SUFFICIENT CONDITIONS FOR RANK-ONE COMMUTATORS AND HYPERINVARIANT SUBSPACES

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Let $\mathscr{X}$ be an infinite dimensional complex Banach space, and let $\mathscr{L}(\mathscr{X})$ denote the algebra of all bounded linear operators on $\mathscr{X}$. In an earlier paper [5] (see also [1]), the authors obtained the following extension of the celebrated theorem of V. Lomonosov [6]:

THEOREM A. *Suppose* T *is an operator in* $\mathscr{L}(\mathscr{X})$ *and there exists a nonzero compact operator* K *in* $\mathscr{L}(\mathscr{X})$ *such that the rank of* TK - KT *is less than or equal to one. Then* T *has a nontrivial hyperinvariant subspace.*

(Recall that a subspace $\mathscr{M}$ of $\mathscr{X}$ is a nontrivial hyperinvariant subspace for an operator T in $\mathscr{L}(\mathscr{X})$ if $(0) \neq \mathscr{M} \neq \mathscr{X}$ and $T'\mathscr{M} \subset \mathscr{M}$ for every operator T' in $\mathscr{L}(\mathscr{X})$ that commutes with T.)

The main purpose of this note is to obtain some results concerning the size of the class of operators to which Theorem A applies. In particular, let $\Delta(\mathscr{X})$ denote the set of all those operators T in $\mathscr{L}(\mathscr{X})$ with the property that there exists a compact operator K such that the rank of TK - KT is *equal to* one. The interest in the class $\Delta(\mathscr{X})$ derives, of course, from Theorem A. It turns out that $\Delta(\mathscr{X})$ is quite large, and in particular, if $\mathscr{X}$ is a separable, infinite dimensional Hilbert space $\mathscr{H}$, we are presently unable to exhibit any nonscalar operator in $\mathscr{L}(\mathscr{H})$ that does not belong to $\Delta(\mathscr{H})$. Thus it is conceivable that the hyperinvariant subspace problem for (separable) Hilbert space can be settled affirmatively by showing that $\Delta(\mathscr{H}) = \mathscr{L}(\mathscr{H}) \setminus \{\lambda\}$.

If $\mathscr{X}$ is, once again, an arbitrary infinite dimensional complex Banach space, and if $f \in \mathscr{X}$ and $\phi \in \mathscr{X}^*$, we shall write $f \otimes \phi$ for the operator of rank one in $\mathscr{L}(\mathscr{X})$ defined as follows: $(f \otimes \phi)(g) = \phi(g) f$, $g \in \mathscr{X}$. Clearly every operator in $\mathscr{L}(\mathscr{X})$ of rank one has the form $f \otimes \phi$ for some choice of nonzero vectors f in $\mathscr{X}$ and ϕ in $\mathscr{X}^*$. Furthermore, for any T in $\mathscr{L}(\mathscr{X})$, an easy calculation shows that

$$T(f \otimes \phi) - (f \otimes \phi)T = (Tf \otimes \phi) - (f \otimes T^*\phi) .$$

This fact will be used several times in what follows. Finally, the spectrum of an operator T in $\mathscr{L}(\mathscr{X})$ will be denoted by $\sigma(T)$.

We begin with the following elementary proposition whose proof we omit.

PROPOSITION 1. *An operator* T *in* $\mathscr{L}(\mathscr{X})$ *belongs to* $\Delta(\mathscr{X})$ *if and only if* $\alpha T + \beta \in \Delta(\mathscr{X})$ *for all scalars* $\alpha \neq 0$ *and* β. *Furthermore, if* $T \in \Delta(\mathscr{X})$ *and if* $S \in \mathscr{L}(\mathscr{X})$ *and is quasisimilar to* T *(that is, if there exist operators* X *and* Y *in* $\mathscr{L}(\mathscr{X})$ *with trivial kernels and cokernels such that* TX = XS, YT = SY), *then* $S \in \Delta(\mathscr{X})$. *Finally, if* $T \in \Delta(\mathscr{X})$, *then* $T^* \in \Delta(\mathscr{X}^*)$.

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It is presently not clear to the authors whether the converse of the last assertion is true; that is, whether the fact that $T^* \in \Delta(\mathscr{X}^*)$ always implies $T \in \Delta(\mathscr{X})$.

