 34].) This was generalized by C. Putnam [9, p. 9] in the following way: If Ω denotes the set of peak points of $R(\sigma(S))$ then $\Omega \cap \sigma_p(N) = \emptyset$. For another related result, consult M. Radjabalipour [10, p. 388].

It was asked in both papers, [4, p. 95] and [9, p. 10], whether a stronger result is true. That is, if S and N are as above, is it true that $\sigma_p(N) \cap \partial\sigma(S) = \emptyset$? (Here ∂ denotes the boundary.) The purpose of this paper is to provide a large class of pure subnormal operators S for which $\sigma_p(N) \cap \partial\sigma(S) \neq \emptyset$.

We close this section by mentioning the well-known fact that the minimal normal extension of a pure subnormal operator can have a nonempty point spectrum. The first example of this is due to J. Wermer. He exhibited [12, Theorems 1 and 2] a pure subnormal operator S such that eigenvectors of N span $\mathscr{K}$. However, the eigenvalues of N in this example all lie in the interior of $\sigma(S)$. Another example of this phenomenon was given by D. Sarason. (See [6], Problem 156 and its solution.) The methods of this paper will provide another proof to Sarason's example.

THEOREM. *Let* K *be a compact set in the plane which contains a point* x *which is not a peak point for* R(K). *Then there exists a pure subnormal operator* S *with minimal normal extension* N *such that*

(i) $\sigma(S) \subset K$,

and

(ii) $\sigma_p(N) \cap \sigma(S) \supset \{x\}$.

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Proof. Without loss of generality one can assume that $x = 0$. Choose a representing measure μ at 0 for $R(K)$ that is not the point-mass measure [13, p. 5] and such that the support of μ is on the boundary of K. Let δ_0 denote the point mass at 0, and set $\lambda = \mu + \delta_0$. Let $R^2(\lambda)$ denote the $L^2(\lambda)$ closure of $R(K)$. If S is the subnormal operator defined by multiplication by z on $R^2(\lambda)$, it is easy to see that N, multiplication by z on $L^2(\lambda)$, is its minimal normal extension. We shall show that these operators satisfy the conclusion of the theorem.

It is easy to see that $\sigma(S) \subset K$ and $0 \in \sigma_p(N)$. Let $R^2(\mu)$ be the $L^2(\mu)$ closure of $R(K)$, and let S_1 be the operator of multiplication by z on $R^2(\mu)$.

We first show that S is similar to S_1. Define $T: R^2(\mu + \delta_0) \to R^2(\mu)$ by inclusion. Now observe that for any $g \in R^2(\lambda)$ we have ${}_{\mu}\|Tg\|_2^2 = \int_{\partial K} |g|^2 d\mu$ and ${}_{\lambda}\|g\|_2^2 = \int_{\partial K} |g|^2 d\mu + |g(0)|^2$. (Here ${}_{\beta}\|f\|_2$ denotes the $L^2(\beta)$ norm of f.)

Let $g \in R^2(\lambda)$ and choose a sequence $\{g_n\}$ of functions in $R(K)$ such that $g_n \to g$ in $L^2(\lambda)$ norm. By dropping to a subsequence if need be, we may assume $g_n \to g$ almost everywhere (λ). Since μ is a representing measure at 0 for $R(K)$, and L^2 convergence implies L^1 convergence, it follows that

$$g(0) = \int_{\partial K} g d\mu . \tag{1}$$

Some simple computations now show that T is an invertible operator between $R^2(\lambda)$ and $R^2(\mu)$, and it induces a similarity between S and S_1.

Remark. Notice that the above arguments show the existence of a bounded ($L^2(\mu)$ norm) evaluation at 0 for functions in $R^2(\mu)$.

J. Stampfli [11] has shown that every subnormal operator similar to a normal operator must be normal. (In fact, he has shown that every spectral subnormal operator is normal.) Hence, similarity preserves purity. Combining this with the fact that similarity also preserves spectrum, it is sufficient to show S_1 is pure and $0 \in \sigma(S_1)$.

Let $\mathcal{M}$ be a reducing subspace of S_1 such that the restriction of S_1 to $\mathcal{M}$ is normal, and let $m \in \mathcal{M}$. Clearly, then, $\bar{z}m$ and zm belong to $\mathcal{M}$. Hence for any continuous function g on ∂K, it follows that $gm \in \mathcal{M}$. By (1) it now follows that $\int gzm d\mu = 0$. Since the continuous functions are dense in $L^2(\mu)$, it follows that $zm = 0$. Therefore, $m = 0$ almost everywhere (μ). This establishes the fact that S_1 is pure.

The only thing left to show is that $0 \in \sigma(S_1)$. This follows because S_1 fails to have dense range; observe that 1 is orthogonal to the closed span of $zR^2(\mu)$. (See the remark in the beginning of the proof.)

Before commenting on the applications of this theorem, the author would like to mention that some of the ideas used in the proof were strongly motivated by Sarason's example [6, solution to problem 156] and by the paper of K. Clancey and C. Putnam [2].

For one application of the theorem above, take K to be the closed unit disk and let $x = 0$. Letting μ be normalized Lebesgue measure on the boundary of the disk, and following the proof above, we obtain Sarason's example.

Now let $\overline{\Delta}$ denote the closed unit disk. For $n = 1, 2, \cdots$, let Δ_n denote the open disk with center at $3/2^{n+1}$ and radius $r_n = 1/2^{4n}$. Setting $K = \overline{\Delta} \setminus \left(\bigcup_{n=1}^{\infty} \Delta_n\right)$ and applying Melnikov's criterion [5, p. 205], we see that 0 is not a peak point of R(K). Therefore, we can find a pure subnormal operator S with minimal normal extension N such that $\sigma_p(N) \cap \partial(\sigma(S)) \neq \emptyset$. (If one follows the proof of the theorem, one sees that $\sigma(N) \subset \partial K$. Since $R(\partial K) = C(\partial K)$ and $\sigma(S) \subset K$, it must be the case that $\sigma(S) = K$. (See [7], Corollary 2, p. 73.))

ADDENDUM

M. Radjabalipour has (independently) discovered the theorem above. I would like to thank him for supplying the following corollary and remark.

COROLLARY. *Let* K *be a compact set in the plane with the following property: if* D *is any open disk with* $K \cap D \neq \emptyset$, *then* $R(K \cap \overline{D}) \neq C(K \cap \overline{D})$. *Then there exists a pure subnormal operator* S *defined on a separable Hilbert space such that* $\sigma(S) = K$ *and the eigenvalues of the minimal normal extension of* S *are dense in* K.

Proof. The proof follows immediately from the theorem and a simple application of Bishop's theorem [5, p. 54].

Remark. If $R(K) \neq C(K)$ and P is the set of peak points of R(K), then by Bishop's theorem $K \setminus P$ has positive area. Therefore, in the corollary, the set of eigenvalues of N cannot contain P. However, if we allow $\mathscr{H}$ to be nonseparable, then this is possible.

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