ARITHMETICALLY EMBEDDABLE LOCAL NOETHER LATTICES

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In [1], Anderson showed that each distributive local Noether lattice can be embedded in the lattice of ideals of a Noetherian ring. However, it is of interest to know when such an embedding can be made into the lattice of ideals of an arithmetical Noetherian ring (that is, a Noetherian ring with a distributive lattice of ideals), and the results of [1] shed no light on this question.

We call a Noether lattice *independently generated* if it satisfies the identity $A(B \wedge C) = AB \wedge AC$, and *basis-annihilating* if it satisfies the identity $(A \vee B)(A \wedge B) = AB$. (Theorems 3 and 9 justify the seemingly strange terminology.) Since the ideals of every arithmetical ring satisfy the stated identities, it is clear that every Noether lattice that is embeddable in the lattice of ideals of an arithmetical ring is both basis-independent and basis-annihilating.

In this paper, we study basis-annihilating and basis-independent Noether lattices, our primary goal being the determination of an answer to the question posed above. However, since the implications of these properties are also interesting outside of the distributive case, we do not limit our considerations to that situation. We note that both properties have received wide attention in rings. (See, for example, [5], [6], [7].)

In [4], K. P. Bogart showed that if E is a principal element in a minimal base of an element A of a distributive local Noether lattice, then E is in every minimal base of A. Since much of our work will be outside of distributive structures, where this obviously need not be the case unless A is principal, it is convenient to introduce the following terminology:

(i) If $\mathscr{L}$ is a local Noether lattice and E is in every minimal base of an element A of $\mathscr{L}$, then E is an *essential generator* of A.

(ii) If $\mathscr{L}$ is local and A has a unique minimal base, then A is *essentially generated.*

(iii) If $\mathscr{L}$ is local and every element is essentially generated, then $\mathscr{L}$ is *essentially generated.*

(iv) If $\mathscr{L}$ is local and no nonprincipal element has an essential generator, then $\mathscr{L}$ is *inessentially generated.*

Finally, we say that a nonlocal Noether lattice is essentially generated or inessentially generated if each of its localizations is. As we have noted, distributive Noether lattices are essentially generated. Lattices of ideals of rings are inessentially generated.

It is easily seen that every independently generated Noether lattice is basis-annihilating (the proof is the same as that given in [6] for rings). And it is fairly easy to see directly that the two conditions are equivalent for Noetherian rings.

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(This also follows from Theorem 6.) Interestingly, the two conditions are not equivalent for Noether lattices, even in the distributive case.

Our main results are (i) a characterization of basis-annihilating distributive local Noether lattices; (ii) an embedding theorem for distributive, independently generated locals into arithmetical Noetherian rings; (iii) a characterization of the distributive Noether lattices as those that are essentially generated.

LEMMA 1. *Let* $(\mathcal{L}, M)$ *be a local Noether lattice, and let* $\{E_1, \cdots, E_k\}$ *be a minimal base for* M *such that* $E_i E_j = 0$ *whenever* $i \neq j$. *If* A *is any element of* $\mathcal{L}$ *such that* $A \leq E_i$, *then either* $A = 0$ *or* A *is a power of* E_i.

Proof. Assume $A \neq 0$, and choose n so that $A \leq M^n E_i$ and $A \not\leq M^{n+1} E_i$. Then $A = A \wedge M^n E_i = A \wedge E_i^{n+1} = (A : E_i^{n+1}) E_i^{n+1}$; therefore $A = E_i^{n+1}$, by the choice of n.

LEMMA 2. *Let* $(\mathcal{L}, M)$ *be a local Noether lattice in which* M *has a minimal base* $\{E_1, \cdots, E_k\}$ *such that* $E_i E_j = 0$ *whenever* $i \neq j$. *Then* $0 : M = \bigvee_i E_i^{s_i}$, *where* $s_i \leq \infty$ *is the least exponent such that* $E_i E_i^{s_i} = 0$.