The class $\Delta(\mathscr{X})$ is not closed under the usual algebraic operations. For example, it is not a linear manifold since it fails to contain 0. Furthermore, if T_1 and T_2 belong to $\Delta(\mathscr{X})$, it does not always follow that $T_1 T_2$ belongs to $\Delta(\mathscr{X})$. Indeed, we shall see later that every nonscalar normal operator on the Hilbert space $\mathscr{H}$ belongs to $\Delta(\mathscr{H})$, but there are many such T for which $T^2 = 1$, and thus $T^2 \notin \Delta(\mathscr{H})$. Nonetheless, $\Delta(\mathscr{X})$ has some algebraic properties. For example, if $T \in \Delta(\mathscr{X})$ and $U \in \mathscr{L}(\mathscr{Y})$ for some complex Banach space $\mathscr{Y}$, it is obvious that $T \oplus U \in \Delta(\mathscr{X} \oplus \mathscr{Y})$. The following result is less obvious. (The space $\mathscr{X}$ continues to be an arbitrary infinite dimensional complex Banach space.)

THEOREM 1. *If* $T \in \mathscr{L}(\mathscr{X})$, *if* r *is a rational function with poles off* $\sigma(T)$, *and if* $r(T) \in \Delta(\mathscr{X})$, *then* $T \in \Delta(\mathscr{X})$.

The proof of the theorem depends upon the following lemma.

LEMMA 1. *If* $T, K, R \in \mathscr{L}(\mathscr{X})$ *with* K *compact, and if* p *is a polynomial such that* $p(T)K - Kp(T) = R$, *then there is another compact operator* K' *such that* $TK' - K'T = R$.

Proof. If p is a constant, then $R = 0$ and we may take $K' = 0$. Suppose next that $p(z) = z^{k+1}$ $(k \geq 0)$. Then

$$T^{k+1}K - KT^{k+1} = T\left(\sum_{i=0}^{k} T^{k-i}KT^i\right) - \left(\sum_{i=0}^{k} T^{k-i}KT^i\right)T = TK' - K'T .$$

The general case now follows by addition.

Proof of Theorem 1. By hypothesis, there is a compact operator K and a rank-one operator R such that

$$r(T)K - Kr(T) = R . \tag{1}$$

Let $r(z) = p(z)/q(z)$, where p and q are polynomials and q has no zeros in $\sigma(T)$. Hence $q(T)$ is invertible. From (1) we have $p(T)Kq(T) - q(T)Kp(T) = q(T)Rq(T)$. Let $R' = q(T)Rq(T)$, which also has rank one since $q(T)$ is invertible. Then, writing P and Q for $p(T)$ and $q(T)$, respectively, we have

$$R' = [P(KQ) - (KQ)P] + [Q(-KP) - (-KP)Q] .$$

Applying Lemma 1 to each term in brackets, we obtain

$$R' = (TK' - K'T) + (TK'' - K''T) = T(K' + K'') - (K' + K'')T ,$$

where K' and K'' are compact, and thus the theorem is proved.

The following theorem gives an interesting sufficient condition that an operator be in $\Delta(\mathscr{X})$, and consequently have a nontrivial hyperinvariant subspace.

THEOREM 2. *Suppose* T *is an invertible operator in* $\mathscr{L}(\mathscr{X})$ *and that there exist nonzero vectors* f *in* $\mathscr{X}$ *and* ϕ *in* $\mathscr{X}^*$ *such that*

$$\sum_{k=0}^{\infty} \|T^k f\| \, \|(T^*)^{-k}\phi\| < +\infty .$$

Then $T \in \Delta(\mathscr{X})$.

Proof. If we define $K = \sum_{k=0}^{\infty} T^k f \otimes (T^*)^{-k}\phi$, then the hypothesis guarantees that the series converges in the norm topology of $\mathscr{L}(\mathscr{X})$, and hence that K is a compact operator. Furthermore, a calculation shows that $TK - KT = -f \otimes T^*\phi$. Since T^* is invertible, this latter operator has rank one, as desired.

The hypotheses of Theorem 2 can be relaxed somewhat and still yield the desired conclusion. For example, the series which defines K need only be convergent, not absolutely convergent. The following results have proofs that are essentially the same as that of Theorem 2 and hence their proofs are omitted.

