THE GENUS OF A CLOSED SIMPLY CONNECTED MANIFOLD

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Let NH be the homotopy category of nilpotent CW-complexes [8]. If $X \in NH$ has finite type (*i.e.*, all its homotopy groups are finitely generated), then define the genus $G(X)$ of X to be the collection of all objects of finite type $Y \in NH$ such that Y_p is homotopy equivalent to X_p for all primes p. Here X_p is the p-localization of X [8, 15]. A homotopy-theoretic property is said to be *generic* if it is shared by all or none of the members of a genus.

In [8], Hilton, Mislin, and Roitberg prove that "being a Poincaré duality space" and "being S-reducible" are both generic properties. This leads them to ask whether "having the homotopy type of a closed manifold" and "having the homotopy type of a closed π-manifold" are generic properties. This paper gives a partial answer to these questions in the simply connected case. The main results are:

THEOREM A. *Let* M^m *be a closed, simply connected, piecewise linear (topological) manifold of dimension* $m \geq 5$, *and let* $X \in G(M)$. *Then* X *is the homotopy type of a closed, piecewise linear (topological) manifold.*

THEOREM B. *Let* M^m *be a closed, simply connected, smooth* (C^∞) *manifold, with* $m \geq 5$ *and* m *odd, and let* $X \in G(M)$. *Then* X *is the homotopy type of a closed smooth manifold.*

THEOREM C. *There exists a closed, simply connected, smooth manifold* B^8 *and a homotopy type* $X \in G(B)$ *such that* X *is not the homotopy type of a closed smooth manifold.*

THEOREM D. *Let* M^m *be a closed, simply connected, smooth* π*-manifold with* $m \geq 5$ *and* $m \neq (2^i - 2)$, *and let* $X \in G(M)$. *Then* X *is the homotopy type of a closed smooth* π*-manifold.*

The proofs of Theorems A, B, and D follow roughly the same plan. A space in the genus of a closed manifold is shown to be a Poincaré duality space whose Spivak normal fibration can be given the structure of an R^m-bundle. This reduces the theorems to a problem of calculating surgery obstructions. The calculation is quite easy in the cases of Theorems A and B, and in that part of Theorem D when $m \not\equiv 2 \pmod 4$. When $m \equiv 2 \pmod 4$, Brown's version of the Kervaire invariant is used to examine the appropriate obstruction.

Section 1 carries out the first part of the program by making minor modifications in the techniques developed by Peter Kahn [10] and independently by the author [1] to examine the mixing of homotopy types of manifolds (see [8, Section II.7] for definitions). In genus questions, one is looking at a space whose p-localizations all agree with the p-localizations of a given space. In mixing questions, one is looking at a space whose localizations agree with those of one space for a given set of primes, and with those of a second space for the complementary set of primes. The similarity of the theorems about mixing in [10] and [1], and those about the genus in this paper, reflects the similarity of these situations. The analysis of the Kervaire

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invariant in Section 4 which is needed to prove Theorem D bears this same similarity to the corresponding work in [1], but is not found in [10].

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1. SPHERICAL FIBRATIONS AND NORMAL SITUATIONS

The notation of [8] for localizations in homotopy theory and group theory will be used throughout this paper. If $X \in NH$, then X_p denotes its p-localization (p is a prime or zero). There are canonical maps $X \to X_p$ and $X_p \to X_0$. If $f: X \to Y$ is a map in NH, then $f_p: X_p \to Y_p$ will be the map making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X_p & \xrightarrow{f_p} & Y_p \end{array}$$

commute up to homotopy.

We will need the following three results, which can be found in [8], [15], and [8], respectively.

THEOREM 1.1. *Let* W *be a connected finite* CW*-complex, and let* X *be a connected nilpotent* CW*-complex of finite type. Then the set of pointed homotopy classes* [W, X] *is the pull-back of the diagram of sets*

$$\{[W, X_p] \to [W, X_0]: p \textit{ is a prime}\}.$$

THEOREM 1.2. *Let* X *be a homotopy commutative* H*-space. Then the group-valued functor* $[\cdot, X] \otimes \mathbb{Z}_p$ *is classified by the space* X_p*; that is, for any* CW*-complex* W, $[W, X_p] \cong [W, X] \otimes \mathbb{Z}_p$ *where* $\mathbb{Z}_p = \{x \in \mathbb{Q}: x = a/b \textit{ with } (b, p) = 1\}$.

