A SELECTOR PRINCIPLE FOR Σ_1^1 EQUIVALENCE RELATIONS

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Let $J = {}^{\omega}2$, the space of functions from the set ω of natural numbers to $2 = \{0, 1\}$ with its topology as a countable product of two-point discrete spaces. We assume familiarity with the hierarchy of Σ_n^1, Π_n^1 and Δ_n^1 subsets of J and its finite Cartesian powers J^k. (See *e.g.*, [13], Chapters 14-16.) We will be interested in equivalence relations E on J which are Σ_1^1 as subsets of J^2. A *selector set* or *transversal* for an equivalence E on J is a subset $S \subseteq J$ containing exactly one element of each E-equivalence class. Our goal is to determine the set-theoretic strength of the following Selector Principle:

(*) Every Σ_1^1 equivalence relation on J has a Δ_2^1 selector.

(Let us note right away that any Σ_2^1 selector S for a Σ_1^1 (or even Σ_2^1) equivalence E is automatically Π_2^1 and hence Δ_2^1, since

$$S = \{x: \neg\exists y(y \in S \ \&\ xEy \ \&\ x \neq y)\}.$$

Thus in (*) we could have written Σ_2^1 for Δ_2^1 without affecting the strength of the principle.)

It has long been known that (*) is consistent with, but independent of, the usual axioms (ZFC) of set theory. Work of D. Myers [12] provides more detailed information. The main contribution of the present paper is as follows: *It is well known that if every real is constructible, then every Σ_1^1 equivalence relation on the reals has a Σ_2^1 selector; the converse is not provable in* ZFC. This result was announced in [2].

In our work we make use of the following Ramsey-style theorem of Galvin: *Let the set* $[J]^2$ *of two-element subsets of* J *be partitioned into finitely many pieces in a nice enough way* (so that for each piece A the corresponding subset

$$\{(x, y): \{x, y\} \in A\}$$

of J^2 has the Baire property). *Then there is a perfect subset* P *of* J *such that all two-element subsets of* P *belong to the same piece of the partition.* Galvin's result was announced in [3] and [4]. Overlooking his work, we rediscovered it and announced it in [2]. Since no proof has so far been published, with Prof. Galvin's kind permission we are including one here.

Section 1 of the present paper contains a survey of known results concerning the status of (*). Section 2 contains a proof of the partition theorem mentioned above, and Section 3 a proof of Con(ZFC + (*) + $\neg$(Every real is constructible)).

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1. SURVEY OF KNOWN RESULTS

Let us begin by collecting some examples of Σ^1_1 equivalences on J.

1.1 *The Vitali Equivalence.* Vitali showed that the equivalence on the real numbers obtained by setting two reals equivalent if their difference is rational has no Lebesgue measurable selector. (See any real analysis text for the proof.) Vitali's example can be adapted to J: Fix a reasonable enumeration $(u_i: i \in \omega)$ of the finite subsets of ω. For $i \in \omega$, let h_i be the autohomeomorphism of J given by:

$$(h_i(x))(n) = \begin{cases} x(n) & \text{if } n \notin u_i, \\ 1 - x(n) & \text{if } n \in u_i. \end{cases}$$

Define an equivalence by

$$xE_V y \leftrightarrow \exists i \in \omega\ h_i(x) = y \leftrightarrow \exists m \in \omega\ \forall n > m\ x(n) = y(n).$$

Clearly E_V is Borel, indeed Σ^0_2, and every E_V class is countable. Lebesgue measure μ on J is so defined that if $s \in {}^n2$ is a finite sequence of 0's and 1's and $J_s = \{x \in J: x \text{ extends } s\}$ then $\mu J_s = 2^{-n}$. (In particular, $\mu J = 1$.) Vitali's argument shows: *No selector for* E_V *is Lebesgue measurable.* An entirely similar argument shows: *No selector for* E_V *has the property of Baire.*

1.2 *The Coding Equivalence.* Let $\pi: \omega \times \omega \to \omega$ be the bijection

$$\pi(i, j) = 2^i(2j + 1) - 1.$$

For $x \in J$, $i \in \omega$, define $(x)_i \in J$ by $((x)_i)(j) = x(\pi(i, j))$, and set

$$\langle x \rangle = \{(x)_i: i \in \omega\}.$$

Define an equivalence by

$$xE_C y \leftrightarrow \langle x \rangle = \langle y \rangle \leftrightarrow \forall i \in \omega\ \exists j \in \omega\ (x)_i = (y)_j \text{ \& vice versa}.$$

Clearly E_C is Borel, indeed Π^0_3. Any Δ^1_2 selector S for E_C would yield a Δ^1_2 selector T for E_V as follows: Let K be the Δ^1_1-definable function such that $(K(x))_i = h_i(x)$ for all $x \in J$, $i \in \omega$, where the h_i are as in Example 1.1 above. Then for all x, y, we have $xE_V y \leftrightarrow K(x) E_C K(y)$, so that

$$T = \{x: \exists y(y \in S \text{ \& } yE_C K(x) \text{ \& } (y)_0 = x)\}$$

defines a Σ^1_2 (hence Δ^1_2) selector.

