A NOTE ON QUATERNIONIC GEOMETRY

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A *quaternionic manifold* is usually thought of as a 4n-dimensional Riemannian manifold (M, g) whose structural group can be reduced to

$$\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) = \mathrm{Sp}(n) \times \mathrm{Sp}(1)/\pm 1 .$$

The group $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ can be considered as the real representation of $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ acting on a quaternionic vector space by $(S_n, S_1)(V) = S_n \cdot V \cdot S_1$ [1, 2, 3]. A quaternionic structure on a manifold is equivalent to the existence of a 3-dimensional vector bundle Q of tensors of type (1, 1) with local bases of almost complex structures I, J, K such that $K = IJ = -JI$, and of a metric g being hermitian with respect to each almost complex structure. It is easy to show that Q is an SO(3) bundle.

Picking local almost complex structures I, J, K, with $IJ = K = -JI$, define

$$\Lambda_{AB} C = g(IA, B) IC + g(JA, B) JC + g(KA, B) KC \tag{1}$$

for local vector fields A, B, C. Then Λ is independent of the choice of I, J, K and thus is a tensor field of type (1, 3) on M. The corresponding tensor field of type (0, 4) is denoted by $\underline{\Lambda}$. Let X be a unit vector field, let S_X be the set of unit vector fields Y orthogonal to X such that $\underline{\Lambda}(X, Y, X, Y) = 1$, and let $[S_X]_m$ be the subspace of the tangent space at m generated by S_X. Then Λ and $\underline{\Lambda}$ have the following properties:

(i) $\Lambda_{XY} = -\Lambda_{YX}$;

(ii) $\underline{\Lambda}(X, Y, Z, W) = \underline{\Lambda}(Z, W, X, Y)$;

(iii) $\Lambda^2_{XY} Z = -\underline{\Lambda}(X, Y, X, Y)Z$;

(iv) $\dim[S_X]_m \geq 2$;

(v) $\Lambda_{X\Lambda_{YZ}X} W = g(X, X)\, \Lambda_{YZ} W$;

(vi) $\Lambda_{YZ} = \Lambda_{XY} \Lambda_{XZ}$.

We will show that a tensor field Λ satisfying (i) to (iv) above determines a reduction of the structural group to $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$, and additionally, if Λ satisfies (v) and (vi), Λ can be recovered by equation (1). Thus Λ is analogous to a tensor field J of type (1, 1) such that $J^2 = -1$, whose existence is equivalent to the reduction of the structural group to U(n).

THEOREM 1. (M, g) *is a quaternionic manifold if and only if* M *admits a global tensor field* Λ *of type* (1, 3) *satisfying the axioms* (i) *to* (iv).

Proof. We first show that $Y \in S_X$ implies that $\Lambda_{XY} X = Y$. Since

$$\underline{\Lambda}(X, Y, X, Y) = g(\Lambda_{XY} X, Y) = 1$$

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and Y is unit, it suffices to show that $\Lambda_{XY}X$ is unit. For this we have by (i), (ii), and (iii),

$$g(\Lambda_{XY}X, \Lambda_{XY}X) = -g(\Lambda^2_{XY}X, X) = g(X, X) = 1.$$

Next we linearize (iii); that is, $\Lambda^2_{X(Y+Z)}V = -\underline{\Lambda}(X, Y+Z, X, Y+Z)V$ gives

$$\Lambda_{XY}\Lambda_{XZ}V + \Lambda_{XZ}\Lambda_{XY}V = -2\underline{\Lambda}(X, Y, X, Z)V. \tag{2}$$

Thus if $\{Y, Z\}$ is an orthonormal pair in S_X, Λ_{XY} and Λ_{XZ} are local almost complex structures which anticommute, and their composition $\Lambda_{XY}\Lambda_{XZ}$ is the third local almost complex structure. Moreover, g is hermitian with respect to these almost complex structures.

Given a unit vector field X, set $Y = IX$ and $Z = JX$. Then (1) gives $I = \Lambda_{XY}$, $J = \Lambda_{XZ}$, and $K = \Lambda_{XY}\Lambda_{XZ}$; but from the axioms (i) to (iv) alone, we cannot obtain a formula like (1) with the third almost complex structure of the form Λ_{XW} for some $W \in S_X$. For example, one could define Λ as in (1) using only two of the almost complex structures; then (i) to (iv) still hold, but (vi) does not.

