REGULAR NEIGHBORHOODS OF ORIENTABLE 3-MANIFOLDS

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1. INTRODUCTION

If M^m and Q^q are PL manifolds with $M \subseteq Q$, any two regular neighborhoods of M in Q are isotopic relative to M [1]. The matter of classifying different regular neighborhoods of a fixed M has been studied by C. P. Rourke and B. J. Sanderson [6], who construct a universal classifying space $BPL_q^{\sim}$; different neighborhoods correspond to homotopy classes of (Δ-) maps of M into $BPL_q^{\sim}$.

In this paper, the different regular neighborhoods of orientable 2- and 3-manifolds will be constructed and compared. As is usually the case, two regular neighborhoods N_1 and N_2 of a manifold M will be considered the same ($N_1 \cong N_2$) if and only if there exists a PL homeomorphism $h: N_1 \to N_2$ such that $h(x) = x$ for all $x \in M$. It will be seen in these two cases that the distinct orientable regular neighborhoods are in one-to-one correspondence with the elements of $H^2(M; \mathbb{Z}_2)$. A similar classification exists for tubular neighborhoods of differentiably embedded closed orientable 2- and 3-manifolds; the techniques are easily adapted to the differentiable case.

The notation and definitions used here will be consistent with those found in J. F. P. Hudson's book [4]. The boundary of a manifold M will be denoted by ∂M, and Δ^n will be the standard n-simplex; further, we write

$$I = [0, 1], \qquad I^1 = [-1, 1], \qquad I^n = I^{n-1} \times I^1, \qquad S^n = \partial I^{n+1}.$$

If L and K are simplicial complexes with $L < K$, $cx(K - L)$ will be used to denote the smallest subcomplex of K that contains $K - L$; by K'' (rel L) we shall mean a second derived subdivision of K relative to L. All maps and manifolds will be PL. In particular, if V and V' are PL manifolds, a *concordance* is a PL homeomorphism $H: V \times I \to V' \times I$ that maps $V \times \{i\}$ homeomorphically to $V' \times \{i\}$ for $i = 0, 1$. Two homeomorphisms $f_0, f_1: V \to V'$ are said to be *concordant relative to* $X \subseteq V$ in case there exists a concordance $H: V \times I \to V' \times I$ with $H_0 = f_0$ and $H_1 = f_1$ such that $H(x, t) = (H_0(x), t)$ for all $x \in X$. In this event, H is said to be *fixed* on X.

Block bundles [6] are a key tool in the construction, as are Δ-sets and their homotopy groups [7]. Of particular importance are the Δ-sets $PL_q^{\sim}$, whose k-simplexes are block isomorphisms of $\Delta^k \times I^q$ onto itself, and $PL_q(I)$, the sub-Δ-set of fibre-preserving block isomorphisms. Each of these sets has two components; however, the symbols $PL_q^{\sim}$ and $PL_q(I)$ will be used here (incorrectly) to represent only the component containing the identity map.

If K is a simplicial complex (with the set of vertices totally ordered), $\underline{K}$ will represent the associated Δ-set. Each n-simplex $A \in K$ will be identified with Δ^n according to the order of its vertices by a map σ^n. This identification induces for

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Received January 16, 1975.
This research was supported in part by the National Science Foundation.

Michigan Math. J. 23 (1976).

each q a unique linear homeomorphism i_A: $A \times I^q \to \Delta^n \times I^q$ defined for $(x, t) \in A \times I^q$ by the equation $i_A(x, t) = (\sigma^n(x), t)$. Hence a Δ-map $\underline{f}$: $\underline{K} \to PL_q^{\sim}$ (respectively, $PL_q(I)$) induces a PL homeomorphism f: $K \times I^q \to K \times I^q$, where for each $A \in K$ the restriction of f to $A \times I^q$ is $i_A^{-1} \circ \underline{f}(A) \circ i_A$. Such a homeomorphism will be called a *block isomorphism* (respectively, a *fibre-preserving block isomorphism*) with base K and blocks I^q.

In Section 2 it will be shown that each orientable regular neighborhood of an orientable 2- or 3-manifold can be written as the union of two product neighborhoods identified by a homeomorphism along certain subsets of their boundary. An equivalence relation will be introduced such that equivalent homeomorphisms define the same neighborhood. In Section 3, these homeomorphisms will be studied; in Sections 4 and 5, orientable regular neighborhoods of orientable 2- and 3- manifolds will be classified.