COROLLARY 1. *Suppose* $T \in \mathscr{L}(\mathscr{X})$, $\mathscr{M}$ *is a nonzero subspace of* $\mathscr{X}^*$ *invariant under* T^*, *and* $T_1^* = T^* | \mathscr{M}$ *is invertible. If there exist nonzero vectors* ϕ *in* $\mathscr{M}$ *and* f *in* $\mathscr{X}$ *such that*

$$\sum_{k=0}^{\infty} \|T^k f\| \, \|(T_1^*)^{-k}\phi\| < +\infty ,$$

then $T \in \Delta(\mathscr{X})$.

COROLLARY 2. *Suppose* S *and* T *are operators in* $\mathscr{L}(\mathscr{X})$, $ST = 1$, *and there exist nonzero vectors* f *in* $\mathscr{X}$ *and* ϕ *in* $\mathscr{X}^*$ *such that*

$$\sum_{k=0}^{\infty} \|T^k f\| \, \|(S^*)^k\phi\| < +\infty .$$

Then $T \in \Delta(\mathscr{X})$.

Theorem 2 can also be combined with Theorem 1 to obtain the following result, which will be used in the proof of Theorem 3.

COROLLARY 3. *If* $T \in \mathscr{L}(\mathscr{X})$, r *is a rational function with poles off* $\sigma(T)$ *such that* r(T) *is invertible, and there exist nonzero vectors* ϕ *in* $\mathscr{X}^*$ *and* f *in* $\mathscr{X}$ *such that*

$$\sum_{k=0}^{\infty} \|r(T)^k f\| \, \|(r(T)^*)^{-k}\phi\| < +\infty ,$$

then $T \in \Delta(\mathscr{X})$.

The above results can also be combined to yield another sufficient condition that an operator have a nontrivial hyperinvariant subspace.

COROLLARY 4. *Suppose* $T \in \mathscr{L}(\mathscr{X})$, S *is an invertible operator in the second commutant of* T (*that is,* S *commutes with every operator that commutes with* T), *and there exist nonzero vectors* f *in* $\mathscr{X}$ *and* ϕ *in* $\mathscr{X}^*$ *such that*

$$\sum_{k=0}^{\infty} \|S^k f\| \, \|(S^*)^{-k}\phi\| < +\infty .$$

Then T *has a nontrivial hyperinvariant subspace.*

Proof. The commutant of T is contained in the commutant of S, and by Theorems 2 and A, S has a nontrivial hyperinvariant subspace.

We turn now to results relating to the size of $\Delta(\mathscr{X})$.

PROPOSITION 2. *If* T *is any nonscalar operator in* $\mathscr{L}(\mathscr{X})$ *such that either* T *or* T^* *has an eigenvector, then* $T \in \Delta(\mathscr{X})$.

Proof. Suppose T has an eigenvector $f \neq 0$. By Proposition 1, we may assume that the associated eigenvalue is 0. Choose ϕ in $\mathscr{X}^*$ such that $T^*\phi \neq 0$, and set $K = f \otimes \phi$. Then $TK - KT = -f \otimes T^*\phi$, which has rank one. The proof is similar if T^* has an eigenvector.

COROLLARY 5. *The class* $\Delta(\mathscr{X})$ *is norm-dense in* $\mathscr{L}(\mathscr{X})$.

The proof of this corollary is an immediate consequence of Proposition 2 and the following proposition, which is well known, at least when $\mathscr{X}$ is a Hilbert space (see [4], Problem 8, Exercise 2, pp. 921-922). For completeness, we give the proof in the general case.

PROPOSITION 3. *The operators in* $\mathscr{L}(\mathscr{X})$ *with nonempty point spectrum are dense in* $\mathscr{L}(\mathscr{X})$.

Proof. Let T be an arbitrary operator in $\mathscr{L}(\mathscr{X})$. Since the approximate point spectrum of T contains $\partial\sigma(T)$, and is therefore nonempty, there exist a complex number λ and a sequence $\{f_n\}$ of unit vectors in $\mathscr{X}$ such that $\|(T - \lambda)f_n\| \to 0$. Choose a sequence $\{\phi_n\}$ of linear functionals in $\mathscr{X}^*$ of norm 1 such that $\phi_n(f_n) = 1$, and for each n, let $\mathscr{M}_n$ be the null space of ϕ_n. If g is any vector in $\mathscr{X}$, then for each n, there exists a vector $h_{n,g}$ in $\mathscr{M}_n$ such that $g = \phi_n(g)f_n + h_{n,g}$. In particular, if $g = f_n$, then $h_{n,f_n} = 0$. We now define a sequence $\{T_n\}$ of linear transformations on $\mathscr{X}$ by

