SOME NEW PROPERTIES OF SUPPORT POINTS FOR COMPACT FAMILIES OF UNIVALENT FUNCTIONS IN THE UNIT DISK

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1. INTRODUCTION

Let $H(U)$ be the linear space of all analytic functions in the unit disk $U = \{z: |z| < 1\}$, with the topology of locally uniform convergence. Let $H'(U)$ be the topological dual space of $H(U)$, and $H_u(U)$ the set of all univalent functions in $H(U)$.

In this article we shall be interested in sets of univalent functions that lie in the intersection of two hyperplanes in $H(U)$; that is, in families

$$\mathscr{F} = \mathscr{F}(U, \ell_1, \ell_2, P, Q) = \{f \in H_u(U): \ell_1(f) = P, \ \ell_2(f) = Q\}$$

for fixed $\ell_1, \ell_2 \in H'(U)$ and $P, Q \in \mathbb{C}$. For example, one easily verifies that the special families

$$S = \{f \in H_u(U): f(0) = 0, \ f'(0) = 1\},$$

$$T = \{f \in H_u(U): f(p) = p, \ f(q) = q\}, \quad p, q \in U, \ p \neq q,$$

are of this form.

In an earlier article [4], we characterized the families $\mathscr{F}$ that are nontrivial and compact. In particular, $\mathscr{F}(U, \ell_1, \ell_2, P, Q)$ is nonempty and compact if and only if

(a) $\ell_1(Q) \neq \ell_2(P)$

and

(b) $\ell_2(1)\, \ell_1(g) \neq \ell_1(1)\, \ell_2(g)$ for all $g \in H_u(U)$.

The families S and T are well known to be compact (the reader may also verify (a) and (b)). More generally, if ℓ_1 is any functional in $H'(U)$ that does not annihilate constants ($\ell_1(1) \neq 0$), we define the families

$$(1.1) \qquad \mathscr{S} = \{f \in H_u(U): \ell_1(f) = P, \ f'(q) = 1\} \qquad P \in \mathbb{C}, \ q \in U,$$

$$(1.2) \qquad \mathscr{T} = \left\{ f \in H_u(U): \ell_1(f) = P, \ \frac{f(p) - f(q)}{p - q} = 1 \right\} \qquad P \in \mathbb{C}, \ p, q \in U \ (p \neq q).$$

Then $\mathscr{S}$ and $\mathscr{T}$ satisfy (a) and (b) and, consequently, are nonempty and compact. Actually, $\mathscr{S}$ is a limiting case of $\mathscr{T}$, corresponding to $p = q$. If $\ell_1(f) = f(0)$ and

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$P = q = 0$, then $\mathscr{S}$ reduces to S. On the other hand, if $\ell_1(f) = f(p)$ and $P = p$, then $\mathscr{T}$ reduces to T. However, there is a wide variety of families of the form $\mathscr{S}$ and $\mathscr{T}$; for example, one may choose $\ell_1(f) = \sum_{n=1}^{N} c_n f(z_n)$, where $z_n \in U$ and $\sum_{n=1}^{N} c_n \neq 0$. We shall use the classes $\mathscr{S}$ and $\mathscr{T}$ as examples in Section 2, before specializing to the class S in Section 3.

Suppose $\mathscr{F} = \mathscr{F}(U, \ell_1, \ell_2, P, Q)$ is compact, $f \in \mathscr{F}$, $L \in H'(U)$ is nonconstant on $\mathscr{F}$, and $\Re L(f) = \max_{\mathscr{F}} \Re L$. That is, f maximizes the real part of a nontrivial linear functional over $\mathscr{F}$. We shall call such functions *support points* of $\mathscr{F}$. It is the purpose of this article to give some general properties of support points of compact families $\mathscr{F}$.