Proof. $0 : E_i = E_i^{s_i} \vee \left(\bigvee_{j \neq i} E_j\right)$, and $E_i \wedge \left(\bigvee_{j \neq i} E_j\right) \leq E_i^{s_i}$, so that $E_i \wedge (0 : E_i) = E_i^{s_i}$. Hence,

$$(0 : E_1) \wedge (0 : E_2) = E_1^{s_1} \vee (E_2 \wedge (0 : E_2)) \vee \bigvee_{j>2} E_j = E_1^{s_1} \vee E_2^{s_2} \vee \bigvee_{j>2} E_j.$$

The induction is clear.

The following gives several alternate characterizations of a basis-annihilating Noether lattice; in particular, it justifies the term "basis-annihilating."

THEOREM 3. *Let* $(\mathcal{L}, M)$ *be a local Noether lattice. Then the following are equivalent:*

(i) $\mathcal{L}$ *is basis-annihilating.*

(ii) *If* $\{F_1, \cdots, F_s\}$ *is a minimal base for an element* $A \in \mathcal{L}$, *then* $F_i F_j = 0$ *whenever* $i \neq j$.

(iii) M *has a minimal base* $\{E_1, \cdots, E_k\}$ *such that* $E_i E_j = 0$ *whenever* $i \neq j$ *and each principal element* $E \neq I$ *is contained in one of the elements* $0 : M, E_1, \cdots, E_k$.

Proof. Assume $\mathcal{L}$ is basis-annihilating, and let E and F be noncomparable principal elements. Since $\mathcal{L}$ is local, so that principal elements are join-irreducible, we may assume $EF = (E \wedge F)(E \vee F) = (E \wedge F)F$. Then $EF = (F : E)EF \leq MEF$, so that $EF = 0$, by the intersection theorem. Hence, (i) implies (ii).

Assume (ii) holds. Let $\{E_1, \cdots, E_k\}$ be a minimal base for M, and let $E \leq M$ be any principal element. Assume $E \not\leq 0 : M$, and choose i so that $EE_i \neq 0$. Then E and E_i do not form a minimal base for the element $E \vee E_i$, and $E_i \not\leq M^2$; therefore $E \leq E_i$. Hence (ii) implies (iii).

Now, assume (iii) holds. Since every element is the join of principal elements, it suffices to show that $(E \vee F)(E \wedge F) = EF$ if E and F are principal. Because $(E \vee F)(E \wedge F) \leq EF$, it suffices to show $EF \leq (E \vee F)(E \wedge F)$. The conclusion follows immediately if $EF = 0$; therefore we may assume $E \leq E_i$ and $F \leq E_i$. But then, by Lemma 1, E and F are comparable, so that the relation

$$EF \leq (E \wedge F)(E \vee F)$$

is clear. Hence, (iii) implies (i), and the proof is complete.

By Theorem 3, it is clear that if $\mathscr{L}$ is a basis-annihilating Noether lattice in which 0 is prime, then each localization of $\mathscr{L}$ is a principal-element lattice, and therefore $\mathscr{L}$ is a principal-element lattice. Since each principal-element lattice is basis-independent and also representable, it follows that in the case of domains, basis-annihilation and basis-independence are equivalent, and each implies distributivity and representability.

As in the following generalization of the case of domains, we shall frequently use decompositions of local Noether lattices by local direct sums. We refer the reader to [3] for a general investigation of the decomposition process. It was first used in [10].