$$T_n g = \phi_n(g)\lambda f_n + Th_{n,g} \qquad (g \in \mathscr{X}).$$

Then $\|(T - T_n)g\| = |\phi_n(g)| \; \|(T - \lambda)f_n\|$, $g \in \mathscr{X}$, so that $\|T - T_n\| \leq \|(T - \lambda)f_n\|$. This proves that the T_n are all bounded and at the same time that $\|T - T_n\| \to 0$. Since $T_n f_n = \lambda f_n$ for each n by construction, the proof is complete.

THEOREM 3. *If* $T \in \mathscr{L}(\mathscr{X})$ *and* $\sigma(T)$ *is disconnected, then* $T \in \Delta(\mathscr{X})$.

Proof. By hypothesis, $\sigma(T)$ can be written as $\sigma(T) = \sigma_1 \cup \sigma_2$ where σ_1 and σ_2 are nonempty disjoint closed subsets of $\sigma(T)$, and hence there exist disjoint open sets $\mathscr{U}_1$ and $\mathscr{U}_2$ in the complex plane such that $\sigma_i \subset \mathscr{U}_i$, i = 1, 2. Let f be the analytic function defined on the open set $\mathscr{U}_1 \cup \mathscr{U}_2$ such that f = 1/2 on $\mathscr{U}_1$ and f = 2 on $\mathscr{U}_2$, and let Σ be a compact neighborhood of $\sigma(T)$ such that $\Sigma \subset \mathscr{U}_1 \cup \mathscr{U}_2$. Then by Runge's theorem (*cf.* [10, p. 288]), there exists a sequence $\{r_n\}$ of rational functions (with prescribed poles off Σ) that converges to f uniformly on Σ. It follows easily that

$$\|r_n(T) - f(T)\| = \|r_n(T)^* - f(T)^*\| \to 0 .$$

Since $\sigma(f(T)) = \sigma(f(T)^*) = f(\sigma(T)) = \{1/2, 2\}$, the Riesz functional calculus tells us that $\mathscr{X}[\mathscr{X}^*]$ splits into the direct sum of two nontrivial subspaces $\mathscr{X}_1$ and $\mathscr{X}_2[\mathscr{Y}_1$ and $\mathscr{Y}_2]$, each hyperinvariant for $f(T)\,[f(T)^*]$ (and therefore for $T[T^*]$) such that

$$\sigma(f(T) \mid \mathscr{X}_1) = \{1/2\}, \qquad \sigma(f(T) \mid \mathscr{X}_2) = \{2\},$$

$$\sigma(f(T)^* \mid \mathscr{Y}_1) = \{1/2\}, \qquad \sigma(f(T)^* \mid \mathscr{Y}_2) = \{2\}.$$

Using the upper semicontinuity of the spectrum, we conclude that there exists a positive integer j such that

$$\sigma(r_j(T) \mid \mathscr{X}_1) \subset \{z: |z - 1/2| < 1/4\},$$

$$\sigma(r_j(T)^* \mid \mathscr{Y}_1) \subset \{z: |z - 1/2| < 1/4\},$$

$$\sigma(r_j(T)^* \mid \mathscr{Y}_2) \subset \{z: |z - 2| < 1/2\}.$$

In particular, $r_j(T)^*$ is invertible, the spectral radius of $r_j(T) \mid \mathscr{X}_1$ is less than 3/4, and the spectral radius of $(r_j(T)^*)^{-1} \mid \mathscr{Y}_2$ is less than 2/3. Hence, if one chooses any unit vectors f in $\mathscr{X}_1$ and ϕ in $\mathscr{Y}_2$, then

$$\sum_{k=0}^{\infty} \|r_j(T)^k f\| \; \|(r_j(T)^*)^{-k}\phi\| < +\infty,$$

and by Corollary 3, T belongs to $\Delta(\mathscr{X})$.