THEOREM 1.3. *Let* G *be a finitely generated abelian group, and let* G_p *denote* $G \otimes \mathbb{Z}_p$*, the* p*-localization of* G*. Then the canonical map* $G \to \prod_p (G_p)$ *is injective.*

Let BSG(k) be the classifying space for oriented S^{k-1}-fibrations, and let BSO(k) be the classifying space for oriented k-dimensional vector bundles. Let BSG and BSO be the classifying spaces for the corresponding stable theories. A CW-complex will be called a *Poincaré duality complex* if there is a class $[X] \in H_n(X; \mathbb{Z})$ such that $[X] \cap: H^q(X; \mathbb{Z}) \to H_{n-q}(X; \mathbb{Z})$ is an isomorphism for all q. By a theorem of Spivak [14], if X is a simply connected Poincaré duality complex, then for large k, there is an S^{k-1}-fibration ν_X, $\pi: E \to X$, whose Thom space $T(\nu) = X \cup_\pi c(E)$ is reducible. This fibration is unique up to fibre homotopy equivalence. For the most general version of this theorem, see [4].

Now let $X \in NH$ be a finite Poincaré duality complex, and suppose $Y \in G(X)$. From [8], we know that Y is a finite Poincaré duality complex. Let ν^k be the Spivak fibration for X, with k chosen large enough so that the map

$$X \xrightarrow{\nu} BSG(k) \longrightarrow BSG(k)_0$$

is homotopically trivial. This is possible since the homotopy groups of BSG are all finite, so BSG_0 is contractible. Here the fibration ν is equated with its classifying map.

We can identify $[Y, BSG(k)_p]$ with $[Y_p, BSG(k)_p]$ since $BSG(k)_p$ is a p-local space [8]. Thus by Theorem 1.1, $[Y, BSG(k)]$ can be identified with the pull-back of the diagram

$$\{[Y_p, BSG(k)_p] \to [Y_0, BSG(k)_0]: p \text{ is a prime}\}. \tag{1.4}$$

For each prime p, let $f(p): Y_p \to X_p$ be a homotopy equivalence. If ν is the fibration $\pi: E \to X$, then let ν_p be the fibration $\pi_p: E_p \to X_p$, with fibre S_p^{k-1}. Since ν_0 is trivial, $\{f(p)^*\nu_p\} \in \prod_p ([Y_p, BSG(k)_p])$ can be thought of as an element of the pull-back of (1.4). Let θ be the element of $[Y, BSG(k)]$ identified with $\{f(p)^*\nu_p\}$ by Theorem 1.1.

PROPOSITION 1.5. *θ is a Spivak fibration for* Y.

Proof. If ξ is any S^{k-1}-fibration $\pi: E \to B$, and ξ_p is the corresponding S_p^{k-1}-fibration $\pi_p: E_p \to B_p$, then it is easy to see that $(T(\xi))_p$ is homotopy equivalent to $T(\xi_p)$. By the way θ was constructed, $(T(\theta))_p$ is homotopy equivalent to $(T(\nu))_p$, so that $T(\theta) \in G(T(\nu))$. But since ν is a Spivak fibration for X, $T(\nu)$ is reducible and so we know that $T(\theta)$ is reducible [8]. Spivak's theorem then says that θ is the desired fibration.

From now on, we will assume that all spaces are simply connected unless otherwise indicated. Let M^m be a closed smooth manifold and let $Y \in G(M)$. Let $\tau(M) \in [M, BSO]$ be the oriented stable tangent bundle to M, and let $\nu(M) = -\tau(M)$ be the oriented stable normal bundle. If ρ: BSO $\to$ BSG is the canonical map induced by "deleting the zero-section", then $\rho_*(\nu(M))$ is a stable Spivak fibration for M.