THEOREM 4. *Let* $(\mathscr{L}, M)$ *be a basis-annihilating local Noether lattice. Then* $\mathscr{L}/(0:M)$ *is the local direct sum of principal-element lattices. In particular, if* M *is not a prime of* 0, *then* $\mathscr{L}$ *is basis-independent and every element of* $\mathscr{L}$ *is join-principal.*

Proof. Let $\{E_1, \cdots, E_k\}$ be a minimal base for M. If

$$E_i \leq \Big(\bigvee_{j \neq i} E_j\Big) \vee (0:M),$$

then $ME_i = E_i^2 = 0$, so that $E_i \leq 0:M$. It follows that in $\mathscr{L}/(0:M)$, M has a minimal base consisting of those $E_i \vee (0:M) \not\leq 0:M$. For notational simplicity, we assume the minimal base consists of the elements $E_i \vee (0:M)$ $(1 \leq i \leq n)$. Now, by Lemma 2,

$$(E_i \vee (0:M)) \wedge \Big(\Big(\bigvee_{j \neq i} E_j\Big) \vee (0:M)\Big) = E_i^{s_i} \vee (0:M) = 0:M\,;$$

therefore the minimal base $E_i \vee (0:M)$ $(1 \leq i \leq n)$ is independent in $\mathscr{L}/(0:M)$. Each principal element of $\mathscr{L}/(0:M)$ is of the form $F \vee (0:M)$, for some principal element F of $\mathscr{L}$, and either F is a power of one of $E_1, \cdots, E_n$ or $F \leq 0:M$. Hence, in $\mathscr{L}/(0:M)$ each principal element is contained in one of the independent minimal-basis elements $E_i \vee (0:M)$ $(1 \leq i \leq n)$. It follows that $\mathscr{L}/0:M$ splits, as desired.

If M is not a prime of 0, then $\mathscr{L}$ is a submultiplicative lattice of a direct sum of local principal-element lattices, so that $\mathscr{L}$ is basis-independent.

It is interesting (though straightforward to prove) that if E is a principal element of a local Noether lattice $(\mathscr{L}, M)$, then $E \vee (0:M)$ is join-principal. This observation yields the following result.

COROLLARY 5. *Let* $(\mathscr{L}, M)$ *be a basis-annihilating local Noether lattice such that* $\mathscr{L}/(0:M)$ *is irreducible. Then* M *is join-principal.*

We note that it follows from the results of [8] that if $(\mathscr{L}, M)$ is a local Noether lattice in which M is join-principal, then $\mathscr{L}/(0:M)$ is independently generated and hence also basis-annihilating.

THEOREM 6. *Let $\mathscr{L}$ be a basis-annihilating Noether lattice. If $\mathscr{L}$ is inessentially generated, then $\mathscr{L}$ is the direct sum of principal-element lattices and trivial-multiplication lattices, and hence is a join-principal-element lattice.*

Proof. Assume $\mathscr{L}$ is basis annihilating, and let $(\mathscr{L}', M)$ be any localization of $\mathscr{L}$. Let $\{E_1, \cdots, E_k\}$ be a minimal base for M in $\mathscr{L}'$. If $ME_1 \neq 0$, then, since each principal element not contained in E_1 annihilates E_1, the element E_1 is necessarily an essential generator of M in $\mathscr{L}'$.

Hence, if $\mathscr{L}$ is inessentially generated, every maximal element is either principal in $\mathscr{L}$ or satisfies the condition $M^2 = 0$ in $\mathscr{L}_M$. Consideration of the nature of a primary decomposition for 0 shows that $\mathscr{L}$ splits as desired.

At this point, it is natural to wonder whether every basis-annihilating Noether lattice is a weak-join principal-element lattice. However, the following examples show that the answer is negative even in the distributive case.

For $i = 1, \cdots, k$, let $(\mathscr{L}_i, E_i)$ be a local principal-element lattice such that $E_i^{n_i+2} = 0$ but $E_i^{n_i+1} \neq 0$ $(n_i \geq 1)$. Let $(Q(n_1, \cdots, n_k), M)$ be the local Noether lattice obtained from the local direct sum of the lattices $\mathscr{L}_i$ by identifying the elements $E_i^{n_i+1}$ $(1 \leq i \leq k)$. If $k \geq 2$, then