If $\mathscr{F}(U, \ell_1, \ell_2, P, Q)$ is compact, then it follows from (a) that we may define in terms of ℓ_1 and ℓ_2 the new functionals

$$\ell_0 = \frac{1}{\ell_1(Q) - \ell_2(P)} [\ell_2(1)\,\ell_1 - \ell_1(1)\,\ell_2]$$

$$\tilde{\ell}_0 = \frac{1}{\ell_1(Q) - \ell_2(P)} [Q\ell_1 - P\ell_2].$$

Note that

$$\ell_0(1) = 0,\ \tilde{\ell}_0(1) = 1, \text{ and } \ell_0(f) = -1,\ \tilde{\ell}_0(f) = 0 \text{ for } f \in \mathscr{F}. \tag{1.3}$$

Suppose now $\mathscr{F}$ is compact, $f \in \mathscr{F}$, $L \in H'(U)$ is nonconstant on $\mathscr{F}$, and $\Re L(f) = \max_{\mathscr{F}} \Re L$. That is, f is a support point of $\mathscr{F}$. It is then a consequence of Schiffer's fundamental lemma [9] and an elementary argument [4, 5, 10] that $\mathbb{C} - f(U)$ consists of a *single* analytic arc extending to ∞ and satisfying the differential equation

$$L_f(1/(f - w))\,(dw)^2 \geq 0,$$

where L_f is the functional $L_f = L + L(f)\ell_0 - L(1)\tilde{\ell}_0$.

Since the functionals in $H'(U)$ may be represented by compactly supported measures (*cf.* [10]), expressions such as $L_f(1/(f - w))$ and $\ell_0(1/(f - w))$ may be extended to analytic functions of w in a neighborhood of $\overline{\mathbb{C}} - f(U)$. Furthermore, (1.3) implies that $\ell_0(1/(f - w))$ has a zero of order 2 at ∞ and $L_f(1/(f - w))$ has a zero of order at least 3 at ∞, but does not vanish identically [4, 10]. Since $\mathbb{C} - f(U)$ lies on the trajectory of an analytic differential, it follows as in [1, 7, 8] that f is analytic on ∂U, except for a pole of order 2 at a point $\zeta \in \partial U$ that corresponds to $w = \infty$. Moreover, f' has a simple zero at a point $\eta \in \partial U$ that corresponds to the finite tip of the slit $\mathbb{C} - f(U)$. In the following section we shall give some relations between ζ, η, and other characteristics of f.

2. PROPERTIES OF SUPPORT POINTS FOR $\mathscr{F}$

In [4, 5] we proved that each support point f of a compact family $\mathscr{F}(U, \ell_1, \ell_2, P, Q)$ has the property that the analytic curve $\mathbb{C} - f(U)$ makes an angle of at most $\pi/4$ with the trajectories of the vector field

$$\operatorname{grad}\left[\Re \int \sqrt{\ell_0(1/(f - w))}\,dw\right]. \tag{2.1}$$

Analytically, this means that $\mathbb{C} - f(U)$ satisfies [10, p. 96]

$$\Re\{\ell_0(1/(f - w))(dw)^2\} \geq 0. \tag{2.2}$$

Example 1. For the class $\mathcal{S}$, the functional $\ell_0(h) = -h'(q)$, and the trajectories of the vector field (2.1) are the rays from $f(q)$. That is, each support point f of $\mathcal{S}$ has the property that the analytic arc $\mathbb{C} - f(U)$ makes an angle of at most $\pi/4$ with the rays emanating from the point $f(q)$.

Example 2. For the class $\mathcal{T}$, the functional

$$\ell_0(h) = -\frac{h(p) - h(q)}{p - q}, \tag{2.3}$$

and the trajectories of the vector field (2.1) are the hyperbolae with foci $f(p)$ and $f(q)$. That is, each support point f of $\mathcal{T}$ has the property that the analytic arc $\mathbb{C} - f(U)$ makes an angle of at most $\pi/4$ with the family of hyperbolae with foci $f(p)$ and $f(q)$.

We shall now use (2.2) to obtain additional information.