PROPOSITION 1.6. *If $\theta \in [Y, BSG]$ is a stable Spivak fibration for* Y, *then there exists a stable vector bundle $\xi \in [Y, BSO]$ such that $\rho_*(\xi) = \theta$.*

LEMMA 1.7. *In the following diagram of abelian groups, the rows are exact and the vertical maps are injective:*

$$\begin{array}{ccccccc} \longrightarrow [Y, BSO] & \xrightarrow{\rho_*} & [Y, BSG] & \xrightarrow{\sigma_*} & [Y, B(G/O)] \longrightarrow \\ \downarrow \ell & & \downarrow \ell & & \downarrow \ell \\ \longrightarrow \prod_p ([Y_p, BSO_p]) & \xrightarrow{\bar\rho_*} & \prod_p ([Y_p, BSG_p]) & \xrightarrow{\bar\sigma_*} & \prod_p ([Y_p, (B(G/O))_p]) \longrightarrow. \end{array} \tag{1.8}$$

Here $\bar\rho_* = \prod_p ((\rho_p)_*)$, $\ell = \prod_p \ell_p$, *etc.*

Proof. The top row is the standard Puppe sequence. By Theorem 1.2, ℓ is essentially the product of the p-localization maps for abelian groups, and so the bottom row is exact because localization is an exact functor on abelian groups. The injectivity of the vertical maps follows from Theorems 1.2 and 1.3.

Proof of Proposition 1.6. From Proposition 1.5,

$$\ell(\theta) = \{\theta_p\} = \{f(p)^*(\rho_{p*}(\nu(M)_p))\} = \bar\rho_*(\{f(p)^*(\nu(M)_p)\}).$$

Thus $\bar{\sigma}_*\ell(\theta) = \bar{\sigma}_*\bar{\rho}_*(\{f(p)^*(\nu(M)_p)\}) = 0$, by exactness. Hence, $\ell\sigma_*(\theta) = 0$. But ℓ is injective, so $\sigma_*(\theta) = 0$, which implies, again by exactness, that $\theta = \rho_*(\xi)$ for some $\xi \in [Y, BSO]$.

This then shows that any space in the genus of a closed smooth manifold is a Poincaré duality complex whose Spivak fibration admits the stronger structure of a vector bundle. Theorem B then follows from the surgery theorem of Browder-Novikov [3]. If BSO is replaced by BSPL or BSTOP and we consider piecewise linear or topological manifolds, then we can get analogous results on strengthening the Spivak fibration. Theorem A will then follow from the surgery theorem of Browder-Hirsch [5].

2. POINCARÉ DUALITY COMPLEXES WITH THREE CELLS

In this section, we will be considering 3-connected spaces which are 8-dimensional Poincaré duality complexes. If X is such a space, then it is easy to see that $H^*(X; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^3)$, where x is a 4-dimensional class. Thus X has the same cohomological structure as quaternionic projective space. Spaces of this type have been studied by Eells-Kuiper [7], Sasao [13] and Kahn [10].

It is easy to see that X must have the homotopy type of $S^4 \cup_\gamma e^8$, where $\gamma \in \pi_7(S^8)$. It is well known [9, p. 329] that $\pi_7(S^8) \cong \mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$, where h, the class of the Hopf map, is a generator of the infinite cyclic summand. Let τ be the element of order 12 in $\pi_7(S^8)$ which satisfies the relation $2h + \tau = [\iota_4, \iota_4]$, where $\iota_4 \in \pi_4(S^4)$ is the class of the identity. Denote by $X_{(a,b)}$ the complex $S^4 \cup_{ah+b\tau} e^8$, where a is an integer and b is an integer mod 12. If $X_{(a,b)}$ is to satisfy Poincaré duality, then $ah + b\tau$ must be a map of Hopf invariant ± 1, so $a = \pm 1$.

Suppose f is a map from $X_{(a,b)}$ to $X_{(c,d)}$. Then the degree of f is defined as usual. The 4-skeleta of the spaces in question can be identified with S^4, and their 4-dimensional homology is isomorphic to $\mathbb{Z}$. The map induced by f on $H_4(\cdot\,; \mathbb{Z})$ is then multiplication by some integer m. Call this integer the *4-degree* of f.

