OPERATOR ALGEBRAS LEAVING COMPACT OPERATOR RANGES INVARIANT

Eric Nordgren, Heydar Radjavi, and Peter Rosenthal

We show that a closed operator whose domain is the range of a compact operator has closed range. This has two easy consequences: a closed operator whose range is contained in that of a compact operator must itself be compact, and a transitive algebra whose only proper invariant operator ranges are the ranges of compact operators must be strongly dense.

We consider operators from any Banach space into any other Banach space. (In all the results other than Theorem 2, the spaces can be real or complex; for Theorem 2, the space must be complex.) A *closed operator* is a linear transformation T with domain $\mathscr{D}(T)$ in some space $\mathscr{X}$ and range in some $\mathscr{Y}$ such that $\{x \oplus Tx: x \in \mathscr{D}(T)\}$ is a closed subspace of $\mathscr{X} \oplus \mathscr{Y}$, where the norm on $\mathscr{X} \oplus \mathscr{Y}$ is any which is equivalent to the norm defined by $\|x \oplus y\| = \|x\| + \|y\|$. Then the projections of $\mathscr{X} \oplus \mathscr{Y}$ onto $\mathscr{X}$ and onto $\mathscr{Y}$ are bounded operators.

We begin with a well-known fact which follows, for example, from Douglas [1].

LEMMA 1. *A bounded operator whose range is contained in the range of a compact operator is itself compact.*

Proof. (Similar to [1].) Let K be the compact operator, with domain $\mathscr{Y}$. By replacing K by its natural quotient on $\mathscr{Y}/\ker K$ if necessary, we can assume that K is injective. Now if T is a bounded operator with $\operatorname{ran} T \subset \operatorname{ran} K$, then $K^{-1}T$ is a closed operator whose domain is a Banach space, so $K^{-1}T = C$ is bounded. Hence, $T = KC$ is compact.

Theorem 1 below generalizes Lemma 1 to closed operators.

LEMMA 2. *A closed operator whose domain is contained in the range of a compact operator must have closed range.*

Proof. Let T be the closed operator; by going to a quotient space if necessary, we can assume that T is injective. Suppose that the domain $\mathscr{D}(T)$ is contained in the range of the compact operator K_0. Then the projection K of the graph of T onto the first coordinate space is a bounded operator with $\operatorname{ran} K \subset \operatorname{ran} K_0$, so K is compact by Lemma 1. Clearly, K is injective and has range equal to $\mathscr{D}(T)$.

Now TK is a closed operator with domain a Banach space $\mathscr{D}(K)$, so TK is a bounded operator C. Clearly,

$$\{x \oplus Tx: x \in \mathscr{D}(T)\} = \{Kz \oplus Cz: z \in \mathscr{D}(K)\}.$$

Thus to show ran T is closed, it suffices to show ran C is closed. Define the mapping U by $Uz = Kz \oplus Cz$ for $z \in \mathscr{D}(K)$. Then U is a bounded injective operator mapping the Banach space $\mathscr{D}(K)$ onto the graph of T, so U is bounded below. We must prove that C is bounded below.

Received March 2, 1976.

Michigan Math. J. 23 (1976).

If C is not bounded below, then there is a sequence $\{z_n\}$ with $\|z_n\| = 1$ for all n and $\{Cz_n\} \to 0$. Since K is compact, there is a subsequence $\{z_{n_i}\}$ such that $\{Kz_{n_i}\}$ converges to some vector x. Then $x \oplus 0$ is in the graph of T, so the assumption that T is injective implies $x = 0$. Thus, $\{Uz_{n_i}\} = \{Kz_{n_i} \oplus Cz_{n_i}\}$ approaches 0, which contradicts the fact that U is bounded below. Hence, C is bounded below.

THEOREM 1. *A closed operator whose range is contained in the range of a compact operator must itself be compact.*

Proof. Let T be the closed operator; by Lemma 1, we need only show that T is bounded. By the closed graph theorem, it suffices to show that T has closed domain, and by taking quotients we can assume that T is injective. (The inverse of the natural projection of the domain of the quotient is the domain of T.) But the domain of T^{-1} is the range of T, so T^{-1} satisfies the hypothesis of Lemma 2. Hence, $\operatorname{ran} T^{-1} = \mathscr{D}(T)$ is closed.

Example. If T is a closed operator from a Banach space into $L^2(0, 1)$ such that Tx is absolutely continuous and has derivative in $L^2(0, 1)$ for all $x \in \mathscr{D}(T)$, then T is compact.

Proof. If V is the Volterra operator defined on $L^2(0, 1)$ by $(Vf)(t) = \int_0^t f(s)\,ds$, then V is compact and has range consisting of all absolutely continuous functions which vanish at 0 and have derivatives in $L^2(0, 1)$. If P is the rank 1 projection of $L^2(0, 1)$ onto the constant functions, then a formula due to Crimmins [3, Theorem 2.2] implies that the range of $\sqrt{VV^* + P}$ is the set of absolutely continuous functions with derivatives in $L^2(0, 1)$. Since $\sqrt{VV^* + P}$ is compact, Theorem 1 applies.