THEOREM 1. *Let* f *be a support point of a compact family* $\mathcal{F}(U, \ell_1, \ell_2, P, Q)$. *Suppose that* $\zeta \in \partial U$ *and* $\eta \in \partial U$ *correspond, respectively, to the infinite and finite endpoints of the arc* $\overline{\mathbb{C}} - f(U)$. *Then*

$$F(z) = \frac{\ell_0\left(\frac{\eta}{\zeta}\left[\frac{z-\zeta}{z-\eta}\right]^2 \frac{[zf'(z)]^2}{[f(z) - f(t)]} + \frac{\eta}{\zeta}\left[\frac{t-\zeta}{t-\eta}\right]^2 \frac{t^2 f'(t)}{t - z}\right)}{+ z\,\ell_0\left(\frac{\eta}{\zeta}\left[\frac{t-\zeta}{t-\eta}\right]^2 \frac{t^2 f'(t)}{1 - \bar{z}t}\right)} \tag{2.4}$$

is analytic in $\overline{U}$ *and* $\Re F \geq 0$. *In particular,*

$$\Re\left\{\frac{\eta}{\zeta}\,\ell_0\left(\left[\frac{z-\zeta}{z-\eta}\right]^2 \frac{[zf'(z)]^2}{[f(z) - f(t)]} + \left[\frac{t-\zeta}{t-\eta}\right]^2 \frac{(1 - |z|^2)t^2 f'(t)}{(t - z)(1 - \bar{z}t)}\right)\right\} \geq 0 \tag{2.5}$$

for all $z \in \overline{U}$.

Remarks. In (2.4) and (2.5), the functional ℓ_0 applies to the function of t. Very much is known about analytic functions with nonnegative real parts. Therefore, Theorem 1 leads to many necessary conditions that a support point must satisfy.

Proof. The functional ℓ_0 may be represented by a measure supported on a compact set $K \subset U$. We may therefore restrict the variable t to K, and it is evident that F is an analytic function of z in $U - K$. Since the apparent singularity when $z = t$ is removable, the function F is actually analytic in U. In verifying that F is even analytic on ∂U, the only term in question is

$$\frac{\eta}{\zeta}\left[\frac{z-\zeta}{z-\eta}\right]^2 [zf'(z)]^2\,\ell_0\left(\frac{1}{f(z) - f(t)}\right).$$

Since f' has a zero at $z = \eta$, the singularity at $z = \eta$ is removable. At $z = \zeta$, the function $[z - \zeta]^2[zf'(z)]^2$ has a pole of order 4. However, by writing

$$\frac{1}{f(z) - f(t)} = \frac{1}{f(z)} + \frac{f(t)}{f(z)^2} + \frac{1}{f(z)^3}\,\frac{f(t)^2}{1 - \frac{f(t)}{f(z)}}$$

and using the relations (1.3), we see that

$$\ell_0\left(\frac{1}{f(z)-f(t)}\right) = \frac{-1}{f(z)^2} + \frac{1}{f(z)^3}\,\ell_0\left(\frac{f(t)^2}{1-\frac{f(t)}{f(z)}}\right)$$

has a zero of order 4 at $z = \zeta$. Consequently, the singularity at $z = \zeta$ is also removable, and F is analytic in $\overline{U}$.

For $z \in \partial U$ the function $\frac{\eta}{\zeta}\left[\frac{z-\zeta}{z-\eta}\right]^2$ is nonnegative. Together with (2.2), this implies that

$$\Re F(z) = \frac{\eta}{\zeta}\left[\frac{z-\zeta}{z-\eta}\right]^2 \Re\left\{\ell_0\left(\frac{1}{f(t)-f(z)}\right)[izf'(z)]^2\right\}$$

is nonnegative on ∂U. By the maximum principle, we have $\Re F(z) \geq 0$ in U also.

The relation (2.5) is just the statement that $\Re F(z) \geq 0$, together with the observation that $\Re(z\overline{\ell_0}) = \Re(\bar{z}\ell_0)$.