PROPOSITION 2.1 ([13]). *There exists a map* f *from* $X_{(a,b)}$ *to* $X_{(c,d)}$ *of 4-degree* m *and degree* s *if and only if:*

(i) $am^2 = sc$; *and*

(ii) $am(m - 1)/2 + mb \equiv sd \pmod{12}$.

For the rest of this section, we will restrict ourselves to the cases where the attaching map γ of $S^4 \cup_\gamma e^8$ is $\pm h + b\tau$. Then in Proposition 2.1, $a^2 = c^2 = 1$, and the conditions can be written as:

$$(2.2)\quad \begin{aligned} &\text{(i) } s = acm^2; \textit{ and} \\ &\text{(ii) } m(2ab - 1) \equiv m^2(2cd - 1) \pmod{24}. \end{aligned}$$

Clearly, f is a homotopy equivalence if and only if $m = \pm 1$. Using this fact and (2.2), it can be shown that the complexes $X_{(\pm 1,b)}$ break up into six different homotopy types, which will be denoted by I, II, $\cdots$, VI. They are listed in Table 2.3.

To decide when $(X_{(\pm 1,b)})_p$ is homotopy equivalent to $(X_{(\pm 1,d)})_p$, recall that all the spaces $X_{(\pm 1,b)}$ are p-universal for every prime p (see [12]).

Homotopy type	Values of (a, b) for which $X_{(a,b)}$ belongs to homotopy type				(2ab - 1) (mod 24)
I	(1, 1)	(1, 0)	(-1, 11)	(-1, 0)	± 1
II	(1, 2)	(1, 11)	(-1, 10)	(-1, 1)	± 3
III	(1, 3)	(1, 10)	(-1, 9)	(-1, 2)	± 5
IV	(1, 4)	(1, 9)	(-1, 8)	(-1, 3)	± 7
V	(1, 5)	(1, 8)	(-1, 7)	(-1, 4)	± 9
VI	(1, 6)	(1, 7)	(-1, 6)	(-1, 5)	± 11

Table 2.3

PROPOSITION 2.4 ([11, Corollary 5.4]). *If* p *is a prime and* A *and* B *are* p-*universal complexes, then* A_p *is homotopy equivalent to* B_p *if and only if there is a map* $g: A \to B$ *which induces isomorphisms on homology with* $\mathbb{Z}/p\mathbb{Z}$ *coefficients.*

LEMMA 2.5. *If* $f: X_{(\pm 1,b)} \to X_{(\pm 1,d)}$ *has* 4*-degree* m, *then* f *induces isomorphisms on homology with* $\mathbb{Z}/p\mathbb{Z}$ *coefficients if and only if* p *and* m *are relatively prime.*

Proof. The only nonzero homology groups are in dimensions 0, 4, and 8; in those dimensions, f induces multiplication by 1, m, and $\pm m^2$, respectively.

Table 2.6 lists some of the possible 4-degrees of maps between complexes $X_{(a,b)}$ and $X_{(c,d)}$. The table may be verified by (2.2).

(a, b)	(c, d)	Some possible 4-degrees of $f: X_{(a,b)} \to X_{(c,d)}$
(1, 1)	(1, 3)	5, 29
(1, 3)	(1, 4)	11, 35
(1, 1)	(1, 6)	11, 35
(1, 2)	(1, 8)	5, 29

Table 2.6

PROPOSITION 2.7. *Homotopy types* I, III, IV, *and* VI *belong to* $G(X_{(1,1)})$, *and homotopy types* II *and* V *belong to* $G(X_{(1,2)})$.

Proof. The proof is immediate from Proposition 2.4, using Lemma 2.5 and the maps of Table 2.6.

PROPOSITION 2.8. $G(X_{(1,1)}) \neq G(X_{(1,2)})$.

Proof. Suppose X is a complex from homotopy type II and Y is a complex from homotopy type I, III, IV, or VI such that X_3 is homotopy equivalent to Y_3. Then by (2.4) and (2.5), there is a map $g: X \to Y$ of 4-degree m, where $(m, 3) = 1$. By (2.2) and (2.3), this means that

$$\pm 3 \equiv \pm km^2 \pmod{24}, \quad \text{where } k = 1, 5, 7, \text{ or } 11.$$

This is impossible since 3 does not divide m. Thus X and Y cannot belong to the same genus.