1.3 *Isomorphism.* Define a homeomorphism $\pi^*: {}^\omega 2 = {}^{\omega \times \omega}2$ by $\pi^*(x) = x\pi$, where π is the pairing function of Example 1.2 above. For $x \in J$, let R_x be the binary relation on ω whose characteristic function is $\pi^*(x)$. Let the group $\omega!$ of permutations of ω act on ${}^{\omega\times\omega}2$ by letting $(gz)(m, n) = z(g(m), g(n))$ for $g \in \omega!$,

$z \in {}^{\omega\times\omega}2$, and m, n $\in \omega$. Define an equivalence on J by

$$xIy \longleftrightarrow (\omega, R_x) \cong (\omega, R_y) \longleftrightarrow \exists g \in \omega!\ g\pi^*(x) = \pi^*(y).$$

I is Σ^1_1. This equivalence is intensively studied in [17].

A Δ^1_2 selector S for I would yield a Δ^1_2 selector T for E_C as follows: Let HF be the family of hereditarily finite sets (sets x such that x itself, all elements of x, all elements of elements of x, *etc.* are finite). Let $D \subseteq J$ be the set of all x such that there exist a nonempty $A \subseteq J$ and an isomorphism

$$\pi: (\omega, R_x) \to (HF \cup A, \epsilon).$$

Note that for $x \in D$ the associated A and π are unique; call them A_x and π_x. It is not hard to see that for all $x \in J$, $x \in D$ if and only if (ω, R_x) is a model of a certain recursive 1st order theory omitting a certain recursive type, and from this it follows that D is Δ^1_1. We can define a Δ^1_1 function $F: D \to J$ by setting, for $x \in D$,

$$(F(x))_i = \begin{cases} \pi_x(i) & \text{if } \pi_x(i) \in J \\ \pi_x(i_x) & \text{otherwise} \end{cases}, \quad \text{where } i_x = \text{the least } i \in \omega \text{ with } \pi_x(i) \in J.$$

Since for every $y \in J$ there is an $x \in D$ with $A_x = \langle y \rangle$, $T = \{F(x): x \in D \cap S\}$ provides a Δ^1_2 selector for E_C.

1.4 *Lebesgue Decomposition.* Let $\Omega = \{x \in J: R_x \text{ wellorders } \omega\}$. It is known that Ω is Π^1_1, and for $\alpha < \omega_1$, $\Omega^\alpha = \{x \in J: R_x \text{ wellorders } \omega \text{ in type } \alpha\}$ is Borel. Define an equivalence by $xE_Ly \longleftrightarrow (x \notin \Omega\ \&\ y \notin \Omega) \vee xIy$. E_L is the simplest example of a Σ^1_1 equivalence with exactly $\aleph_1$ equivalence classes, and is essentially due to Lebesgue. Clearly any Δ^1_2 selector for I would yield one for E_L.

Let us call $S \subseteq J$ a *semi-selector* for an equivalence E on J if for each E-equivalence class C, $S \cap C \neq \emptyset$, but $S \cap C$ contains no perfect subset. Then *any semi-selector for* E_L *is an uncountable set with no perfect subset.* To see this, suppose S is a semi-selector for E_L and $P \subseteq S$ is perfect. Let C be an E_L equivalence class (so $C = J - \Omega$ or $C = \Omega^\alpha$ for some α), $D = P \cap C$. Then D is analytic, and being a subset of $C \cap S$ contains no perfect subset. So D is countable. Applying this in the case of $C = J - \Omega$, we see $B = P \cap \Omega = P - (P \cap (J - \Omega))$ is an uncountable Borel set. But since $B \cap \Omega^\alpha$ is countable for every α, $B \cap \Omega^\alpha$ must be nonempty for uncountably many α. This contradicts the classical Boundedness Theorem: Any analytic $A \subseteq \Omega$ has nonempty intersection with only countably many Ω^α.

A trick of Solovay's shows that any Σ^1_2 selector S for E_L would actually yield a Π^1_1 selector T: By the Kondo-Addison Uniformization Theorem, we may assume that S is the projection to the 1st coordinate of a Π^1_1 set $P \subseteq J^2$ such that for each $x \in S$ there is a unique $y \in J$ with $(x, y) \in P$. Let $F: J^2 \to J$ be the function such that for any $x, y \in J$, if $R = R_{F(x,y)}$ then: (i) R linearly orders the odd elements of ω in their natural order, with the exception that the positions of $4n + 1$ and $4n + 3$ are switched for precisely those n with $y(n) = 0$, (ii) $(i, j) \in R$ for all pairs with i odd, j even, and for no pairs with i even, j odd, and (iii) for all $i, j \in \omega$, $(2i, 2j) \in R$ if and only if $(i, j) \in R_x$. It is not hard to see that F is a recursive injection with a Π^0_1 range, and that there exists a recursive $G: J \to J^2$ such that $F^{-1} = G \restriction \text{range } F$. Let $T_1 = \{F(x, y): (x, y) \in P\} = \{x \in \text{range } F: G(x) \in P\}$. Let

T_0 be a Δ^1_1 set containing one element of $J - \Omega$, and containing for each $n \in \omega$ one $x \in \Omega$ coding a wellordering of type $\omega + n$, and containing no other elements. Then $T = T_0 \cup T_1$ is a Π^1_1 selector for E_L.