$$ME_1^{n_1} = ME_2^{n_2}, \qquad 0:M = E_1^{n_1+1} = E_2^{n_2+1}, \quad \text{and} \quad E_1^{n_1} \vee 0:M \neq E_2^{n_2} \vee 0:M,$$

so that M is not weak-join principal. For example, $Q(1, 2)$ is the lattice in Figure 1. Actually, it is easy to see that if $(\mathscr{L}, M)$ is a distributive, basis-annihilating, local Noether lattice of altitude 0, and if, in the notation of Lemma 2, $E_1^{s_1} = E_2^{s_2} = \cdots = E_k^{s_k}$, then either $k = 1$, or $s_i \geq 2$ and $\mathscr{L} \cong Q(n_1, \cdots, n_k)$, where $n_i = s_i - 1$. The following theorem shows that the lattices $Q(n_1, \cdots, n_k)$ come close to characterizing basis-annihilating distributive local Noether lattices.

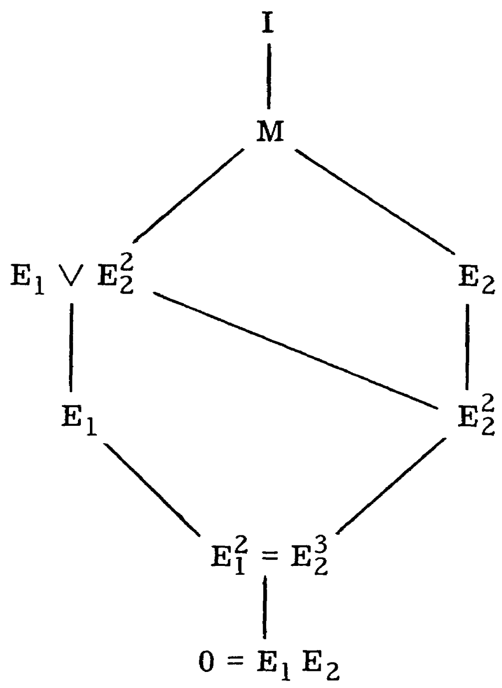

Figure 1

THEOREM 7. *Let $(\mathscr{L}, M)$ be a basis-annihilating local Noether lattice. Then $\mathscr{L}$ is distributive if and only if $0:M$ is essentially generated. If $\mathscr{L}$ is distributive, then $\mathscr{L}$ is the local direct sum of local principal-element lattices and local Noether lattices of the form $Q(n_1, \cdots, n_k)$.*

Proof. Assume $0:M$ is essentially generated, and let $\{E_1, \cdots, E_k\}$ be a minimal base for M.

We note that if A is an essentially generated element and if $\{F_1, \cdots, F_n\}$ is the minimal base for A, then every element that is the join of a subcollection of $\{F_1, \cdots, F_n\}$ is also essentially generated. Since, in the notation of Lemma 2, $0:M = \bigvee_i E_i^{s_i}$, it follows that the unique minimal base for $0:M$ consists of the distinct, nonzero elements $E_i^{s_i}$, and that every element generated by a subset of these elements is also essentially generated.

Now assume that $E \leq 0:M$ is a nonzero principal element and that $\{E_{i_1}, \cdots, E_{i_n}\}$ is a minimal subcollection of the elements $E_1, \cdots, E_k$ such that $E \leq E_{i_1} \vee \cdots \vee E_{i_n}$. For notational simplicity, we assume $i_j = j$ $(1 \leq j \leq n)$. Then by [9, Theorem 2.1] and by Lemma 1, it follows that there exist exponents $m_1, \cdots, m_n$ such that

$$E \vee \Big(\bigvee_{\substack{1 \leq i \leq n \\ i \neq r}} E_i^{m_i} \Big) = \bigvee_{1 \leq i \leq n} E_i^{m_i}.$$

Moreover, the elements $E_i^{m_i}$ must be distinct and nonzero, by the minimality of the collection $\{E_1, \cdots, E_n\}$. Since $E \leq 0:M$, $E \vee \Big(\bigvee_{\substack{1 \leq i \leq n \\ i \neq r}} E_i^{m_i} \Big)$ is annihilated by E_r; therefore $m_i = s_i$ for $i = 1, \cdots, n$. Since E can be used in a minimal base for $E_1^{s_1} \vee \cdots \vee E_n^{s_n}$, it follows that $E = E_i^{s_i}$ for some i $(1 \leq i \leq n)$.

