ON AUTOMORPHIC FORMS AND CARLESON SETS

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1. INTRODUCTION

Let Γ be a Fuchsian group in the unit disk $D \subset \mathbb{C}$ and let $L \subset \partial D$ be its limit set. An automorphic form of weight q $(q = 0, \pm 1, \cdots)$ is an analytic function $f(z)$ $(z \in D)$ such that

$$(1.1) \qquad f(\gamma(z))\,\gamma'(z)^q \equiv f(z) \qquad (\gamma \in \Gamma).$$

Let $A_2^\infty(\Gamma)$ be the space of automorphic forms of weight 2 with

$$(1.2) \qquad \sup_{z \in D} (1 - |z|^2)^2 \, |f(z)| < \infty .$$

This space was introduced by L. Bers [1] and has applications, for instance, in Teichmüller space theory [2, p. 272]. The theory of the related spaces $A_q^p(\Gamma)$ $(1 \le p \le \infty,\ q = 2, 3, \cdots)$ is described, for instance, in the book of Kra [5].

The *Eichler integral* of $f \in A_2^\infty(\Gamma)$ is defined by

$$(1.3) \qquad h(z) = \frac{1}{2}\int_0^z (\zeta - z)^2 f(\zeta)\, d\zeta \qquad (z \in D);$$

that is, by $h'''(z) = f(z)$ and $h(0) = h'(0) = h''(0) = 0$. It follows from (1.1) that

$$(1.4) \qquad h(\gamma(z))/\gamma'(z) = h(z) + c_\gamma(z) \qquad (\gamma \in \Gamma),$$

where the *Eichler period* $c_\gamma(z)$ is a polynomial of degree ≤ 2. The Eichler periods are elements of the Eichler cohomology group $H^1(\Gamma, \Pi_2)$ [5, pp. 148, 196], and (1.4) defines a homomorphism from $A_2^\infty(\Gamma)$ into $H^1(\Gamma, \Pi_2)$. Bers [1] has shown that this homomorphism is injective for groups of the first kind (that is, $L = \partial D$). We shall prove that it is injective if and only if L is not a Carleson set.

A closed set $E \subset \partial D$ is called a *Carleson set* if

$$(1.5) \qquad \sum_n \ell_n = 2\pi, \qquad \sum_n \ell_n \log \frac{2\pi}{\ell_n} < \infty,$$

where ℓ_n are the lengths of the component arcs of $\partial D \setminus E$. It was proved by L. Carleson [3] that if a function is analytic in D and belongs to $\operatorname{Lip}\alpha$ for some $\alpha > 0$, then its zero set on ∂D is a Carleson set; conversely, every Carleson set is the zero set on ∂D of an analytic function even with bounded derivative. We shall use results of Taylor and Williams [9] and of Nelson [7] on Carleson sets.

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THEOREM 1. *Let* Γ *be a Fuchsian group without elliptic elements and with limit set* L.

(a) *If* L *is a Carleson set then there exist infinitely many linearly independent functions* $f \in A_2^\infty(\Gamma)$ *with* $c_\gamma(z) \equiv 0$ *for* $\gamma \in \Gamma$.

(b) *If* L *is not a Carleson set then there exists no function* $f(z) \not\equiv 0$ *in* $A_2^\infty(\Gamma)$ *with* $c_\gamma(z) \equiv 0$ *for* $\gamma \in \Gamma$.

T. A. Metzger [6] has shown for all Bers spaces $A_q^p(\Gamma)$ $(1 \le p \le \infty,\ q = 2, 3, \cdots)$ that the existence of a non-trivial function with vanishing Eichler periods implies that L is a Carleson set; the converse is still open in the general case. I want to thank him for our discussions on this subject.

The theorem shows that the limit set of a finitely generated Fuchsian group of the second kind is a Carleson set. This is, in general, not true for infinitely generated groups of the second kind because then the limit set may be of positive measure (see for instance [8], Example 2).

2. AUTOMORPHIC FORMS OF WEIGHT -1

The condition that $c_\gamma(z) \equiv 0$ $(\gamma \in \Gamma)$ is, by (1.4), equivalent to

$$h(\gamma(z))\,\gamma'(z)^{-1} \equiv h(z) \qquad (\gamma \in \Gamma), \tag{2.1}$$

so that $h(z)$ is an automorphic form of weight -1.