1.5 *A Universal Equivalence.* Every $x \in J$ can be regarded as coding an integer $i(x)$, the least i with $(x)_0(i) = 0$ if such exists, and 0 otherwise. Fix a reasonable enumeration $(\sigma_i: i \in \omega)$ of the Σ^1_1 formulas of 2nd order Peano arithmetic with three free variables for elements of J. Let $U = \{(t, x, y): \sigma_{i(t)}((t)_1, x, y)\}$. Then every analytic binary relation T on J has the form of a cross-section $U_t = \{(x, y): (t, x, y) \in U\}$ of U. We call such a t an *index* for T. The smallest equivalence relation E_T containing the binary relation $T = U_t$ is:

$$\{(x, y): \exists n \in \omega\, \exists z_0, \cdots, z_n \in J\, (x = z_0 \;\&\; y = z_n \;\&\; \forall i < n\, (z_i T z_{i+1} \text{ or vice versa}))\}.$$

We leave it to the reader to write down an explicit definition of a Δ^1_1 function $F: J \to J$ such that $F(t)$ is an index for E_T whenever t is an index for T. Let E^* be the equivalence:

$$\{(x, y): (x)_0 = (y)_0 \;\&\; (F((x)_0), (x)_1, (y)_1) \in U\}.$$

Then E^* is a Σ^1_1 equivalence which is "at least as complicated as" any other such equivalence. In particular, a Δ^1_2 selector for E^* would yield one for E_V, E_C, E_L, I or any other Σ^1_1 equivalence. (Miller [10] constructs a similar "universal" equivalence.)

Let us now turn to the question of the status of the Selector Principle (*). A Σ^1_2-*good* wellordering of J is a wellordering $<$ of J in order type ω_1 such that the relation $\{(x, y): \langle y \rangle = \{x': x' < x\}\}$ is Σ^1_2. (It then follows that both this relation and the order $<$ are Δ^1_2.) Addison [1] shows that if every real is constructible, then the natural order $<_L$ on the constructible universe provides Σ^1_2-good wellordering of J. (Recent work of Mansfield, improving on earlier results of Friedman, shows that the existence of *any* Σ^1_2 wellordering of J implies, and hence is equivalent to, the constructibility of every real.) The following establishes the consistency of (*) relative to ZFC.

1.6 PROPOSITION (Folklore). *If every real is constructible, then for every* $n > 1$, *every* Δ^1_n (Σ^1_n) (Π^1_n) *equivalence relation on* J *has a* Δ^1_n (respectively, Π^1_n) (respectively, Σ^1_n) *selector.*

Proof. Assume every real is constructible, so $<_L$ is a Σ^1_2-good wellordering of J. Let E be an equivalence on J. Then the following set is a selector for E which, for $n > 1$, is Π^1_n if E is Σ^1_n, and Σ^1_n if E is Π^1_n:

$$\{x: \neg\exists y(y <_L x \;\&\; xEy)\} = \{x: \exists y(\langle y \rangle = \{x': x' <_L x\} \;\&\; \forall i \in \omega\, \neg(y)_i Ex)\}.$$

1.7 PROPOSITION. (a) $\omega_1{}^L < \omega_1$ *implies* E_V *has no* Σ^1_2 *selector.*

(b) Con(ZFC) *implies* Con(ZFC + $\omega_1{}^L = \omega_1$ + E_V has no Σ^1_2 selector).

(c) Con(ZFC + There exists an inaccessible cardinal) *implies* Con(ZFC + E_V has no ordinal-definable (OD) selector).

(d) *The following are equivalent:*

(i) $\omega_1{}^L = \omega_1$.

(ii) E_L *has a* Π^1_1 *selector.*

(iii) *There exists an uncountable* Σ^1_2 *set having no perfect subset.*

(iv) E_L *has a* Σ^1_2 *semi-selector.*

(e) Con(ZFC + There exists an inaccessible cardinal) *implies* Con(ZFC + E_L has no OD semi-selector).

Proof. (a) Solovay has shown that if the constructible ω_1, $\omega_1{}^L$, is less than the real ω_1, then every Σ^1_2 set has the Baire property. (For a neat version of this proof see [6].) But as we have noted, no selector for E_V has the Baire property.