The next theorem is a slight generalization of the well-known result of Foiaş [4; 7, Theorem 8.9]. We use $\mathscr{B}(\mathscr{X})$ to denote the algebra of bounded linear operators mapping the complex Banach space $\mathscr{X}$ into itself.

THEOREM 2. *If $\mathscr{A}$ is a subalgebra of $\mathscr{B}(\mathscr{X})$, and if every proper operator range invariant under $\mathscr{A}$ is the range of a compact operator, then either $\mathscr{A}$ is strongly dense in $\mathscr{B}(\mathscr{X})$ or $\mathscr{A}$ has a finite-dimensional invariant subspace other than* $\{0\}$.

Proof. If $\mathscr{A}$ has a nontrivial closed invariant subspace, then it must be finite-dimensional, since the range of a compact operator cannot be closed unless it is finite-dimensional. Thus we can assume that $\mathscr{A}$ is transitive; *i.e.*, $\mathscr{A}$ has no nontrivial invariant subspaces.

We use the notation and techniques described in [7, Chapter 8]. As in the proof of Theorems 8.9 and 9.7 of [7], it suffices to show that each graph transformation for $\mathscr{A}$ is bounded. Let $\{x \oplus T_1 x \oplus \cdots \oplus T_n x : x \in \mathscr{D}\}$ be an invariant graph subspace for $\mathscr{A}^{(n+1)}$; we assume (by induction) that all graph transformations arising from $\mathscr{A}^{(n)}$ are bounded. Let T be the map from $\mathscr{D}$ into $\mathscr{X}^{(n)}$ defined by

$$Tx = T_1 x \oplus \cdots \oplus T_n x \;;$$

we must show that T is bounded.

Case (i): T_n is not injective. Let $\hat{\mathscr{D}} = \{x \in \mathscr{D} : T_n x = 0\}$; since $\hat{\mathscr{D}} \neq \{0\}$ and $\hat{\mathscr{D}}$ is invariant under $\mathscr{A}$, $\hat{\mathscr{D}}$ is dense in $\mathscr{X}$. The inductive hypothesis implies that the restriction $\hat{T}$ of T to $\hat{\mathcal{D}}$ is bounded and thus extends to $\mathcal{X}$. Hence, $\hat{\mathcal{D}} = \mathcal{X}$, so $\mathcal{D} = \mathcal{X}$ and T is bounded.

Case (ii): T_n is injective. If $\mathcal{D} \neq \mathcal{X}$, $\mathcal{D}$ is the range of a compact operator, so Lemma 2 implies that $\operatorname{ran} T = \{T_1 x \oplus \cdots \oplus T_n x \colon x \in \mathcal{D}\}$ is closed. Also,

$$\operatorname{ran} T = \{T_1 T_n^{-1} y \oplus \cdots \oplus T_{n-1} T_n^{-1} y \oplus y \colon y \in T_n \mathcal{D}\},$$

so the inductive hypothesis implies $T_i T_n^{-1}$ is bounded for $i = 1, \cdots, n-1$. Hence, $T_n \mathcal{D} = \mathcal{X}$,

$$\{x \oplus T_1 x \oplus \cdots \oplus T_n x \colon x \in \mathcal{D}\} = \{T_n^{-1} y \oplus T_1 T_n^{-1} y \oplus \cdots \oplus y \colon y \in \mathcal{X}\},$$

and T_n^{-1} is a bounded transformation with range $\mathcal{D}$, so Lemma 1 implies T_n^{-1} is compact. Since T_n^{-1} obviously commutes with $\mathcal{A}$, Lomonosov's theorem [6; 7, Corollary 8.25] implies $\mathcal{A}$ has a nontrivial invariant subspace. This contradiction proves that $\mathcal{D} = \mathcal{X}$ and T is bounded.

It seems that the theory of operator ranges on Banach spaces has not been seriously investigated. Which of the results of [3] hold on Banach spaces? As E. Azoff has kindly informed us, the paper [5] of S. Grabiner gives some results along these lines.

REFERENCES

1. R. G. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space.* Proc. Amer. Math. Soc. 17 (1966), 413-415.

2. A. Feintuch and P. Rosenthal, *Remarks on reductive operator algebras.* Israel J. Math. 15 (1973), 130-136.

3. P. A. Fillmore and J. P. Williams, *On operator ranges.* Advances in Math. 7 (1971), 254-281.

4. C. Foiaş, *Invariant para-closed subspaces.* Indiana Univ. Math. J. 21 (1972), 887-906.

5. S. Grabiner, *Ranges of products of operators.* Canad. J. Math. 26 (1974), 1430-1441.

6. V. I. Lomonosov, *Invariant subspaces for the family of operators which commute with a completely continuous operator.* Functional Anal. and Appl. 7 (1973), 213-214.

7. H. Radjavi and P. Rosenthal, *Invariant subspaces.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 77. Springer-Verlag, Berlin-Heidelberg-New York, 1973.

Eric Nordgren:
Department of Mathematics
University of New Hampshire
Durham, New Hampshire 03824

Peter Rosenthal:
Department of Mathematics
University of Toronto
Toronto, Canada M5S 1A1

Heydar Radjavi:
Department of Mathematics
Dalhousie University
Halifax, Nova Scotia
Canada B3H 3J5