Examples. In view of (2.3) and (2.5), a support point of the class $\mathscr{T}$ must satisfy

$$\Re\left\{\frac{\eta}{\zeta}\left(\left[\frac{z-\zeta}{z-\eta}\right]^2 \frac{[zf'(z)]^2}{[f(z)-f(p)][f(z)-f(q)]} + \frac{1-|z|^2}{p-q}\left[\left(\frac{p-\zeta}{p-\eta}\right)^2 \frac{p^2 f'(p)}{(p-z)(1-\bar{z}p)} - \left(\frac{q-\zeta}{q-\eta}\right)^2 \frac{q^2 f'(q)}{(q-z)(1-\bar{z}q)}\right]\right)\right\} \leq 0$$

for all $z \in \overline{U}$. Since F in (2.4) has nonnegative real part, the inequality

$$|F(p) - F(q)| \leq \left|\frac{p-q}{1-p\bar{q}}\right| \, |F(p) + \overline{F(q)}|$$

contains further information of value. We leave its explicit determination to the reader.

Similarly, for the class $\mathscr{S}$, (2.5) leads to the relation

$$\Re\left\{\frac{\eta}{\zeta}\left(\left[\frac{z-\zeta}{z-\eta}\right]^2 \frac{[zf'(z)]^2}{[f(z)-f(q)]^2} + (1-|z|^2)\frac{\partial}{\partial q}\left[\left(\frac{q-\zeta}{q-\eta}\right)^2 \frac{q^2 f'(q)}{(q-z)(1-\bar{z}q)}\right]\right)\right\} \leq 0 \quad (z \in \overline{U}) \tag{2.6}$$

for a support point. In this case the inequality

$$(1-|q|^2)\,|F'(q)| \leq 2\,\Re F(q) \tag{2.7}$$

contains valuable additional information. Again we leave to the reader the explicit determination both of (2.6) and of (2.7) as $z \to q$. However, we shall observe a special case in Theorem 2.

3. THE CLASS S

Since the class S is a special case of the class $\mathscr{S}$, Example 1 implies that each support point f of S has the property that $\mathbf{C}$ - f(U) is a single analytic arc whose tangent at each point makes an angle of at most $\pi/4$ with the radial direction (see also [1, 7]). We now note some additional properties that follow from Theorem 1.

THEOREM 2. *Let* f *be a support point of* S. *Suppose that* $\zeta \in \partial U$ *and* $\eta \in \partial U$ *correspond, respectively, to the infinite and finite endpoints of the arc* $\overline{\mathbf{C}}$ - f(U). *Then*

$$F(z) = -\frac{\eta}{\zeta}\left(\frac{z-\zeta}{z-\eta}\right)^2\left[\frac{zf'(z)}{f(z)}\right]^2 \tag{3.1}$$

is analytic in $\overline{U}$ *and* $\Re F > 0$ *in* U. *In particular,*

$$\Re \zeta\bar{\eta} < 0. \tag{3.2}$$

Furthermore,

$$\frac{1-|z|}{1+|z|} < \left|\frac{2}{1+(\zeta\bar{\eta})^2}\left(\frac{z-\zeta}{z-\eta}\right)^2\left[\frac{zf'(z)}{f(z)}\right]^2 + \frac{1-(\zeta\bar{\eta})^2}{1+(\zeta\bar{\eta})^2}\right| < \frac{1+|z|}{1-|z|} \tag{3.3}$$

for $0 < |z| < 1$; *and if* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, *then*

$$|a_2 - \bar{\zeta} + \bar{\eta}| < -\Re\zeta\bar{\eta}, \tag{3.4}$$

$$|4a_3 - a_2^2 - 4a_2(\bar{\zeta} - \bar{\eta}) + \bar{\zeta}^2 - 4\bar{\zeta}\bar{\eta} + 3\bar{\eta}^2| < -2\Re\zeta\bar{\eta}. \tag{3.5}$$

Moreover,

$$|a_2| > 1. \tag{3.6}$$

Proof. For the class S, the functional $\ell_0(h) = -h'(0)$. Therefore, the function (3.1) is precisely the function (2.4) of Theorem 1. It follows that F is analytic in $\overline{U}$ and $\Re F \geq 0$. If $\Re F = 0$ at a point of U, then the maximum principle and an integration lead to the explicit representation

$$f(z) = z\exp[(\zeta\bar{\eta} - 1)\log(1 - \bar{\zeta}z)] \quad \text{where } \zeta\bar{\eta} = \pm i.$$

Note that the singularity at $z = \zeta$ is not a pole. Consequently, $\Re F > 0$ in U.