(b) Martin and Solovay [9] construct a model where $\omega_1{}^L = \omega_1$, CH fails, and Martin's Axiom (MA) holds. But as is shown in [9], MA + $\neg$CH implies every Σ^1_2 set has the Baire property.

(c) uses Solovay's result [16] that in the model obtained by collapsing an inaccessible, every OD set is Lebesgue measurable.

(d) Mansfield [8] and Solovay [15] establish the equivalence of (i), (iii), and the existence of a Σ^1_2 selector for E_L. This last trivially implies (iv). By our remarks on Example 1.4 above, it also implies (ii). By the same remarks, (iv) implies (iii).

(e) again uses Solovay's model [16].

1.8 COROLLARY (Myers [12]). (*) *implies* $\omega_1{}^L = \omega_1$, *but* Con(ZFC) *implies* Con(ZFC + $\omega_1{}^L = \omega_1$ + $\neg$(*)).

Proof. The first implication follows from 1.7(a) or (d), the second from (b).

Though Myers [12] is explicitly concerned only with the equivalence I of Example 1.3, implicitly [12] contains many observations akin to 1.7(a)-(e) above. (Note that since the existence of selectors for I implies the existence of selectors for E_V, E_C, and E_L, 1.7(a)-(e) have negative consequences for the existence of selectors for I.)

Myers also makes an argument which may be paraphrased as follows: Consider a model of ZFC of the form L[x] where x is Cohen-generic over L. With h_i as in Example 1.1, it is readily seen that $L[x] = L[h_i(x)]$ and $h_i(x)$ is Cohen-generic, for every $i \in \omega$. Standard forcing arguments using the homogeneity of the Cohen forcing conditions establish that any OD set S in L[x] containing one $h_i(x)$ contains them all. This establishes the following improvement of 1.7(b) and (c):

(f) (Myers [12]). Con(ZFC) *implies* Con(ZFC + $\omega_1{}^L = \omega_1$ + E_V has no OD selector).

By similar arguments we have established:

(g) Con(ZFC) *implies* Con(ZFC + $\omega_1{}^L = \omega_1$ + E_C has no Σ^1_2 semi-selector).

(h) Con(ZFC) *implies*

$$\text{Con}(\text{ZFC} + \omega_1{}^L = \omega_1 + \neg\text{CH}$$
$$+ \neg\exists \text{ OD set } S \ \forall E_C\text{-equivalence class } D \ (0 < \text{card}(S \cap D) < 2^{\aleph_0}))\,.$$

To entirely clear up this area, it would be desirable to show Con(ZFC) implies Con(ZFC + $\omega_1{}^L = \omega_1 \pm$ CH + E_C has no OD semi-selector). (We omit proofs of (g) and (h) on account of the partial and technical character of these results.)

2. PARTITION THEOREMS

Let X be a Polish space (separable topological space admitting a complete metric). For $n \leq \omega$, we mean by $\langle X \rangle^n$ the n^{th} *deleted power* of X $\{(x_i: i < n): \forall i<j<n\ (x_i \neq x_j)\}$ with its topology as a subspace (open for $n < \omega$ and G_δ for $n = \omega$) of the Cartesian power X^n. By $[X]^n$ we mean the set of all n-membered subsets of X with the following topology: Let $\mathscr{S}_n$ be the group of permutations of n. Let $\mathscr{S}_n$ act on $\langle X \rangle^n$ by setting $g((x_i: i < n)) = (x_{g(i)}: i < n)$ for $g \in \mathscr{S}_n$, $x_i \in X$. This action induces an equivalence; $(x_i: i < n)\, E\, (y_i: i < n)$ if and only if $(y_i: i < n) = g(x_i: i < n)$ for some $g \in \mathscr{S}_n$. It is natural to identify an n-element subset A of X with the E-equivalence class of some/any enumeration $(x_i: i < n)$ of A. Then we can impose on $[X]^n$ the quotient topology $\langle X \rangle^n/E$. Each $\langle X \rangle^n$ is clearly Polish; we also have:

2.1 PROPOSITION. *For any Polish space* X *and* $n < \omega$, $[X]^n$ *is Polish.*

Proof. This can be proved in several ways. Put the usual order on the unit interval [0, 1], and the lexicographic product of copies of this order on the Hilbert Cube $[0, 1]^\omega$. Any Polish X can be embedded as a subspace of the Hilbert Cube, and so inherits a linear ordering < which is G_δ (as a subspace of X^2). Using this order we obtain a G_δ selector $S = \{(x_i: i < n): \forall i<j<n\ (x_i < x_j)\}$ for the equivalence E on $\langle X \rangle^n$ for any $n < \omega$. S is Polish in its own right and clearly $[X]^n \cong S$.

Alternatively we could obtain a G_δ, hence Polish, selector S by appealing to a general selector theorem of Kuratowski and Ryll-Nardzewski [7], or we could directly define a complete metric inducing the topology on $[X]^n$ given one for X.