PROPOSITION 2.9 ([7], [13]). *The homotopy types* I, IV, *and* V *can be represented by a smooth manifold, and the homotopy types* II, III, *and* VI *cannot.*

To prove this proposition, one notes that all the homotopy types may be represented by the Thom spaces of orthogonal S^3-bundles over S^4 and hence by closed piecewise linear manifolds. An analysis of the Pontrjagin classes of these bundles and manifolds yields the result.

Now let B^8 be a smooth manifold in homotopy type I, and let Y be a complex in homotopy type III or VI. By (2.7), $Y \in G(B)$; but by (2.9), Y cannot have the homotopy type of a smooth manifold. This proves Theorem C. More examples of this form can be obtained by further use of (2.7) and (2.9). A similar program can be carried out with complexes of the form $S^8 \cup e^{16}$.

3. π-MANIFOLDS

Let M^m be a closed smooth π-manifold, and let $Y \in G(M)$. Choose $k > m$ large enough so that the normal bundle of M in S^{m+k} is trivial. Then by the arguments of Section 1, the Spivak fibration θ for Y (of fibre dimension k - 1) will be fibre homotopy trivial. Thus the trivial bundle ε^k over Y will be fibre homotopy equivalent to θ. This equivalence can be used to produce a normal situation

(3.1)
$$\begin{array}{ccc} \nu_N & \xrightarrow{b} & \varepsilon \\ \downarrow & & \downarrow \\ N & \xrightarrow{f} & Y, \end{array}$$

where N is a closed smooth manifold, ν_N is the normal bundle of N in S^{m+k}, and f is a degree-one map [3].

The Browder-Novikov theorem then implies that Y has the homotopy type of a π-manifold if m is odd and $m \geq 5$. If $m \equiv 0 \pmod 4$ and $m \geq 5$, then the obstruction to making f into a homotopy equivalence is $(\langle \mathscr{L}(\varepsilon), [Y] \rangle - I(Y))/8$, where $\mathscr{L}(\varepsilon)$ is the inverse Hirzebruch polynomial in the Pontrjagin classes of ε and I(Y) is the classical index or signature of Y. Since Y_p is homotopy equivalent to M_p, then Y_0 is homotopy equivalent to M_0. Thus Y and M have isomorphic rational cohomology algebras and I(Y) = I(M). I(M) = 0 by the Hirzebruch formula, since M is a π-manifold and $\langle \mathscr{L}(\varepsilon), [Y] \rangle = 0$, so the obstruction vanishes.

As observed in [8], these facts prove Theorem D when $m \not\equiv 2 \pmod 4$. The remaining cases of Theorem D will be dealt with in the next section.

4. THE KERVAIRE INVARIANT AND TRIVIAL FIBRATIONS

In this section, all homology and cohomology will be with $\mathbb{Z}/2\mathbb{Z}$ coefficients unless otherwise indicated. K_n will denote an Eilenberg-MacLane space $K(\mathbb{Z}/2\mathbb{Z}, n)$. Let $\mathscr{W}(n)$ denote the spectrum $\{W_\ell(n)\}$, where $W_\ell(n)$ is the total space of the fibration over K_ℓ with fibre $K_{n+\ell}$ and k-invariant $\chi(Sq^{n+1})\iota_\ell$. Here

$$\iota_\ell \in H^\ell(K_\ell) \cong \mathbb{Z}/2\mathbb{Z}.$$

A $\mathscr{W}(n)$-*oriented Poincaré duality complex* of dimension m will be a quadruple (X, ξ, α, W) where:

(i) X is a Poincaré duality complex of dimension m;

(ii) ξ^k is a Spivak fibration for X;

(iii) $\alpha \in \pi_{m+k}(T(\xi))$ satisfies $H(\alpha) \cap U_\xi = [X]$, where H is the Hurewicz homomorphism and $U_\xi \in H^k(T(\xi); \mathbb{Z})$ is the Thom class of ξ;

(iv) $W: T(\xi) \to W_k(n)$ is a map such that $W^*: H^k(W_k(n)) \to H^k(T(\xi))$ is an isomorphism.