THEOREM 2. *Let* Γ *be a Fuchsian group without elliptic elements. Then* L *is a Carleson set if and only if there exists an automorphic form* $h(z) \not\equiv 0$ *of weight* -1 *such that*

$$\sup_{z \in D} |h'(z)| < \infty . \tag{2.2}$$

Remark. Let $0, a_0, a_1, \cdots, a_n$ be given non-equivalent points in D. If L is a Carleson set we shall actually construct the function $h(z)$ such that it has fourfold zeros at $0, a_1, \cdots, a_n$ and satisfies $h(a_0) \neq 0$.

We derive now Theorem 1 from Theorem 2.

(a) Let z_μ $(\mu = 1, 2, \cdots)$ be non-equivalent points $\neq 0$. According to the above remark we can construct automorphic forms $h_m(z)$ $(m = 1, 2, \cdots)$ of weight -1 that satisfy (2.2),

$$h_m(z_\mu) = 0 \ (\mu = 1, \cdots, m-1), \qquad h_m(z_m) \neq 0, \tag{2.3}$$

and $h_m(0) = h_m'(0) = h_m''(0) = 0$. Then h_m is the Eichler integral of $f_m = h_m'''$, and we see from (1.4) and (2.1) that all Eichler periods vanish. Differentiating (2.1) three times, we obtain $f_m(\gamma(z))\,\gamma'(z)^2 = f_m(z)$ [5, p. 197], and (2.2) implies (1.2) by a standard argument. Hence $f_m \in A_2^\infty(\Gamma)$, and it follows from (1.3) and (2.3) that the functions f_m $(m = 1, 2, \cdots)$ are linearly independent.

(b) Conversely, let there exist $f(z) \not\equiv 0$ in $A_2^\infty(\Gamma)$ with $c_\gamma(z) \equiv 0$ for $\gamma \in \Gamma$. Then the Eichler integral (1.3) satisfies (2.1). Now

$$h'''(z) = f(z) = O((1 - |z|)^{-2}) \qquad (|z| \to 1)$$

implies $h \in \mathrm{Lip}\,\alpha$ $(0 < \alpha < 1)$ by a result of Hardy and Littlewood [4, p. 74]. In particular, $h(z)$ is continuous in $\overline{D}$. Given $\zeta \in L$, we can find $\gamma_n \in \Gamma$ with $\gamma_n(0) \to \zeta$ $(n \to \infty)$, and it follows from (2.1) that

$$h(\gamma_n(0)) = \gamma_n'(0)\,h(0) \to 0 \qquad (n \to \infty). \tag{2.4}$$

Hence $h(\zeta) = 0$ for $\zeta \in L$, and we conclude from Carleson's theorem that L is a Carleson set.

The proof of one direction of Theorem 2, namely that L has to be a Carleson set under the given condition, is contained in the last paragraph (where we did not use $h(0) = 0$); we have $h \in \mathrm{Lip}\,1$ because of (2.2). The proof of the other direction will be postponed to Section 4.

3. GROUPS OF WIDOM TYPE

Let Γ be a Fuchsian group with limit set L and let

$$u(z) = \sum_{\gamma \in \Gamma} |\gamma'(z)| \qquad (z \in \overline{D}); \tag{3.1}$$

the group is, by definition, of convergence type if $u(z) < \infty$ $(z \in D)$. We call Γ of *Widom type* if

$$\int_{\partial D} \log u(z)\,|dz| < \infty. \tag{3.2}$$

These groups were first considered by H. Widom [10] in his investigations of bounded character-automorphic functions; see [8] for a discussion of these groups. We shall need:

LEMMA 1 [8, Theorem 4]. *Let Γ be of Widom type. Then the analytic function*

$$w(z) = \exp\left\{\frac{1}{2\pi}\int_{\partial D} \frac{\zeta + z}{\zeta - z}\log u(\zeta)\,|d\zeta|\right\} \qquad (z \in D) \tag{3.3}$$

satisfies

$$|w(\gamma(z))\,\gamma'(z)| = |w(z)| \qquad (\gamma \in \Gamma), \tag{3.4}$$

$$1 \le u(z) \le |w(z)| \qquad (z \in D). \tag{3.5}$$

We will consider only groups of the second kind (that is, $\partial D \setminus L \neq \emptyset$), and these are of convergence type.