Let X be a Polish space. $P \subseteq X$ is perfect if it is nonempty, closed, and dense in itself. Thus X itself is perfect if it has no isolated points. Any perfect set has cardinality $2^{\aleph_0}$. Recall that a subset $B \subseteq X$ is *Baire* (has the *property of Baire*, is *almost open*) if there is an open $U \subseteq X$ such that the symmetric difference $S \triangle U = (S - U) \cup (U - S)$ is meager (1st category). The class of Baire sets contains all analytic sets and is closed under complementation and countable union. It is readily verified that $B \subseteq [X]^n$ is meager (Baire) if and only if

$$\{(x_i: i < n): \{x_i: i < n\} \in B\}$$

is meager (respectively, Baire) in $\langle X \rangle^n$ and equivalently X^n. The following lemma is in Mycielski [11], but we include a proof for completeness.

2.2 LEMMA. *Let* X *be a perfect Polish space,* $n < \omega$, $N \subseteq X^n$ *meager. Then there is a perfect* $P \subseteq X$ *such that* $\langle P \rangle^n \cap N = \emptyset$.

Proof. Fix a complete metric δ inducing the topology of X. By an *open disk* we mean a nonempty open subset of X of the form $\{y: \delta(x, y) < \rho\}$ for some $x \in X$ and $\rho > 0$. The topology of X^n is generated by products of open disks. Let $N = \bigcup_{j\in\omega} N_j$, where each N_j is nowhere dense (NWD). This means for any open disks U_i, $i < n$, there exist open disks V_i, $i < n$, with $V_i \subseteq U_i$ for each i and: $(V_0 \times \cdots \times V_{n-1}) \cap N_j = \emptyset$.

For each k-tuple $s \in {}^k2$ of 0's and 1's and i = 0 or 1, let $s*i$ be the (k + 1)-tuple extending s with last term i. We wish to assign to each $s \in {}^{\underset{\smile}{\omega}}2$ $(= \bigcup_{k\in\omega} {}^k2)$ an open disk U_s in such a way that:

(i) δ-diameter $U_s < 2^{-k}$, for $s \in {}^k2$.

(ii) closure $U_{s*0} \cap$ closure $U_{s*1} = \emptyset$, for all s.

(iii) closure $U_{s*0} \cup$ closure $U_{s*1} \subseteq U_s$, for all s.

(iv) $(U_{f(0)} \times \cdots \times U_{f(n-1)}) \cap N_j = \emptyset$, for any k, any injective function $f: n \to {}^k2$, and any $j < k$.

This is easily accomplished. Having U_s for all $s \in {}^k2$, choose U_t for $t \in {}^{(k+1)}2$ to satisfy (i), (ii), (iii), and then use the NWD property of the N_j, $j < k$, to refine these U_t if necessary to satisfy (iv).

Let $P = \bigcap_{k \in \omega} \bigcup_{s \in {}^k2} U_s$. It is readily verified that P is a perfect subset of X. (This claim is a special case of the "Fusion Lemma" of Sacks; see [5].) If $x_0, \cdots, x_{n-1} \in P$ are distinct, there is an $m \in \omega$ such that for all $k > m$, the $s \in {}^k2$ to which the x_i belong are distinct. By (iv), $(x_0, \cdots, x_{n-1}) \notin N_j$ for every j, and $\langle P \rangle^n \cap N = \emptyset$.

2.3 COROLLARY. *Let* X *be a perfect Polish space,* Y *a topological space with a countable base,* $n < \omega$, $f: \langle X \rangle^n \to Y$ *a function such that the inverse image of any open set is Baire. Then there is a perfect* $P \subseteq X$ *such that the restriction of* f *to* $\langle P \rangle^n$ *is continuous.*

2.4 COROLLARY. *Let* X *be an uncountable Polish space,* Y *a Polish space,* $B \subseteq Y$ *Borel,* $n < \omega$, $f: B \to \langle X \rangle^n$ *a Borel measurable surjection. Then there exist perfect* $P \subseteq X$ *and continuous* $g: \langle P \rangle^n \to B$ *such that* fg = identity.

Proofs. For 2.3, pick for each V in a countable base for the topology of Y an open $U \subseteq \langle X \rangle^n$ such that $U \triangle f^{-1}(V)$ is meager. The union N of these symmetric differences is meager, and by 2.2 there is a perfect $P \subseteq X$ with $\langle P \rangle^n \cap N = \emptyset$. Clearly f is continuous on $\langle P \rangle^n$.

For 2.4 we may assume X perfect, since by the Cantor-Bendixson Theorem, X has a perfect subspace X', and $f^{-1}(\langle X' \rangle^n)$ is still Borel. By a lemma of von Neumann (*cf.* [18], 448 ff.), there is a $G: \langle X \rangle^n \to B$ with fG = identity, such that the inverse image of any open set under G belongs to the σ-algebra generated by the analytic sets. Such sets are Baire, so 2.3 applies to yield the required P and $g = G \mid \langle P \rangle^n$.