If $m = 2n$, a *quadratic form* on $H^n(X)$ will be a map $\phi: H^n(X) \to \mathbb{Z}/4\mathbb{Z}$ satisfying

$$\phi(u + v) = \phi(u) + \phi(v) + j(\langle u \cup v, [X] \rangle),$$

where $j: \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ is multiplication by 2. From the work of Brown [6], we can associate to any 2n-dimensional $\mathscr{W}(n)$-oriented Poincaré duality complex (X, ξ, α, W) a quadratic form $\phi(X, \xi, \alpha, W)$ on $H^n(X)$. Brown also defines an invariant σ which maps nonsingular quadratic forms of this kind into $\mathbb{Z}/8\mathbb{Z}$. Following [6], let $K(X, \xi, \alpha, W)$ denote $\sigma(\phi(X, \xi, \alpha, W)) \in \mathbb{Z}/8\mathbb{Z}$.

Let X be a Poincaré duality complex of dimension m, and let the total Wu class $V(X) = \sum_{i \geq 0} v_i(X)$ be defined by $\langle Sq(u), [X] \rangle = \langle V(X) \cup u, [X] \rangle$ for all $u \in H^*(X)$. It can be shown [3] that $v_i(X) = 0$ when $2i > m$. Furthermore, if ξ^k is a Spivak fibration for X and $m = 2n$, then [3] $v_{n+1}(X) \cup U_\xi = \chi(Sq^{n+1}) U_\xi$. Thus $\chi(Sq^{n+1}) U_\xi = 0$, and so there exists at least one $\mathscr{W}(n)$-orientation for ξ in this case.

The effect on the quadratic form $\phi(X, \xi, \alpha, W)$ of a change in the homotopy class α or in the orientation W is studied in [6]. The results needed here are summarized in the next three propositions, whose proofs may be found in [6].

If $W, W': T(\xi) \to W_k(n)$ are $\mathscr{W}(n)$-orientations, then they differ by a map from $T(\xi)$ into K_{n+k}; that is, by a class $d(W, W') \in H^{n+k}(T(\xi)) \cong H^n(X)$.

PROPOSITION 4.1 ([6]). *If* $\phi = \phi(X, \xi, \alpha, W)$ *and* $\phi' = \phi(X, \xi, \alpha, W')$, *then* $\phi(u) = \phi'(u) + j(\langle u \cup d(W, W'), [X] \rangle)$.

Let (X, ξ, α, W) and (X', ξ', α', W') be 2n-dimensional $\mathscr{W}(n)$-oriented Poincaré duality complexes. Let

$$\begin{array}{ccc} \xi & \xrightarrow{g} & \xi' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & X' \end{array}$$

be a map of fibre spaces. Let $\phi = \phi(X, \xi, \alpha, W)$ and $\phi' = \phi(X', \xi', \alpha', W')$.

PROPOSITION 4.2 ([6]). *If* $W = T(g)^* W'$ *and* $T(g)_* \alpha = \alpha'$, *then* $\phi(f^* u) = \phi'(u)$ *for all* $u \in H^n(X')$.

Let (X, ξ, α, W) and (X, ξ, α', W) be 2n-dimensional $\mathscr{W}(n)$-oriented Poincaré duality complexes. By the uniqueness part of Spivak's theorem, there is a fibre homotopy equivalence $g: \xi \to \xi$ such that $T(g)_* \alpha = \alpha'$. Then by Proposition 4.2, $\phi(X, \xi, \alpha, T(g)^* W) = \phi(X, \xi, \alpha', W)$. Thus, to compare $\phi(X, \xi, \alpha, W)$ and $\phi(X, \xi, \alpha', W)$, it suffices to compute $d(W, T(g)^* W)$.

PROPOSITION 4.3 ([6]). *There are classes* $z_i \in H^{2^i - 1}(X)$ *such that* $d(W, T(g)^* W) = \sum_i v_{n-(2^i-1)}(X) \cup z_i$.

If these formulae are applied when ξ is fibre homotopy trivial, then it can be shown that, to a certain extent, $K(X, \xi, \alpha, W)$ is independent of α and W in this case.