LEMMA 2. *Let*

$$\delta(z) = \inf_{\gamma \in \Gamma} |z - \gamma(0)|. \tag{3.6}$$

If $z = e^{i\theta} \in \partial D \setminus L$, *then*

$$u(z) \le \frac{u(0)}{\delta(z)^2}, \qquad \left|\frac{\partial u(z)}{\partial \theta}\right| \le \frac{2u(z)}{\delta(z)} \le \frac{2u(0)}{\delta(z)^3}, \qquad \left|\frac{\partial^2 u(z)}{\partial \theta^2}\right| \le \frac{6u(0)}{\delta(z)^4}.$$

Proof. If $\gamma \in \Gamma$ we write

$$\gamma(z) = e^{i\alpha}\frac{z-a}{1-\bar{a}z}, \qquad a = \gamma^{-1}(0). \tag{3.7}$$

Let $z = e^{i\theta} \in \partial D \setminus L$. We obtain from (3.6) that

$$|\gamma'| = \frac{1-|a|^2}{|z-a|^2} \leq \frac{|\gamma'(0)|}{\delta(z)^2}, \qquad \left|\frac{\gamma''}{\gamma'}\right| = \left|\frac{2a}{z-a}\right| \leq \frac{2}{\delta(z)}.$$

Hence it follows from (3.1) that

$$u(z) \leq \frac{1}{\delta(z)^2}\sum_{\gamma}|\gamma'(0)| = \frac{u(0)}{\delta(z)^2},$$

$$\left|\frac{\partial u}{\partial \theta}\right| = \left|\sum_{\gamma}|\gamma'|\,\Im\left[z\frac{\gamma''}{\gamma'}\right]\right| \leq \frac{2u(z)}{\delta(z)} \leq \frac{2u(0)}{\delta(z)^3}.$$

Using $[\gamma''/\gamma']' = (\gamma''/\gamma')^2/2$, we see that

$$\left|\frac{\partial^2 u}{\partial \theta^2}\right| = \left|\sum_{\gamma}|\gamma'|\left(\frac{1}{2}\left|\frac{\gamma''}{\gamma'}\right|^2 + \Re\left[z\frac{\gamma''}{\gamma'}\right]\right)\right| \leq \frac{6u(0)}{\delta(z)^4}.$$

LEMMA 3. *If* Γ *is of Widom type then*

$$\frac{|w'(z)|}{|w(z)|^2} \leq \frac{K}{\delta(z)^4} \qquad (z \in D) \tag{3.8}$$

for some constant K.

Proof. Making a suitable rotation, we see that it is sufficient to prove (3.8) for $z = r$, $0 < r < 1$. It follows from (3.5) that $|1/w(z)| < 1$ for $z \in D$, hence that

$$\frac{|w'(r)|}{|w(r)|^2} \leq \frac{1}{1-r^2} < \frac{1}{1-r}.$$

Thus (3.8) holds if $\delta(r) \leq 3(1-r)$.

We may therefore assume that

$$\beta = \delta(r)/3 > 1-r. \tag{3.9}$$

We consider the function

$$v(t) = \log u(e^{it}) \geq 0 \qquad (e^{it} \in \partial D \setminus L). \tag{3.10}$$

If $|t| \leq \beta$ then $\delta(e^{it}) \geq \beta$ by (3.9). Hence it follows from Lemma 2 that, for $|t| \leq \beta$,

$$v(t) \leq \frac{K_1}{\beta^2}, \qquad |v'(t)| \leq \frac{K_1}{\beta^3}, \qquad |v''(t)| \leq \frac{K_1}{\beta^4}. \tag{3.11}$$

Taking the logarithmic derivative in (3.3), we obtain

(3.12)
$$\frac{w'(r)}{w(r)} = \frac{1}{\pi}\int_0^{2\pi} \frac{e^{it}}{(e^{it} - r)^2}\, v(t)\, dt\,.$$

Using (3.2) we see that

(3.13)
$$\left|\int_\beta^{2\pi-\beta} \frac{e^{it}}{(e^{it} - r)^2}\, v(t)\, dt\right| \le \frac{4}{\beta^2}\int_\beta^{2\pi-\beta} v(t)\, dt \le K_2\,\beta^{-2}\,.$$