The inequalities (3.2) to (3.5) are just the well-known estimates $\Re F(0) > 0$,

$$\frac{1-|z|}{1+|z|} < \left|\frac{F(z) - i\,\Im F(0)}{\Re F(0)}\right| < \frac{1+|z|}{1-|z|}, \quad 0 < |z| < 1,$$

$|F'(0)| < 2\Re F(0)$, and $|F''(0)| < 4\Re F(0)$ for an analytic function with positive real part. The strict inequalities reflect the fact that F is bounded.

Finally, the lower bound (3.6) can be obtained from (3.4) as follows:

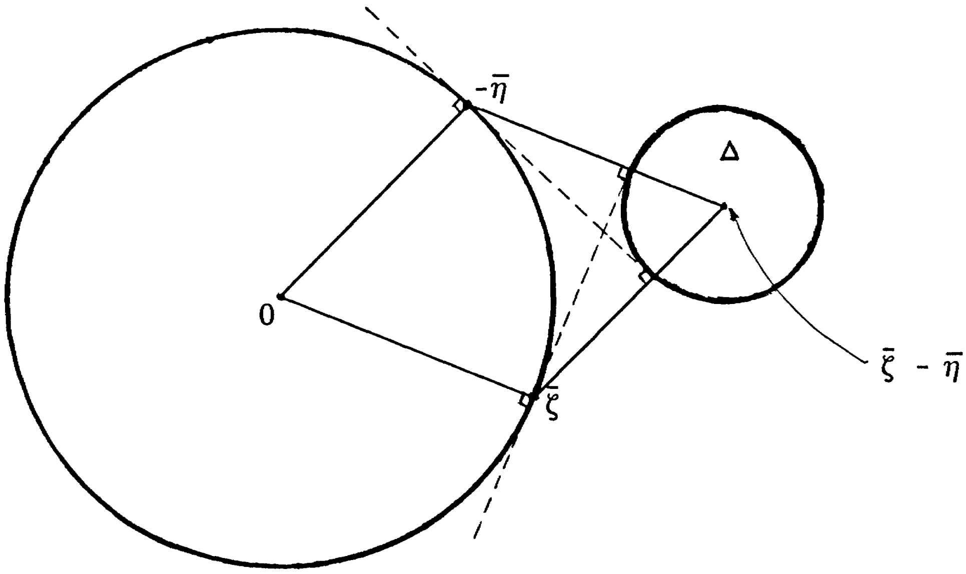
$$|a_2| \geq \Re a_2\zeta > \Re(a_2\zeta + \zeta\bar{\eta}) + |a_2 - \bar{\zeta} + \bar{\eta}| = \Re(a_2\zeta + \zeta\bar{\eta}) + |-a_2\zeta - \zeta\bar{\eta} + 1| \geq 1.$$

Remarks. Even the elementary inequality (3.2) has an interesting geometric interpretation. It says that the points $\zeta, \eta \in \partial U$ corresponding to the endpoints of $\overline{\mathbf{C}}$ - f(U) are separated by an angle of more than 90 degrees.

A number of results about functions in the class S depend on the value of the second coefficient. Since $|a_2| \leq 2$ for every function in S, it follows from (3.6) that each support point of S has the remarkable property that

$$1 < |a_2| \leq 2.$$

Moreover, the inequality (3.4) restricts a_2 further to an open disk Δ of radius $-\Re\,\zeta\bar{\eta}$ with center $\bar{\zeta} - \bar{\eta}$, as shown in the figure.