Theorems 2.5 and 2.6 below were proved a number of years ago by Galvin, and were later rediscovered by Simpson [14] and the author [2], respectively; we had overlooked the announcement [3], [4].

2.5 THEOREM (Galvin [3], [4]; Simpson [14]). *Let* X *be a perfect Polish space,* $n < \omega$. *Then:*

(a) *If* $A \subseteq \langle X \rangle^n$ *is Baire and nonmeager* (*2nd category*), *then there exist perfect* $P_i \subseteq X$, $i < n$, *such that* $\prod_{i<n} P_i \subseteq A$.

(b) *If* $\langle X \rangle^n = \bigcup_{k \in \omega} A_k$, *where each* A_k *is Baire, then there exist* $k \in \omega$ *and perfect* $P_i \subseteq X$, $i < n$, *such that* $\prod_{i<n} P_i \subseteq A_k$.

2.6. THEOREM (Galvin [3], [4]; Burgess [2]). *Let* X *be a perfect Polish space,* $K < \omega$. *If* $[X]^2 = \bigcup_{k<K} A_k$, *where each* A_k *is Baire, then there exist a perfect* $P \subseteq X$ *and a* $k < K$ *such that* $[P]^2 \subseteq A_k$.

Proofs. For 2.5, see [14].

For 2.6, we may without loss of generality assume the A_k are disjoint. We may also assume that the sets $B_k = \{(x, y)\colon \{x, y\} \in A_k\}$ are open in $\langle X \rangle^2$. (Otherwise, apply 2.3 to the map sending $(x, y) \in \langle X \rangle^2$ to the point k in the discrete space $\{0, 1, \cdots, K-1\}$ with $\{x, y\} \in A_k$, to obtain a perfect $X' \subseteq X$ such that this map is continuous on $\langle X' \rangle^2$; *i.e.*, each $B_k \cap \langle X' \rangle^2$ is open. Then replace X by X'.) We begin by considering the case K = 2.

So let $[X]^2 = A_0 \cup A_1$ (disjoint), where for k = 0 or 1 $\{(x, y)\colon \{x, y\} \in A_k\}$ has the form $\langle X \rangle^2 \cap V_k$ for some open $V_k \subseteq X^2$. We must find a perfect $P \subseteq X$ and a k such that $\langle P \rangle^2 \subseteq V_k$. If for some nonempty open $U \subseteq X$, $U^2 \subseteq V_0$, then any perfect subset of U will do. If no such U exists, then for any nonempty open $W \subseteq X$, $\langle W \rangle^2 \cap V_1$ will be a nonempty open set, and there will exist disjoint nonempty open $W', W'' \subseteq W$ with $W' \times W'' \subseteq V_1$. In this case, proceeding as in the proof of Lemma 2.2, we may fix a complete metric δ on X and define for each $s \in {}^{\omega}2$ an open disk U_s so that:

(i), (ii), (iii) of the proof of Lemma 2.2 hold.

(iv) $U_{s*0} \times U_{s*1} \subseteq V_1$, for every s.

Then $P = \bigcap_{k \in \omega} \bigcup_{s \in {}^k2} U_s$ is a perfect set with $\langle P \rangle^2 \subseteq V_1$ and $[P]^2 \subseteq A_1$.

To extend this result to a partition into more than two pieces $A_0, A_1, \cdots, A_{K-1}$, proceed by induction: First consider the partition into two pieces A_0 and $(A_1 \cup \cdots \cup A_{K-1})$, and if we get a perfect Q with $[Q]^2$ contained in the second piece, consider the obvious partition of this $[Q]^2$ into K - 1 pieces.

The Axiom of Choice (AC) is not needed for the proof of Theorem 2.6, since the open sets "chosen" in its proof can all be taken from a countable, hence wellorderable, base for the topology of the Polish space X. Solovay [16] shows

ZF (without AC) + (Every set of reals is Baire)

is consistent relative to ZF + (There exists an inaccessible cardinal). From 2.6, it follows that

ZF + (The cardinal $\kappa = 2^{\aleph_0}$ (which is not an aleph)

satisfies the Erdös property $\kappa \to (\kappa)^2_2$)

is consistent relative to ZF + (There exists an inaccessible cardinal). By contrast, under AC this Erdös property implies that κ is (strongly) inaccessible, hyperinaccessible, Mahlo, *etc.* The Axiom of Determinateness (AD) also implies this Erdös property for $2^{\aleph_0}$, since it implies that all sets of reals are Baire.