THEOREM 2. *Let* (M, g) *be a quaternionic manifold with tensor field* Λ *satisfying* (i) *to* (iv). *Then* Λ *satisfies* (v) *and* (vi) *if and only if there exists an orthonormal triple* $\{Y, Z, W\} \subset S_X$ *such that* $\Lambda_{XW} = \Lambda_{XY}\Lambda_{XZ}$ *and*

$$\Lambda_{AB}C = g(\Lambda_{XY}A, B)\Lambda_{XY}C + g(\Lambda_{XZ}A, B)\Lambda_{XZ}C + g(\Lambda_{XW}A, B)\Lambda_{XW}C. \tag{3}$$

Proof. Assume (v) and (vi) hold. Now if $\{Y, Z\}$ is an orthonormal pair in S_X, set $W = \Lambda_{XY}Z$. We then claim that $W \in S_X$ and $\Lambda_{XW} = \Lambda_{XY}\Lambda_{XZ}$. First of all,

$$g(\Lambda_{XY}Z, X) = -g(\Lambda_{XY}X, Z) = -g(Y, Z) = 0$$

and

$$g(\Lambda_{XY}Z, \Lambda_{XY}Z) = -g(\Lambda^2_{XY}Z, Z) = g(Z, Z) = 1,$$

so we must show that $\underline{\Lambda}(X, \Lambda_{XY}Z, X, \Lambda_{XY}Z) = 1$. Linearizing (v), we have

$$\Lambda_{X\Lambda_{YZ}V}W + \Lambda_{V\Lambda_{YZ}X}W = 2g(X, V)\Lambda_{YZ}W \tag{4}$$

for arbitrary vector fields. Thus in the present case, by (4) and (vi),

$$\Lambda_{XW} = \Lambda_{X\Lambda_{XY}Z} = -\Lambda_{Z\Lambda_{XY}X} = -\Lambda_{ZY} = \Lambda_{XY}\Lambda_{XZ}$$

and

$$\underline{\Lambda}(X, W, X, W) = g(\Lambda_{XY}\Lambda_{XZ}X, \Lambda_{XY}Z) = g(\Lambda_{XZ}X, Z) = 1,$$

as desired.

It is also easy to check that W is orthogonal to Y and Z, so that $\dim[S_X]_m \geq 3$. In fact, $\dim[S_X]_m = 3$, as can be easily seen in various ways; we shall show that $\Lambda_{XV} = 0$ for V orthogonal to X, Y, Z, and W. By equation (2), Λ_{XV} anticommutes with each of Λ_{XY}, Λ_{XZ}, Λ_{XW}; but $\Lambda_{XW} = \Lambda_{XY}\Lambda_{XZ}$, so Λ_{XV} commutes with Λ_{XW}. Therefore, $\Lambda_{XW}\Lambda_{XV} = 0$, but Λ_{XW} is nonsingular, so $\Lambda_{XV} = 0$. Moreover, if V is a unit vector field orthogonal to X, Y, Z, and W, it is straightforward to check that $\{\Lambda_{XY}V, \Lambda_{XZ}V, \Lambda_{XW}V\}$ is an orthonormal triple in S_V.

Finally, we prove the formula (3). It suffices to give the proof for A and B belonging to an orthonormal basis. In view of the preceding paragraph, if $B \notin S_A$, both sides of (3) vanish. Also, if $B \in S_A$, then

$$B = g(\Lambda_{XY} A, B)\Lambda_{XY} A + g(\Lambda_{XZ} A, B)\Lambda_{XZ} A + g(\Lambda_{XW} A, B)\Lambda_{XW} A.$$

Putting this into $\Lambda_{AB} X$ and using (v), we obtain

$$\Lambda_{AB} X = g(\Lambda_{XY} A, B) Y + g(\Lambda_{XZ} A, B) Z + g(\Lambda_{XW} A, B) W.$$

Using (v) again, we have

$$\begin{aligned}\Lambda_{AB} C &= \Lambda_{X\Lambda_{AB}X} C \\ &= g(\Lambda_{XY} A, B)\Lambda_{XY} C + g(\Lambda_{XZ} A, B)\Lambda_{XZ} C + g(\Lambda_{XW} A, B)\Lambda_{XW} C,\end{aligned}$$

completing the proof.