Since $\{e^{it}\colon |t| \le \beta\} \subset \partial D \setminus L$, Taylor's formula shows that, with suitable $|\tau| \le |t| \le \beta$,

$$\int_{-\beta}^{\beta} \frac{e^{it}}{(e^{it} - r)^2}\, v(t)\, dt = v(0)\int_{-\beta}^{\beta} \frac{e^{it}}{(e^{it} - r)^2}\, dt + v'(0)\int_{-\beta}^{\beta} \frac{t\,e^{it}}{(e^{it} - r)^2}\, dt$$

$$+\frac{1}{2}\int_{-\beta}^{\beta} \frac{t^2 e^{it}}{(e^{it} - r)^2}\, v''(\tau)\, dt\,.$$

We substitute $z = e^{it}$ in the first two integrals on the right-hand side and then deform the integration path to the outer circular arc from $e^{-i\beta}$ to $e^{i\beta}$ of center 1. Thus we see from (3.11) that these integrals are bounded by $K_3\,\beta^{-3}$. Since $|e^{it} - r| \ge |\sin t|$, it follows from (3.11) that the last integral is bounded by $K_4\,\beta^{-4}$. Hence we conclude that

$$\left|\int_{-\beta}^{\beta} \frac{e^{it}}{(e^{it} - r)^2}\, v(t)\, dt\right| \le K_5\,\beta^{-4},$$

and we see from (3.12), (3.13) and (3.9) that

$$\frac{|w'(r)|}{|w(r)|^2} \le \left|\frac{w'(r)}{w(r)}\right| \le \frac{K_6}{\beta^4} = \frac{K_7}{\delta(r)^4}\,.$$

4. CONSTRUCTION OF THE AUTOMORPHIC FORM

We need two results on Carleson sets.

LEMMA 4 (Taylor and Williams [9]). *Let* $Z \subset \overline{D}$ *be closed and let*

(4.1)
$$\sum_{z \in D \cap Z} (1 - |z|) < \infty,$$

(4.2)
$$\int_{\partial D} \log \frac{2}{\operatorname{dist}(z, Z)}\, |dz| < \infty\,.$$

Then there exists a function analytic in D *whose derivatives of all orders are continuous in* $\overline{D}$, *that has* Z *as its zero set in* $\overline{D}$.

LEMMA 5 (Nelson [7]). *Let* $Z \subset \overline{D}$ *be closed. If* $E = Z \cap \partial D$ *is a Carleson set, if* (4.1) *holds, and if*

$$(4.3) \qquad \sum_{z \in Z \cap D} \left[\operatorname{dist}\left(\frac{z}{|z|}, E \right) \right]^{\lambda} < \infty$$

for some $\lambda \geq 1$, *then* (4.2) *is satisfied.*

The next lemma is the only place where we use that Γ has no elliptic elements. Our theorems are probably true without this assumption.

LEMMA 6. *Let* Γ *be a Fuchsian group without elliptic elements whose limit set* L *is a Carleson set. If* $b \in D$ *and* $B = \{\gamma(b): \gamma \in \Gamma\}$, *then*

$$(4.4) \qquad \int_{\partial D} \log \frac{2}{\operatorname{dist}(z, B)} |dz| < \infty$$

and Γ *is of Widom type.*

Proof. Every Moebius transformation $\gamma \in \Gamma$ is either hyperbolic or parabolic. Hence its fixed points lie in L. We can choose the fixed point ζ such that

$$(4.5) \qquad \frac{\gamma'(z)}{(\gamma(z) - \zeta)^2} = \frac{c}{(z - \zeta)^2}, \qquad |c| \geq 1.$$

Since $\zeta \in L$ we deduce that

$$(4.6) \qquad \operatorname{dist}\left(\frac{\gamma(b)}{|\gamma(b)|}, L \right)^2 \leq \left| \frac{\gamma(b)}{|\gamma(b)|} - \zeta \right|^2 \leq \frac{|\gamma(b) - \zeta|^2}{|\gamma(b)|} \leq 4 \left| \frac{\gamma'(b)}{\gamma(b)} \right| .$$

We now apply Lemma 5 with $Z = \overline{B}$ and $\lambda = 2$. Condition (4.1) holds because Γ is of convergence type and (4.3) follows from (4.6). Finally $\overline{B} \cap \partial D = L$ is a Carleson set. We conclude that (4.4) is satisfied. If $b = 0$ then

$$|\gamma'(z)| \leq (1 - |\gamma^{-1}(0)|^2)[\operatorname{dist}(z, B)]^{-2},$$

and we see from (3.1) and (4.4) that Γ is of Widom type.