PROPOSITION 4.4. *Let* ε^k, $\pi: X \times S^{k-1} \to X$, *be the trivial* S^{k-1}*-fibration over a Poincaré duality complex of dimension* $2n$. *Then there is a canonical* $\mathcal{W}(n)$*-orientation* $W_X: T(\varepsilon) \to W_k(n)$ *which is natural with respect to maps of the base space.*

Proof. There is a map of fibre spaces

$$\begin{array}{ccc} X \times S^{k-1} & \xrightarrow{\hat{c}} & * \times S^{k-1} \\ \downarrow & & \downarrow \\ X & \xrightarrow{c} & * \end{array}$$

If ε_*^k denotes the S^{k-1}-fibration over a point, then it is well known that $T(\varepsilon_*^k)$ is homotopy equivalent to S^k and $T(\varepsilon^k)$ is homotopy equivalent to $\Sigma^k(X^+)$, where Σ denotes suspension and X^+ is X with a disjoint base point. Let $\overline{W}: S^k \to W_k(n)$ be a map representing the nonzero homotopy class in dimension k. Then $W_X = T(\hat{c})^*\overline{W}$ is the desired orientation.

Let X be a $2n$-dimensional Poincaré duality complex whose Spivak fibration ξ^k is fibre homotopy trivial. A $\mathcal{W}(n)$-orientation W for ξ is said to be induced by a trivialization if there exists a fibre homotopy equivalence $\rho: \xi \to \varepsilon^k$ such that $W = T(\rho)^*W_X$.

LEMMA 4.5. *Let* (X, ξ, α, W) *and* (X, ξ, α, W') *be* $(4\ell + 2)$*-dimensional* $\mathcal{W}(2\ell + 1)$*-oriented Poincaré duality complexes with* ξ *fibre homotopy trivial. Suppose further that* $(2\ell + 1) \neq (2^j - 1)$ *for any* j *and that both* W *and* W' *are induced by trivializations. Then* $K(X, \xi, \alpha, W) = K(X, \xi, \alpha, W')$.

Proof. Let $\rho, \rho': \xi^k \to \varepsilon^k$ be trivializations such that $T(\rho)^*W_X = W$ and $T(\rho')^*W_X = W'$. Then $\rho'\rho^{-1}$ is a fibre homotopy equivalence of ε^k, and Proposition 4.3 implies that

$$d(W_X, T(\rho'\rho^{-1})^*W_X) = \sum_j v_{(2\ell+1)-(2^j-1)}(X) \cup z_j .$$

But since ξ is fibre homotopy trivial, $v_i(X) = 0$ for all $i \neq 0$; and since $(2\ell + 1) \neq (2^j - 1)$, this case never occurs. Thus $d(W_X, T(\rho'\rho^{-1})^*W_X) = 0$. Hence, by Proposition 4.2,

$$\begin{aligned} K(X, \xi, \alpha, T(\rho)^*W_X) &= K(X, \xi, T(\rho)_*\alpha, W_X) \\ &= K(X, \xi, T(\rho)_*\alpha, T(\rho'\rho^{-1})^*W_X) = K(X, \xi, \alpha, T(\rho')^*W_X) . \end{aligned}$$

Now recall the following result of Browder, which also deals with orientations coming from trivializations.

THEOREM 4.6 ([2]). *Let* M^{4k+2} *be a smooth closed manifold with trivial normal bundle* ν^ℓ. *Let* $\beta \in \pi_{4k+2+\ell}(T(\nu))$ *be the class of the Thom collapse, and let* W *be any* $\mathcal{W}(2k + 1)$*-orientation for* ν *induced by a trivialization of* ν. *Then if* $(4k + 2) \neq (2^j - 2)$, $K(M, \nu, \beta, W) = 0$.