Now we partition the elements $E_1, \cdots, E_k$ by the equivalence relation $\sim$ defined by the rule $E_i \sim E_j$ if and only if $E_i^{s_i} = E_j^{s_j}$ and either $i = j$ or $E_i^{s_i} \neq 0$. Denote the equivalence class of E_i by $\langle E_i \rangle$. We may assume that the distinct equivalence classes are $\langle E_1 \rangle, \cdots, \langle E_n \rangle$. If E is a nonzero principal element and

$$E \leq \Big(\bigvee \langle E_i \rangle \Big) \wedge \Big(\bigvee \Big(\bigcup_{\substack{1 \leq j \leq n \\ j \neq i}} \langle E_j \rangle \Big) \Big),$$

where $1 \leq i \leq n$, then clearly $E \leq 0:M$. Hence, by the argument above, $E = E_i^{s_i} = E_j^{s_j}$ for some $j \neq i$ $(1 \leq j \leq n)$, which contradicts the nature of the partition. Hence, the elements $\bigvee \langle E_i \rangle$ $(1 \leq i \leq n)$ are independent. Moreover, since each nonzero element is either contained in an E_i or is usable in a minimal base for $0:M$, it follows that every principal element is contained in one of the elements $\bigvee \langle E_i \rangle$, and hence that $\mathscr{L}$ is the local direct sum of the local Noether lattices $(\mathscr{L}_i, M_i) = \big[0, \bigvee \langle E_i \rangle \big] \cup \{I\}$. Now, each of the Noether lattices $\mathscr{L}_i$ satisfies the hypothesis on $\mathscr{L}$ and has the additional property that $(0:M_i)$ is principal and either 0 or a power of each of the minimal-basis elements of M_i. It follows in either case that $\mathscr{L}/(0:M_i)$ is distributive (Theorem 4), that $[0, 0:M_i]$ is distributive, and that $0:M_i$ is a distributive element in $\mathscr{L}_i$. Hence, by the discussion preceding the theorem, each $(\mathscr{L}_i, M_i)$ is either a principal-element lattice or of the form $Q(n_1, \cdots, n_s)$.

Figure 2 shows that an independently generated Noether lattice need not be distributive.

As we pointed out earlier, a basis-annihilating Noether lattice need not be a weak-join principal-element lattice. In contrast to this, we have the following proposition.

THEOREM 8. *Let* $(\mathscr{L}, M)$ *be a local Noether lattice, and let* B *be an element of* $\mathscr{L}$ *such that* $B(E \wedge A) = BE \wedge BA$, *for all* A *and for all principal* E. *Then*

(i) B *is weak-join principal, and*

(ii) $0:B$ *is a distributive element.*

Proof. Assume B satisfies the stated condition. If A is an element of $\mathscr{L}$ and E is principal, with $BE \leq BA$, then

$$BE = BE \wedge BA = B(E \wedge A) = BE(A:E),$$

so that either $E \leq A$ or $BE = 0$. In either case, $E \leq A \vee (0:B)$, so that B is weak-join principal. To see that $(0:B)$ is a distributive element, let E be any principal element, and let A be any element. Then