The following theorem gives some additional analytic and geometric information about support points of the class S. For $|\omega| = 1$ let $R_\omega = \{r\omega\colon 0 < r < 1\}$ be the radius from 0 to ω.

THEOREM 3. *Let* f *be a support point of* S, *and suppose that* $\zeta, \eta \in \partial U$ *correspond, respectively, to the infinite and finite endpoints of the arc* $\overline{\mathbb{C}} - f(U)$. *Then*

$$G(z) = \left(\left[\frac{zf'(z)}{f(z)}\right]^2 + \frac{z+\zeta}{z-\zeta}\right)\frac{(z-\zeta)^2}{\zeta z} \tag{3.7}$$

and

$$H(z) = \left(\left[\frac{f(z)}{zf'(z)}\right]^2 + \frac{z+\eta}{z-\eta}\right)\frac{(z-\eta)^2}{\eta z} \tag{3.8}$$

are analytic in $\overline{U}$, *and*

$$\Re G > 0 \quad \textit{and} \quad \Re H > 0 \quad \textit{in } U. \tag{3.9}$$

In particular, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, *then*

$$\Re\, a_2\zeta > 1, \qquad \Re\, a_2\eta < -1, \tag{3.10}$$

$$|(4a_3 - a_2^2)\zeta - 4a_2 + 2\bar{\zeta}| < 4\Re(a_2\zeta) - 4, \tag{3.11}$$

$$|(4a_3 - 5a_2^2)\eta - 4a_2 - 2\bar{\eta}| < -4\Re(a_2\eta) - 4. \tag{3.12}$$

Moreover,

$$|4a_3 - a_2^2| > 2 \tag{3.13}$$

and

$$|a_3| > 3/8. \tag{3.14}$$

In addition, the arcs $f(R_\zeta)$ *and* $f(R_\eta)$ *have the property that their tangents at each point make an angle less than* $\pi/4$ *with the radial direction.*

Furthermore, the functions

$$\begin{aligned} g_\zeta(z) &= \int_0^z \left(\left[\frac{tf'(t)}{f(t)}\right]^2 + \frac{t+\zeta}{t-\zeta}\right)\frac{dt}{t}, \quad & h_\eta(z) &= \int_0^z \left(\left[\frac{f(t)}{tf'(t)}\right]^2 + \frac{t+\eta}{t-\eta}\right)\frac{dt}{t}, \\ g(z) &= \int_0^z \left(\left[\frac{tf'(t)}{f(t)}\right]^2 - 1\right)\frac{dt}{t}, \quad & h(z) &= \int_0^z \left(\left[\frac{f(t)}{tf'(t)}\right]^2 - 1\right)\frac{dt}{t}, \end{aligned} \tag{3.15}$$

are univalent and map U *onto domains that are convex and unbounded in the* u-*direction* (*i.e.*, *the intersection with each horizontal line either is empty or is a ray in the positive direction*).

Proof. The proof is very similar to the proofs of Theorems 1 and 2. In this case the functions

$$\left[\frac{zf'(z)}{f(z)}\right]^2 \frac{(z-\zeta)^2}{\zeta z} \quad \text{and} \quad \left[\frac{f(z)}{zf'(z)}\right]^2 \frac{(z-\eta)^2}{\eta z}$$

are analytic in $\overline{U}$, except for simple poles at the origin, with residues ζ and η, respectively. Therefore

$$G(z) = \left[\frac{zf'(z)}{f(z)}\right]^2 \frac{(z-\zeta)^2}{\zeta z} - \frac{\zeta}{z} + \frac{z}{\zeta} \quad \text{and} \quad H(z) = \left[\frac{f(z)}{zf'(z)}\right]^2 \frac{(z-\eta)^2}{\eta z} - \frac{\eta}{z} + \frac{z}{\eta}$$

are analytic in $\overline{U}$. If $z \in \partial U$, then $(z-\zeta)^2/(\zeta z) \leq 0$, $(z-\eta)^2/(\eta z) \leq 0$, and

$$\Re\, G(z) = \frac{(z-\zeta)^2}{\zeta z} \Re\left[\frac{zf'(z)}{f(z)}\right]^2, \quad \Re\, H(z) = \frac{(z-\eta)^2}{\eta z} \Re\left[\frac{f(z)}{zf'(z)}\right]^2.$$