2.7 COROLLARY. *Let* X *be an uncountable Polish space,* R *a transitive binary relation on* X *which is Baire as a subset of* X^2, E *the induced equivalence* (xEy *if and only if* xRy & yRx). *Then either* (i) *there exists an* E-*equivalence class of cardinality* $2^{\aleph_0}$, *or* (ii) *there exists a Borel subset* B *of* X *linearly ordered by* R *in the order type of the real numbers, or else* (iii) *there exists a perfect set of pairwise* R-*incomparable elements of* X.

Proof. We sketch the proof for X = the real numbers, leaving it to the interested reader to reduce the general case to this case. Partition $[X]^2$ into four Baire pieces as follows: If $x < y$ in the natural order on the reals, put $\{x, y\}$ in A_0 if xEy. Put it in A_1 if xRy & ¬yRx, and in A_2 if the opposite happens. Finally, put it in A_3 if x, y are R-incomparable. Apply Theorem 2.6 to get a perfect $P \subseteq X$ and a $k < 4$ with $[P]^2 \subseteq A_k$. If $k = 0$ we get alternative (i) of the Corollary, and if $k = 3$ we get (iii). Since any perfect set contains a Borel subset whose natural order is order-isomorphic to the natural order on all of X, if $k = 1$, we get (ii). We also get (ii) if $k = 2$, since the reverse of the natural order on X is order-isomorphic to the natural order.

Note that if X is, say, J, and R is the relation of Turing reducibility (relative recursiveness), then alternatives 2.7(i) and (ii) are impossible. Thus 2.7 generalizes the old theorem of Sacks that there exists a set of $2^{\aleph_0}$ pairwise incomparable Turing degrees.

Two examples (apparently known to Galvin, pointed out to us by Kunen) rule out certain extensions of 2.6.

2.8 *Example.* Unlike 2.5, 2.6 does not extend to partitions into infinitely many pieces. For consider the partition of $[J]^2$ into pieces

$$A_k = \{\{x, y\}: x \mid k = y \mid k \ \& \ x(k) \neq y(k)\}.$$

Clearly any perfect $P \subseteq J$ must have $[P]^2 \cap A_k \neq \emptyset$ for arbitrarily large k. A. Taylor has an (unpublished) result to the effect that this is in some sense the "only" counterexample to the extension of 2.6 to infinite partitions.

2.9 *Example.* Unlike 2.5, 2.6 does not extend to X^3. For let X be the real numbers, and partition $[X]^3$ into two pieces as follows: If $x < y < z$ in the natural order on X, put $\{x, y, z\}$ in A_0 if $y - x < z - y$, and in A_1 otherwise. Clearly any perfect subset of X contains triples of both types. Galvin has proved that this is in some sense the "only" counterexample to the extension of 2.6 to X^3. Recently A. Blass has proved the following general theorem (due to Galvin for $n = 2$ or 3): For any partition $[X]^n = \bigcup_{k<K} A_k$ with the A_k Baire, there exist a perfect $P \subseteq X$ and a subset $M \subseteq \{0, \cdots, K-1\}$ of cardinality less than or equal to $(n-1)!$ such that $[P]^n \subseteq \bigcup_{k \in M} A_k$.

3. THE SET-THEORETIC STRENGTH OF THE SELECTOR PRINCIPLE (*)

We have seen in Section 1 that (*) implies, but is not implied by, $(\omega_1^L = \omega_1)$; also (*) is implied by the hypothesis that every real is constructible. To complete the picture, we have the following result, which was announced in [2].

3.1 THEOREM. Con(ZFC) *implies* Con(ZFC + (*) Every Σ^1_1 equivalence relation on J has a Δ^1_2 selector + There is no OD wellordering of J).

Proof. We make use of Sacks' perfect set forcing, for the main properties of which see, *e.g.*, [5]. A *perfect tree* is a subset $T \subseteq {}^{\omega}2$ such that: $t \mid n \in T$ whenever $t \in T$ and $n <$ length t, and for every $t \in T$ there exist $n >$ length t and distinct $t', t'' \in {}^n2$ extending t such that $t', t'' \in T$. $x \in J$ is a *branch* through a tree T if $x \mid n \in T$ for all $n \in \omega$. If T is a perfect tree,

$$\mathscr{P}(T) = \{x \in J: x \text{ is a branch through } T\}$$

is a perfect subset of J; while if $P \subseteq J$ is perfect,

$$\mathcal{J}(P) = \{x \mid n : x \in P,\ n \in \omega\}$$

is a perfect tree. Clearly $\mathcal{P}(\mathcal{J}(P)) = P$ and $\mathcal{J}(\mathcal{P}(T)) = T$.

Let us start from a countable, transitive model M of ZF + V = L. Let $\mathfrak{P} \in M$ be, in M, the Sacks forcing conditions, the set of all perfect trees partially ordered by inclusion. Let G be $\mathfrak{P}$-generic over M, and let N = M[G]. It is known that: In N, there is no OD wellordering of J.