Note that Λ is not unique on M. For example, let $\mathscr{I}$, $\mathscr{J}$, and $\mathscr{K}$ denote the following matrices, respectively:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

On $M = R^8$, we can define triples of almost complex structures $\{I, J, K\}$ and $\{I', J', K'\}$ by the following matrices:

$$I = \begin{pmatrix} \mathscr{I} & 0 \\ 0 & \mathscr{I} \end{pmatrix}, \quad J = \begin{pmatrix} \mathscr{J} & 0 \\ 0 & \mathscr{J} \end{pmatrix}, \quad K = \begin{pmatrix} \mathscr{K} & 0 \\ 0 & \mathscr{K} \end{pmatrix},$$

$$I' = \begin{pmatrix} \mathscr{I} & 0 \\ 0 & \mathscr{K} \end{pmatrix}, \quad J' = \begin{pmatrix} \mathscr{J} & 0 \\ 0 & \mathscr{K} \end{pmatrix}, \quad K' = \begin{pmatrix} \mathscr{K} & 0 \\ 0 & \mathscr{I} \end{pmatrix}.$$

Now define Λ and Λ' as in equation (1) using $\{I, J, K\}$ and $\{I', J', K'\}$, respectively. Now it is easy to choose vectors X and Y such that $\Lambda_{XY} \neq \Lambda'_{XY}$.

Finally, we note the conditions under which different triples $\{X, Y, Z\}$ give the same local quaternionic structure. For a triple $\{X, Y, Z\}$, denote the pair (or bivector) $X \wedge Y$ by 1, $X \wedge Z$ by 2, and $Y \wedge Z$ by 3.

PROPOSITION. *Let* $\{X, Y, Z\}$ *and* $\{X', Y', Z'\}$ *be ordered orthonormal triples with* $Y, Z \in S_X$ *and* $Y', Z' \in S_{X'}$. *Then* $\Lambda_{XY} = \Lambda_{X'Y'}$, $\Lambda_{XZ} = \Lambda_{X'Z'}$, *and* $\Lambda_{XW} = \Lambda_{X'W'}$, *where* $W = \Lambda_{XY} Z$ *and* $W' = \Lambda_{X'Y'} Z'$, *if and only if* $\underline{\Lambda}(i, j') = \delta_{ij}$.

Proof. The result is almost immediate from equation (3); the only work is involved with the terms involving the W's. We give two of the computations here, the other three being similar:

$$g(\Lambda_{XW} X', Y') = g(\Lambda_{X\Lambda_{XY}Z} X', Y') = -g(\Lambda_{Z\Lambda_{XY}X} X', Y') = \underline{\Lambda}(3, 1');$$

$$g(\Lambda_{XW}X', W') = g(\Lambda_X \Lambda_{XY} {}_Z X', \Lambda_{X'Y'Z'}) = -g(\Lambda_{ZY}X', \Lambda_{X'Y'}Z')$$

$$= -g(\Lambda_{X'}\Lambda_{X'Y'} {}_{Z'} Z, Y) = \underline{\Lambda}(3, 3') .$$

For example, if X' is a unit vector field orthogonal to X, Y, Z, and W, $\{X', \Lambda_{XY}X', \Lambda_{XZ}X'\}$ is a triple related to $\{X, Y, Z\}$ as in the proposition.

Remarks. In [5], M. Obata announced the following theorem.

THEOREM. *Let* (M, g) *be a complete, connected, simply connected Riemannian manifold. Then* M *admits a nontrivial solution* f *of*

$$(\nabla\nabla\omega)(Z, X, Y) + k(2\omega(Z)g(X, Y) + \omega(X)g(Y, Z) + \omega(Y)g(X, Z)) = 0 ,$$

where $\omega = df$ *and* k *is a positive constant, if and only if* M *is globally isometric to a Euclidean sphere of radius* $k^{-1/2}$.

Here ∇ denotes the Riemannian connection of g and

$$(\nabla\nabla\omega)(X, Y, Z) = (\nabla_X\nabla_Y\omega)(Z) - (\nabla_{\nabla_X Y}\omega)(Z) .$$

The important thing to note is that the vector field grad f is an infinitesimal projective transformation on the sphere (see *e.g.* [5]).

The corresponding result for quaternionic projective space was obtained by Maeda [4].

THEOREM. *Let* (M, g) *be a complete, connected quaternionic Kähler manifold of dimension* $4n \geq 8$. *Then* M *admits a nontrivial solution* f *of*

$$(\nabla\nabla\theta)(Z, X, Y) + k(2\theta(Z)g(X, Y) + \theta(X)g(Y, Z) + \theta(Y)g(X, Z)$$

$$- \theta(\Lambda_{ZY}X) - \theta(\Lambda_{ZX}Y)) = 0 ,$$

where $\theta = df$ *and* k *is a positive constant, if and only if* M *is globally isometric to a quaternionic projective space with metric induced from the metric of constant curvature* k *on the* $(4n+3)$*-sphere by the Hopf fibration* $\pi: S^{4n+3} \to PH^n$.

A solution to this equation on PH^n may be found by choosing a projectable solution to the equation of Obata on S^{4n+3} and projecting it to PH^n; it can be shown that such a solution exists and that its projection satisfies the equation of Maeda (see [4] for details).

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