For the class S the relation (2.2) says that $\Re\left[\frac{zf'(z)}{f(z)}\right]^2 \leq 0$. It follows that $\Re G \geq 0$ and $\Re H \geq 0$ on ∂U, hence in U also, by the maximum principle.

If $\Re G = 0$ at a point of U, then $G \equiv i\sigma$, where σ is a real constant, and an integration leads to the representation

$$f(z) = z \exp\left(\int_0^z \left[\frac{\sqrt{\zeta^2 - t^2 + i\sigma\zeta t}}{\zeta - t} - 1\right]\frac{dt}{t}\right).$$

However, this function does not have the required pole since $\lim_{r \to 1} (1-r)^2 f(r\zeta)$ does not exist unless $\sigma = 0$, in which case f is bounded. Consequently, $\Re G > 0$ in U. After a similar analysis for H, we have (3.9).

The estimates (3.10) follow by substituting $z = 0$ into (3.9). They are also consequences of (3.4). In addition, the well-known estimates

$$|G'(0)| < 2\,\Re\, G(0) \quad \text{and} \quad |H'(0)| < 2\,\Re\, H(0)$$

yield (3.11) and (3.12). As before, strict inequality holds since G and H are bounded. The estimate (3.11) implies that

$$\Re\,[(4a_3 - a_2^2)\zeta^2] > 2, \tag{3.16}$$

from which (3.13) follows.

To verify (3.14), we shall first estimate $\Re(a_2\zeta)^2$. From (3.4) we can write

$$a_2\zeta = 1 + e^{i\theta} + r(\cos\theta)e^{i\phi},$$

where $0 \le r < 1$, $e^{i\theta} = -\zeta\bar{\eta}$, $\cos\theta > 0$, and $0 \le \phi < 2\pi$. Therefore,

$$\Re(a_2\zeta)^2 = (2 - r^2)\cos^2\theta + 2r\cos\theta\,\cos\phi\,(r\cos\theta\,\cos\phi + 1)$$
$$+ 2\cos\theta\,(1 + r\cos(\theta + \phi)).$$

Since the first and last terms are positive and $2x(x + 1) \ge -1/2$, we have

$$\Re(a_2\zeta)^2 > -\frac{1}{2}. \tag{3.17}$$

Now (3.16) and (3.17) imply

$$\Re(a_3\zeta^2) > \frac{3}{8},$$

from which (3.14) follows.

If $0 < r < 1$, then

$$0 < \frac{\Re\, G(r\zeta)}{r(1 - r)^2} = \Re\left(\left[\frac{\zeta f'(r\zeta)}{f(r\zeta)}\right]^2 - \frac{1 + r}{r^2(1 - r)}\right) < \Re\left[\frac{\zeta f'(r\zeta)}{f(r\zeta)}\right]^2 = \Re\left[\frac{\frac{\partial f}{\partial r}(r\zeta)}{f(r\zeta)}\right]^2$$

and, similarly,

$$0 < \frac{\Re\, H(r\eta)}{r(1 - r)^2} < \Re\left[\frac{f(r\eta)}{\frac{\partial f}{\partial r}(r\eta)}\right]^2.$$

Therefore, $\left|\arg\left[\frac{\partial f}{\partial r}/f\right]\right| < \pi/4$ for $z = r\zeta$ and $z = r\eta$ (and appropriate branches of the argument). Consequently, tangents to the arcs $f(R_\zeta)$ and $f(R_\eta)$ make angles less than $\pi/4$ with the radial direction.