If (X, ξ, α, W) is a 2n-dimensional $\mathcal{W}(n)$-oriented Poincaré duality complex, then it is relatively easy to "2-localize" Brown's program to produce a quadratic form $\phi(X_2, \xi_2, \alpha_2, W_2)$ on $H^n(X_2)$. X_2 is the 2-localization of X, ξ_2 is the S_2^{k-1}-fibration obtained by 2-localizing ξ as in Section 1, $\alpha_2 \in \pi_{2n+k}(T(\xi_2))$ is the 2-localization of α, and W_2: $T(\xi_2) \to W_k(n)$ is the 2-localization of W. The last statement makes sense because $W_k(n)$ is already a 2-local space in the sense of [8], so $(W_k(n))_2$ is homotopy equivalent to $W_k(n)$. It can also be shown that $K(X_2, \xi_2, \alpha_2, W_2) = K(X, \xi, \alpha, W)$. Essentially, the point is that Brown's program depends only on mod 2 information and can be carried out for spaces which have the same "2-type" as a $\mathcal{W}(n)$-oriented Poincaré duality space. Details of this may be found in [1].

Suppose that

$$\text{(4.7)}\qquad \begin{array}{ccc} \nu_N & \xrightarrow{b} & \xi \\ \downarrow & & \downarrow \\ N^{4k+2} & \xrightarrow{f} & Y \end{array}$$

is a normal situation; that is, Y is a $(4k+2)$-dimensional Poincaré duality complex, ξ^{ℓ} is a vector bundle over Y, N^{4k+2} is a smooth manifold, and ν_N is the normal bundle of N in $S^{4k+2+\ell}$. Let $\beta \in \pi_{4k+2+\ell}(T(\nu_N))$ be the class of the Thom collapse, and let W be any $\mathcal{W}(2k+1)$-orientation for ξ.

LEMMA 4.8 ([6]). *If* $k > 0$, *then the obstruction to making* (4.7) *cobordant to a homotopy equivalence is*

$$c(f, b) = K(N, \nu_N, \beta, T(b)^*W) - K(Y, \xi, T(b)_*\beta, W).$$

Now we can complete the proof of Theorem D. Let M^{4k+2} be a smooth closed π-manifold, $(4k+2) \neq (2^j - 2)$, $k > 0$. Suppose $Y \in G(M)$. From the considerations of Section 3, we can produce a normal situation

$$\begin{array}{ccc} \nu_N & \xrightarrow{b} & \varepsilon \\ \downarrow & & \downarrow \\ N & \xrightarrow{f} & Y\,. \end{array}$$

The obstruction to making f into a homotopy equivalence is

$$c(f, b) = K(N, \nu_N, \beta, T(b)^*W_Y) - K(Y, \varepsilon, T(b)_*\beta, W_Y).$$

By Theorem 4.6, $K(N, \nu_N, \beta, T(b)^*W_Y) = 0$, so $c(f, b) = -K(Y, \varepsilon, T(b)_*\beta, W_Y)$. Also, by Proposition 4.2,

$$\begin{aligned} K(Y, \varepsilon, T(b)_*\beta, W_Y) &= K(Y_2, \varepsilon_2, (T(b)_*\beta)_2, (W_Y)_2) \\ &= K(M_2, \varepsilon_2, T(f(2))_*(T(b)_*\beta)_2, (W_M)_2)\,, \end{aligned}$$

where $f(2)$: $Y_2 \to M_2$ is a homotopy equivalence. Since M is a π-manifold, there is a normal situation

$$\begin{array}{ccc} \nu_M^{\ell} & \xrightarrow{\bar{b}} & \varepsilon^{\ell} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\text{id}} & M \end{array} .$$

Let $\alpha \in \pi_{4k+2+\ell}(T(\nu_M))$ be the class of the Thom collapse. Then, by Proposition 4.3,

$$K(M, \nu_M, \alpha, T(\bar{b})^* W_M) = K(M, \varepsilon, T(\bar{b})_* \alpha, W_M) = K(M_2, \varepsilon_2, (T(\bar{b})_* \alpha)_2, (W_M)_2)$$

$$= K(M_2, \varepsilon_2, T(f(2))_* (T(b)_* \beta)_2, (W_M)_2) = -c(f, b) .$$

But by Theorem 4.6, $K(M, \nu_M, \alpha, T(\bar{b})^* W_M) = 0$. Thus $c(f, b) = 0$, and the proof of Theorem D is complete.

Notice that the assumption $(4k + 2) \neq (2^j - 2)$ is used in two different ways. It is used to apply Brown's formula (4.3), and it is used to apply Browder's theorem (4.6).

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