We prove now the converse part of Theorem 2 and establish the Remark following that theorem. We assume that L is a Carleson set. Let

$$(4.7) \qquad A = \{\gamma(z): z = 0, a_0, \cdots, a_n; \gamma \in \Gamma\} \setminus \{a_0\}.$$

We apply Lemma 4 with $Z = \overline{A} = A \cup L$; it follows easily from Lemma 6 that (4.2) is satisfied.

We can thus find a function $g_0(z)$ analytic in D with

$$(4.8) \qquad |g_0'(z)| < \frac{1}{4} \qquad (z \in D)$$

that has A as its zero set in D. It follows from (3.6), (4.7) and (4.8) that $|g_0(z)| \leq \delta(z)$ for $z \in D$. Hence the analytic function

$$(4.9) \qquad g(z) = g_0(z)^4 \qquad (z \in D)$$

satisfies $g(a_0) \neq 0$ and

$$|g'(z)| < 1, \qquad |g(z)| \leq \delta(z)^4 \qquad (z \in D) . \tag{4.10}$$

We consider the Poincaré theta series (see Lemma 1)

$$h(z) = \sum_{\gamma \in \Gamma} \frac{g(\gamma(z))}{w(\gamma(z))^2 \gamma'(z)} . \tag{4.11}$$

We see from (4.10) and (3.4) that its terms are bounded by

$$|w(\gamma(z))|^{-2} |\gamma'(z)|^{-1} = |w(z)|^{-2} |\gamma'(z)| .$$

Thus (4.11) converges absolutely and locally uniformly in D because Γ is of convergence type, and direct calculation shows that $h(z)$ satisfies (2.1) and is therefore an automorphic form of weight -1. Since $g(z)$ has a fourfold zero at every point of A, we obtain from (4.7) and (4.11) that $h(z)$ has a fourfold zero at $0, a_1, \cdots, a_n$, whereas

$$h(a_0) = g(a_0) w(a_0)^{-2} \neq 0 .$$

It remains to prove (2.2). We obtain from (4.11) that, for $z \in D$,

$$h'(z) = \sum_{\gamma \in \Gamma} \left(- \frac{2w'(\gamma) g(\gamma)}{w(\gamma)^3} + \frac{g'(\gamma)}{w(\gamma)^2} - \frac{\gamma'' g(\gamma)}{\gamma'^2 w(\gamma)^2} \right) . \tag{4.12}$$

We deduce from Lemma 3 and from (4.10) that

$$\sum_{\gamma \in \Gamma} \frac{|w'(\gamma) g(\gamma)|}{|w(\gamma)|^3} \leq \sum_{\gamma} \frac{K}{\delta(\gamma)^4} \frac{\delta(\gamma)^4}{|w(\gamma)|} = K \sum_{\gamma} \left|\frac{\gamma'(z)}{w(z)}\right| = K \frac{u(z)}{|w(z)|} \leq K , \tag{4.13}$$

where we have used (3.4), (3.1) and (3.5). It follows similarly from (4.10) that

$$\sum_{\gamma \in \Gamma} \frac{|g'(\gamma)|}{|w(\gamma)|^2} \leq \frac{1}{|w(z)|} \sum_{\gamma} \left|\frac{\gamma'(z)}{w(z)}\right| |\gamma'(z)| \leq 1 . \tag{4.14}$$

Using the notation (3.7) we see that, for $\gamma \in \Gamma$,

$$\left|\frac{\gamma''(z)}{\gamma'(z)}\right| = \frac{2|a|}{|1 - \bar{a}z|} = \frac{2|az| |\gamma'(z)|}{|\gamma(z) - \gamma(0)|} \leq \frac{2|\gamma'(z)|}{\delta(\gamma(z))} .$$

Hence we obtain from (4.10), (3.4) and (3.5) that

$$\sum_{\gamma \in \Gamma} \frac{|\gamma''| |g(\gamma)|}{|\gamma' w(\gamma)|^2} \leq \frac{2}{|w(z)|^2} \sum_{\gamma} |\gamma'(z)|^2 \leq 2 .$$

Therefore we conclude from (4.12), (4.13) and (4.14) that $|h'(z)| \leq 2K + 3$ for $z \in D$, which is the assertion (2.2) of Theorem 2.

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