For all $x \in J \cap N$, either $x \in M$, or else

$$G_x = \{T \in \mathfrak{P} : x \text{ is a branch through } T\}$$

is $\mathfrak{P}$-generic over M, and N = M[x]. In the latter case, x is said to be a *Sacks real* (over M).

Now let σ be a Σ^1_1 formula of 2nd order arithmetic which, in N, defines an equivalence relation E^* on J. Define:

$$D_0 = \{T \in \mathfrak{P} : \forall x, y \in N \ (x, y \text{ are branches through } T \rightarrow xE^*y)\};$$

$$D_1 = \{T \in \mathfrak{P} : \forall x, y \in N \ (x, y \text{ are branches through } T \ \& \ x \neq y \rightarrow \neg xE^*y)\}.$$

By standard absoluteness considerations, $E = E^* \cap M$ is defined in M by the same formula σ, and:

$$D_0 = \{T \in \mathfrak{P} : \forall x, y \in M \ (x, y \text{ are branches through } T \rightarrow xEy)\};$$

$$D_1 = \{T \in \mathfrak{P} : \forall x, y \in M \ (x, y \text{ are branches through } T \ \& \ x \neq y \rightarrow \neg xEy)\}.$$

(We use the fact that membership in D_i can be expressed as a Π^1_2 condition, and apply Shoenfield's Theorem.)

CLAIM. $D_0 \cup D_1$ *is dense in* $\mathfrak{P}$.

To show this, we work "inside" M. Let $T \in \mathfrak{P}$, and let P be the Polish space $\mathcal{P}(T) \subseteq J$. Apply Theorem 2.6 to the partition $[P]^2 = A_0 \cup A_1$, where $A_0 = \{\{x, y\} : xEy\}$, $A_1 = \{\{x, y\} : \neg xEy\}$, to obtain a perfect $Q \subseteq P$ and an i with $[Q]^2 \subseteq A_i$. Then $\mathcal{J}(Q) \leq T$ in $\mathfrak{P}$ and $\mathcal{J}(Q) \in D_i$, proving the Claim.

From the Claim it follows that if $x \in J \cap N$ is a Sacks real, then x is a branch through some $T \in D_0 \cup D_1$. "Inside" N we thus have: Every element of J is either (i) constructible ($\in L = M$), or else a Sacks real, in which case it is either (ii) a branch through some $T \in D_0$, hence E^*-equivalent to some constructible element (*viz.*, any constructible branch through T), or finally (iii) a branch through some $T \in D_1$; *i.e.*, through some constructible perfect tree, no two branches through which are E^*-equivalent. To recapitulate, still inside N, for every $x \in J$ there exists an object $u \in L$ such that either (a) $u \in J$ and xE^*u, or else (b) u is a perfect tree, no two branches through which are E^*-equivalent and having some branch E^*-equivalent to x. Let us call the $<_L$-least such object u(x); and let f(x) be u(x) in case (a) or the (unique) branch through u(x) E^*-equivalent to x in case (b); and finally let S = range f = $\{x \in J : f(x) = x\}$. It is readily verified that, inside N, S is a Δ^1_2 selector for E^*. (The verification uses the fact that $<_L$ yields a Σ^1_2-good wellordering of J, plus a tedious but routine coding of trees by elements of J.)

Theorem 3.1 can be slightly extended as follows: A subset E of J^2 is *absolutely* Δ^1_2 if it has Σ^1_2 and Π^1_2 definitions which are not merely equivalent, but which also remain equivalent in any forcing extension of the universe V of set theory (*i.e.*, which are forced to remain equivalent by any partially ordered set of forcing conditions). *E.g.*, any finite Boolean combination of Σ^1_1 and Π^1_1 sets is absolutely Δ^1_2. Solovay (unpublished) has shown that any absolutely Δ^1_2 set is Baire. It can be shown that in the model N of the proof of 3.1, every absolutely Δ^1_2 equivalence has a (not necessarily absolutely) Δ^1_2 selector. We do not know whether the hypothesis that *every* Δ^1_2 equivalence on J has a Δ^1_2 selector implies that every real is constructible.

The results of Section 1 hold in "boldface" form. Indeed, Myers [12] states all his results in this form. Thus the assumption:

(A) $$\exists t \in J\ \forall x \in J\ x \in L[t]$$

(which is equivalent to the existence of a (good) PCA wellordering of J) implies:

(**) Every analytic equivalence relation on J has a PCA selector.

This in turn implies, but is not implied by:

(B) $$\exists t \in J\ \omega_1^{L[t]} = \omega_1 .$$

(It is worth noting that (*) implies (**): Simply apply (*) to the universal equivalence of Example 1.5.)

We do not, however, have a proof of the boldface analogue of Theorem 3.1, Con(ZFC + (**) + $\neg$(A)), and so the question whether (**) implies (A) remains open. A suitable generalization of Theorem 2.6 might yield an answer (negative), but as the examples at the end of Section 2 show, it is not so easy to find (and prove!) correct generalizations of 2.6.

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