$$(0:B) \vee (A \wedge E) = (B(A \wedge E)):B$$

$$= (BA \wedge BE):B = (A \vee (0:B)) \wedge (E \vee (0:B)).$$

Hence, also

$$(0:B) \wedge (A \vee E) = ((0:B) \wedge A) \vee ((0:B) \wedge E).$$

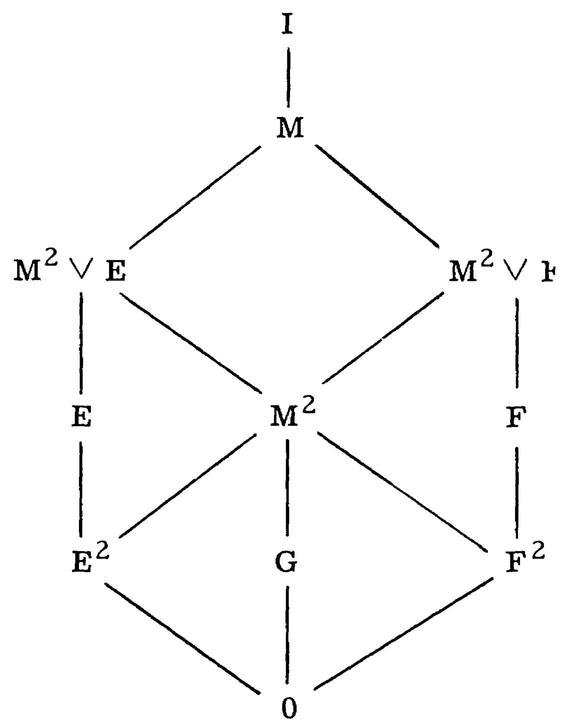

Figure 2

Since every element is the finite join of principal elements, it follows that $(0:B) \wedge (A \vee C) = ((0:B) \wedge A) \vee ((0:B) \wedge C)$ for all $A, C \in \mathscr{L}$.

The following gives alternate characterizations of basis-independent Noether lattices, and it justifies the terminology:

THEOREM 9. *Let* $(\mathscr{L}, M)$ *be a local Noether lattice. Then the following are equivalent:*

(i) $\mathscr{L}$ *is basis-independent.*

(ii) M *is weak-join principal and* $\mathscr{L}$ *is basis-annihilating.*

(iii) *If* $\{F_1, \cdots, F_n\}$ *is a minimal base for an element* $A \in \mathscr{L}$, *then* $F_1, \cdots, F_n$ *are independent.*

Proof. If E and F are noncomparable principal elements such that $M(E \wedge F) = ME \wedge MF$, then from $(E:F)F = E \wedge F = (F:E)E$ it follows that $E \wedge F \leq ME \wedge MF \leq M(E \wedge F)$, and therefore that $E \wedge F = 0$. Hence (i) implies (ii).

It was shown in [2] that if M is weak-join principal and has a minimal base $E_1, \cdots, E_k$ satisfying the condition $E_iE_j = 0$ whenever $i \neq j$, then $E_1, \cdots, E_k$ are independent. Hence, assume that $\mathscr{L}$ is basis-annihilating and that $\{E_1, \cdots, E_k\}$ is an independent minimal base for M. Let $\{A_1, \cdots, A_s\}$ be a minimal base for an element $A \leq M$. By the minimality of the base $\{A_1, \cdots, A_s\}$ and the basis-annihilation property of $\mathscr{L}$, we may assume that $A_i = E_i^{n_i}$ for $1 \leq i \leq r$ and that $MA_i = 0$ for $i > r$. Consider the element $Z = A_i \wedge (A_1 \vee \cdots \vee \hat{A}_i \vee \cdots \vee A_s)$. Since $Z \leq MA_i$, clearly $Z = 0$ if $i > r$. On the other hand, if $i \leq r$, then Z is a power of E_i. In this case, by Lemma 1 and [9, Theorem 2.1], there exist principal elements $A'_j \leq A_j$ $(j \neq i)$ such that $Z \vee \left(\bigvee_{j \neq t} A'_j\right) = \bigvee_j A'_j$ for each $t \neq i$. By the minimality of the base $\{A_1, \cdots, A_s\}$, it is clear that $A'_j \leq MA_j$, so that

$Z \leq E_i \wedge \left(\bigvee_{\substack{j \neq i \\ 1 \leq j \leq r}} E_j \right) = 0$. Hence (ii) implies (iii).