Using the definitions (3.15), we observe that

$$0 < \Re\, G = \Re\,\frac{g_\zeta'}{\phi_\zeta'} \quad \text{and} \quad 0 < \Re\, H = \Re\,\frac{h_\eta'}{\phi_\eta'},$$

where $\phi_\zeta(z) = z/(\zeta - z)$ and $\phi_\eta = z/(\eta - z)$ are univalent mappings of U onto convex domains (half-planes). Consequently, g_ζ and h_η are close-to-convex, hence univalent [6]. Furthermore, for $z \in U$

$$0 < \Re G + 2\,\Re(1 - \bar{\zeta}z) = \Re\frac{g'}{\phi'_\zeta} \quad\text{and}\quad 0 < \Re H + 2\,\Re(1 - \bar{\eta}z) = \Re\frac{h'}{\phi'_\eta}.$$

Consequently, g and h are also close-to-convex, hence univalent. To see that g_ζ, h_η, g, h map U onto domains that are convex and unbounded in the u-direction, one may use an argument similar to that in [2, Section 6].

4. CONCLUDING REMARKS

The results of this article are really a consequence of the geometric $\pi/4$-property described analytically in (2.2). For the boundary values of an analytic function to have this property, it is not necessary that the function be univalent. Therefore, similar results also apply to many nonunivalent functions. We have confined our attention to support points of the univalent families $\mathscr{F}(U, \ell_1, \ell_2, P, Q)$ because of their obvious interest.

Functions with nonnegative real part corresponding to G and H in Theorem 3 can be constructed more generally in the framework of the families $\mathscr{F}(U, \ell_1, \ell_2, P, Q)$. We have further restricted our attention to the class S since the applications are more appealing. In particular, it is interesting that the $\pi/4$-property of the boundary arc $\mathbb{C} - f(U)$ extends inside to the arcs $f(R_\zeta)$ and $f(R_\eta)$. In addition, the $\pi/4$-property of $\mathbb{C} - f(U)$ implies that it is a monotone arc. As a consequence of earlier work [3, 10], the functions $\log[f(z)/z]$ and $f(z)/z$ are univalent and have geometric properties. This should be compared with the univalence and other properties of the functions (3.15).

For a support point $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ of the class S, we have shown that the coefficients satisfy $|a_2| > 1$, $|4a_3 - a_2^2| > 2$, and $|a_3| > 3/8$. It would be interesting to determine additional properties of the coefficients. In particular, what additional information can be gleaned from the fact that the functions (3.1), (3.7), and (3.8) have positive real part?

REFERENCES

1. L. Brickman and D. Wilken, *Support points of the set of univalent functions.* Proc. Amer. Math. Soc. 42 (1974), 523-528.

2. W. Hengartner and G. Schober, *On schlicht mappings to domains convex in one direction.* Comment. Math. Helv. 45 (1970), 303-314.

3. ———, *Extreme points for some classes of univalent functions.* Trans. Amer. Math. Soc. 185 (1973), 265-270.

4. ———, *Compact families of univalent functions and their support points.* Michigan Math. J. 21 (1974), 205-217.

5. ———, *Propriétés des points d'appui des familles compactes de fonctions univalentes.* C. R. Acad. Sci. Paris Sér. A 279 (1974), 551-553.

6. W. Kaplan, *Close-to-convex schlicht functions.* Michigan Math. J. 1 (1952), 169-185.

7. A. Pfluger, *Lineare Extremalprobleme bei schlichten Funktionen.* Ann. Acad. Sci. Fenn. Ser. AI 489 (1971), 32 pp.

8. A. C. Schaeffer and D. C. Spencer, *Coefficient regions for schlicht functions*. American Mathematical Society Colloquium Publications, Vol. 35. American Mathematical Society, Providence, R.I., 1950.

9. M. Schiffer, *A method of variation within the family of simple functions*. Proc. London Math. Soc. (2) 44 (1938), 432-449.

10. G. Schober, *Univalent Functions--Selected Topics*. Lecture Notes in Mathematics, Vol. 478. Springer-Verlag, Berlin-New York, 1975.

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