We now turn our attention to the case that $\mathscr{X}$ is a separable infinite dimensional Hilbert space $\mathscr{H}$, and we remark first that it is easy to see that all of the above results are valid in this context, where T^* is interpreted as the Hilbert-space adjoint of an operator T in $\mathscr{L}(\mathscr{H})$. (The reason this is not completely automatic is that there is a minor difference between the Banach-space adjoint of an operator on Hilbert space and the Hilbert-space adjoint of that operator.)

That several much-studied classes of operators on $\mathscr{H}$ are contained in $\Delta(\mathscr{H})$ is a consequence of the following result.

THEOREM 4. *Let* $T \in \mathscr{L}(\mathscr{H})$, *and suppose that* $\mathscr{H}$ *can be decomposed as an orthogonal direct sum* $\mathscr{H} = \mathscr{M}_1 \oplus \cdots \oplus \mathscr{M}_n$ $(n > 1)$, *of nonzero subspaces in such a way that the matrix* (T_{ij}) *for* T *corresponding to this decomposition is in upper triangular form and satisfies* $\sigma(T_{11}) \cap \sigma(T_{nn}) = \emptyset$. *Then* $T \in \Delta(\mathscr{H})$.

Proof. Let R: $\mathscr{M}_n \to \mathscr{M}_1$ be any operator of rank one, and consider the operator equation $T_{11}X - XT_{nn} = R$, where X: $\mathscr{M}_n \to \mathscr{M}_1$. It follows from [9] that there exists a compact operator $\tilde{X}$: $\mathscr{M}_n \to \mathscr{M}_1$ satisfying this equation. An easy calculation shows that the commutator $(T_{ij})(X_{ij}) - (X_{ij})(T_{ij})$, where $X_{1n} = \tilde{X}$ and $X_{ij} = 0$ for all other pairs (i, j), is the matrix (R_{ij}) where $R_{1n} = R$ and $R_{ij} = 0$ for all other pairs (i, j). Since the matrix (R_{ij}) clearly has rank one, the theorem is proved.

COROLLARY 6. *Suppose* $T \in \mathscr{L}(\mathscr{H})$ *and there exist nonzero orthogonal subspaces* $\mathscr{M}$ *and* $\mathscr{N}$ *of* $\mathscr{H}$ *such that* $T\mathscr{M} \subset \mathscr{M}$, $T^*\mathscr{N} \subset \mathscr{N}$, *and*

$$\sigma(T \mid \mathscr{M}) \cap \sigma((T^* \mid \mathscr{N})^*) = \emptyset.$$

Then $T \in \Delta(\mathscr{H})$.

Proof. If $\mathscr{H}$ is decomposed as $\mathscr{H} = \mathscr{M} \oplus (\mathscr{H} \ominus (\mathscr{M} \oplus \mathscr{N})) \oplus \mathscr{N}$, then the matrix (T_{ij}) for T relative to this decomposition satisfies the conditions of Theorem 4.

Operators T in $\mathscr{L}(\mathscr{H})$ that satisfy a somewhat weaker hypothesis than that of Corollary 6 are known to have nontrivial hyperinvariant subspaces (*cf.* [3]).

Recall that an operator T in $\mathscr{L}(\mathscr{H})$ is said to be n-normal (for some positive integer n) if T is unitarily equivalent to an $n \times n$ operator matrix (T_{ij})

(i, j = 1, $\cdots$, n) acting on the direct sum of n copies of $\mathscr{H}$ such that the T_{ij}, $1 \leq i, j \leq n$, are mutually commuting normal operators.

COROLLARY 7. *All nonscalar normal operators and all nonscalar* n-*normal operators* $(n \geq 2)$ *in* $\mathscr{L}(\mathscr{H})$ *belong to* $\Delta(\mathscr{H})$.

Proof. That every nonscalar normal operator in $\mathscr{L}(\mathscr{H})$ belongs to $\Delta(\mathscr{H})$ is an easy consequence of the spectral theorem and either Theorem 2 or Corollary 6; one simply uses the fact that T possesses orthogonal nonzero spectral subspaces corresponding to disjoint subsets of $\sigma(T)$. That every nonscalar n-normal operator in $\mathscr{L}(\mathscr{H})$ $(n \geq 2)$ belongs to $\Delta(\mathscr{H})$ is an easy consequence of the well known fact [2] that every such operator is unitarily equivalent to an operator matrix (N_{ij}) (i, j = 1, $\cdots$, n) acting on the direct sum of n copies of $\mathscr{H}$ that is in upper triangular form and satisfies the additional condition that the N_{ij}, $1 \leq i, j \leq n$, are mutually commuting normal operators. Indeed, if $\sigma(N_{11})$ is a single point, then the result follows from Proposition 2. If $\sigma(N_{11})$ is not a singleton, then it is easy to find spectral subspaces $\mathscr{M}_1$ and $\mathscr{M}_n$ of N_{11} and N_{nn}, respectively, such that

$$\sigma(N_{11} \mid \mathscr{M}_1) \cap \sigma(N_{nn} \mid \mathscr{M}_n) = \emptyset,$$

and the result then follows from Corollary 6.