Now, assume $\mathscr{L}$ satisfies (iii). Let $\{E_1, \cdots, E_k\}$ be a minimal base for M, and let $\{B_1, \cdots, B_s\}$ and $\{C_1, \cdots, C_t\}$ be minimal bases for B and C, respectively. Because it suffices to show that $AB \wedge AC \leq A(B \wedge C)$, we may obviously assume that $AB \wedge AC \neq 0$ and that B and C are not comparable.

Since every principal element is either a power of an E_i or is annihilated by M, we assume $B_i = E_{\phi(i)}^{m_i}$ and $C_j = E_{\psi(j)}^{n_j}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ and that $MB_i = MC_j = 0$ for $i > m$ and $j > n$. Then the elements AB_i $(1 \leq i \leq m)$ are powers of distinct elements E_j, as are the elements AC_i $(1 \leq i \leq n)$; thus, by the independence of $E_1, \cdots, E_k$, we see that

$$AB \wedge AC = \left(\bigvee_{1 \leq i \leq m} AB_i \right) \wedge \left(\bigvee_{1 \leq j \leq n} AC_j \right) = \bigvee_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} (AB_i \wedge AC_j) .$$

On the other hand, for $1 \leq i \leq m$ and $1 \leq j \leq n$, B_i and C_j are either comparable or independent. Hence

$$\bigvee_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} (AB_i \wedge AC_j) = \bigvee_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} A(B_i \wedge C_j),$$

and therefore $AB \wedge AC \leq A(B \wedge C)$. The proof is now complete.

THEOREM 10. *Let* ($\mathscr{L}$, M) *be a local Noether lattice. Then the following are equivalent.*

(i) *$\mathscr{L}$ is embeddable in the lattice of ideals of an arithmetical ring.*

(ii) *$\mathscr{L}$ is distributive and basis-independent.*

Proof. It is obvious that (i) implies (ii). Hence, assume $\mathscr{L}$ is distributive and basis-independent. It is obvious from Theorem 9 that $\mathscr{L}$ is the local direct sum of principal-element lattices. By definition, $\mathscr{L}$ is then a submultiplicative lattice of a direct sum of principal-element lattices; therefore (i) follows from (ii).

By similar methods, one obtains the following alternate characterization:

THEOREM 11. *Let* ($\mathscr{L}$, M) *be a local Noether lattice. Then the following are equivalent:*

(i) *$\mathscr{L}$ is embeddable in the lattice of ideals of an arithmetical ring.*

(ii) *$\mathscr{L}$ is distributive, and* $M(E \wedge F) = ME \wedge MF$ *for all principal elements* E, F.

As we noted earlier, every element of a distributive local Noether lattice is essentially generated. We close with the converse, which has been conjectured for some time.

THEOREM 12. *Let $\mathscr{L}$ be an essentially generated Noether lattice. Then $\mathscr{L}$ is distributive.*

Proof. We may assume $\mathscr{L}$ is local and has a maximal element M.

Let E and F be principal elements, and let B be arbitrary. Assume $E \leq F \vee B$, $E \not\leq B$, and $F \not\leq B$. Let $n \geq 0$ be any integer such that $E \leq F \vee M^n B$. Since $F \vee M^n B$ is essentially generated and has a minimal base consisting of F and some principal elements $B_i \leq M^n B$, it follows that E cannot be used in a minimal base for $F \vee M^n B$, and hence that $E \leq M(F \vee M^n B) \leq F \vee M^{n+1} B$. Hence $E \leq \bigwedge_n (F \vee M^n B) = F$. Since every element A is the finite join of principals, it follows that $E \leq A \vee B$ implies $E \leq A$ or $E \leq B$, and hence that $\mathscr{L}$ is distributive.

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