The foregoing results make it clear that the class $\Delta(\mathscr{H})$ is really quite large, and it seems likely to the authors that the possibility of showing that an operator in $\mathscr{L}(\mathscr{H})$ has a nontrivial hyperinvariant subspace by showing that it belongs to $\Delta(\mathscr{H})$ will lead to the discovery of new classes of operators with nontrivial hyperinvariant subspaces. (That an operator can belong to $\Delta(\mathscr{H})$ and have hyperinvariant subspaces for nontrivial reasons is clear from the above results; due to an unfortunate choice of phraseology, this was not made clear in [5].) We close this note by proving two additional propositions and by setting forth what seem to be some interesting problems. The following proposition shows that a question we asked at the conclusion of [5] has a negative answer. (We gratefully acknowledge a note from Mr. Mihai Pimsner of the University of Bucharest who independently made the same observation.)

PROPOSITION 4. *Let* $\mathscr{X}$ *be an arbitrary infinite dimensional complex Banach space, let* $T \in \mathscr{L}(\mathscr{X})$, *and let* n *be a positive integer. If no polynomial of degree less than or equal to* n *annihilates* T, *then there is a compact operator* K *in* $\mathscr{L}(\mathscr{X})$ *of rank* $n+1$ *such that* TK - KT *has rank at most* 2.

Proof. Since neither the hypothesis nor the conclusion is affected if a scalar is subtracted from T, we may assume that T is invertible. By a theorem of Kaplansky (see [8], Theorem 4.8 and its proof), it follows that there exists a vector f in $\mathscr{X}$ such that the set

$$\{f, Tf, \cdots, T^n f\}$$

is linearly independent. Also, since T^* cannot be annihilated by any polynomial of degree less than or equal to n (for, otherwise, T^{**} would be, and hence, restricting to the image of $\mathscr{X}$ in $\mathscr{X}^{**}$, T would be), there exists a vector ϕ in $\mathscr{X}^*$ such that the set

$$\{\phi, T^*\phi, \cdots, (T^*)^n \phi\}$$

is linearly independent. Hence the operator

$$K = (f \otimes \phi) + (Tf \otimes (T^*)^{-1}\phi) + \cdots + (T^n f \otimes (T^*)^{-n}\phi)$$

has rank $n+1$. But $TK - KT = (T^{n+1} f \otimes (T^*)^{-n} \phi) - f \otimes T^* \phi$, so the proposition is proved.

The following result is due to Sheldon Axler. It shows that an operator on Hilbert space may be in $\Delta(\mathscr{H})$, but the sufficient condition of Corollary 3 may fail to be satisfied. He uses the unilateral shift operator T which may be represented as multiplication by z on the Hardy space H^2. Hence if $r(z)$ is a rational function with no poles on the closed unit disc, then $r(T)$ may be represented as multiplication by $r(z)$.

PROPOSITION 5 (S. Axler). *Let* T *denote the unilateral shift operator on* H^2. *Let* $r(z)$ *be any rational function such that* $r(T)$ *is invertible. Let* $f, g \in H^2$ *be arbitrary nonzero vectors. Then*

$$\sum_{k=0}^{\infty} \|r(T)^k f\| \; \|(r(T)^*)^{-k} g\| = +\infty .$$

Remark. $T \in \Delta(\mathscr{H})$ by Proposition 2.

Proof. If h is any vector, then

$$\|(r(T)^*)^{-k} g\| \geq |((r(T)^*)^{-k} g, \, r(z)^k h)| / \|r(z)^k h\| = |(g, h)| / \|r(z)^k h\| .$$

Hence

$$\sum \|r(T)^k f\| \; \|(r(T)^*)^{-k} g\| \geq \sum |(g, h)| \; \|r(z)^k f\| / \|r(z)^k h\| .$$

If $(g, f) \neq 0$, choose $h = f$ to complete the proof. Otherwise, let f_1 denote the outer factor of f, choose n such that $(g, z^n f_1) \neq 0$, and let $h = z^n f_1$.

We conclude by stating some problems.

Problem 1. Is every nonzero compact operator in $\mathscr{L}(\mathscr{H})$ also in $\Delta(\mathscr{H})$?

Problem 2. Proposition 1 shows that the property of belonging to $\Delta(\mathscr{H})$ is preserved by quasisimilarity transforms. Is every operator that is a quasiaffine transform of an operator in $\Delta(\mathscr{H})$ also in $\Delta(\mathscr{H})$?

Problem 3. Is every operator in $\mathscr{L}(\mathscr{H})$ of the form $N + R$, where N is normal and R has rank 1, in $\Delta(\mathscr{H})$? It is not even known whether every such operator has a proper invariant subspace.

Problem 4. Is every bilateral weighted shift in $\mathscr{L}(\mathscr{H})$ also in $\Delta(\mathscr{H})$? (See [11], Section 10, p. 109 and Section 12, p. 119 for information on the existence of hyperinvariant subspaces for bilateral weighted shift operators.)

Problem 5. Is every nonscalar Toeplitz operator in $\mathscr{L}(\mathscr{H})$ also in $\Delta(\mathscr{H})$? Clearly every analytic Toeplitz operator belongs to $\Delta(\mathscr{H})$ by virtue of Proposition 2. We note that even for Toeplitz operators with continuous symbol, the question of the existence of nontrivial hyperinvariant subspaces is open in general.

Problem 6. Is every nonscalar subnormal operator in $\mathscr{L}(\mathscr{H})$ also in $\Delta(\mathscr{H})$?

Problem 7. Suppose $(N_{ij})_{i,j=1}^{\infty}$ is a nonscalar operator matrix such that the N_{ij} are mutually commuting normal operators on $\mathscr{H}$ with $\sum_{i,j} \|N_{ij}\|^2 < \infty$. (It follows from this that $(N_{ij}) \in \mathscr{L}(\mathscr{H} \oplus \mathscr{H} \oplus \cdots)$.) Is (N_{ij}) in $\Delta(\mathscr{H} \oplus \mathscr{H} \oplus \cdots)$?

Problem 8. Suppose $(N_{ij})_{i,j=1}^{\infty}$ is a nonscalar operator matrix that belongs to $\mathcal{L}(\mathcal{H} \oplus \mathcal{H} \oplus \cdots)$ and has the property that the N_{ij}, $1 \leq i, j \leq \infty$, are mutually commuting normal operators on $\mathcal{H}$ such that $N_{ij} = 0$ whenever $i > j$. Is (N_{ij}) in $\Delta(\mathcal{H} \oplus \mathcal{H} \oplus \cdots)$?

In both Problem 7 and in Problem 8 it is not known whether (N_{ij}) must have nontrivial hyperinvariant subspaces. (If $\dim \mathcal{H} > 1$, then (N_{ij}) does have reducing subspaces.)

Problem 9. Let T be any nonscalar operator in $\mathcal{L}(\mathcal{H})$ with the property that there exists an invariant subspace $\mathcal{M} \neq (0)$ for T such that $T \mid \mathcal{M}$ is normal. Is every such operator in $\Delta(\mathcal{H})$? Whether every such operator has a nontrivial hyperinvariant subspace is not known at present. Clearly an affirmative answer to Problem 9 implies an affirmative answer to Problem 8.

Problem 10. If $T \in \mathcal{L}(\mathcal{H})$ and $\mathcal{M}$ is an invariant subspace for T such that $\sigma(T \mid \mathcal{M})$ is disconnected, is T in $\Delta(\mathcal{H})$? L. Fialkow has asked whether such operators have nontrivial hyperinvariant subspaces. An affirmative answer to Problem 10 clearly implies an affirmative answer to Problem 9.

Perhaps the most important question from the standpoint of determining the size and utility of the class $\Delta(\mathcal{H})$ is the following.

Problem 11. Does there exist a nonscalar operator T in $\mathcal{L}(\mathcal{H})$ that does not belong to $\Delta(\mathcal{H})$?

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