The author is pleased to acknowledge the help received from Professor T. M. Price, who supervised the dissertation from which this article is taken.

2. PRELIMINARY LEMMAS

Let $M^m \subseteq Q^{q+3}$ be closed orientable connected PL manifolds ($m = 2$ and $q \geq 2$ or $m = 3$ and $q \geq 3$), and write $M = M_1 \cup_f M_2$, where M_1 and M_2 are orientable discs (or cubes) with handles having connected boundaries. For $i = 1, 2$, let N_i be a regular neighborhood of M_i relative to ∂M_i in Q such that $N = N_1 \cup N_2$ is a regular neighborhood of M and such that $N_1 \cap N_2 = N_i' \subseteq \partial N_i$ is a regular neighborhood of ∂M_i in ∂N_i. Since M_1 and M_2 collapse to a wedge of circles, N_1 and N_2 do also. In fact, as a consequence of J. F. P. Hudson's unknotting theorem and the theory of regular neighborhoods, there exists a homeomorphism

$$G_i: (M_i \times I^q, M_i \times \{0\}) \to (N_i, M_i)$$

such that

$$(1)\quad G_i(x, 0) = x \qquad \text{for all } x \in M_i$$

and

$$(2)\quad G_i \mid \partial M_i \times I^q: \partial M_i \times I^q \cong N_i'.$$

Therefore $G_1 \cup G_2$: $(M_1 \times I^q) \cup (M_2 \times I^q) \to N$ is a homeomorphism if $\partial M_1 \times I^q$ is identified with $\partial M_2 \times I^q$ by the homeomorphism $G_2^{-1} G_1 \mid \partial M_1 \times I^q$.

Hence, up to homeomorphism, every orientable regular neighborhood of M is of the form $(M_1 \times I^q) \cup_F (M_2 \times I^q)$, where F: $\partial M_1 \times I^q \to \partial M_2 \times I^q$ is a homeomorphism that extends f.

Notation. (a) Let M_1 and M_2 be manifolds with boundary, let f: $\partial M_1 \to \partial M_2$ be a homeomorphism, and let $M = M_1 \cup_f M_2$. If F: $\partial M_1 \times I^q \to \partial M_2 \times I^q$ is a homeomorphism extending f, we shall use either $N_q(M_1, M_2; F)$ or $N_q(M_1, M; F)$ or simply $N_q(M; F)$ to denote $(M_1 \times I^q) \cup_F (M_2 \times I^q)$.

(b) If G: $N_q(M; F_0) \to N_q(M; F_1)$ is a homeomorphism such that $G(x, 0) = (x, 0)$ for all $x \in M$, we shall say that G is *0-preserving*.

Now, while every orientable neighborhood of M is of the form $N_q(M_1, M_2; F)$, it is also clear that different extensions of f may define the same neighborhood.

The next two lemmas dismiss the obvious cases and suggest a useful equivalence relation. In both cases, M_1 and M_2 are manifolds with boundary, and $f: \partial M_1 \to \partial M_2$ is a homeomorphism.

LEMMA 1. *Let* $F_0, F_1: \partial M_1 \times I^q \to \partial M_2 \times I^q$ *be homeomorphisms extending* f. *If there exists a concordance*

$$H: \partial M_1 \times I^q \times I \to \partial M_1 \times I^q \times I,$$

fixed on $\partial M_1 \times \{0\}$, *such that* $H_0 = F_1^{-1} \circ F_0$ *and* $H_1 =$ *identity, then there exists a* 0-*preserving homeomorphism* $G: N_q(M; F_0) \to N_q(M; F_1)$.

Proof. Let $c: \partial M_1 \times I \to M_1$ be a boundary collar with $c(x, 0) = x$ for all $x \in \partial M_1$, and let $C: \partial M_1 \times I^q \times I \to M_1 \times I^q$ be the map $C(x, y, t) = (c(x, t), y)$. The desired homeomorphism G is given by

$$G(x, y) = \begin{cases} (x, y) & \text{for } (x, y) \in \mathrm{cl}(M - c(\partial M_1 \times I)) \times I^q, \\ CHC^{-1}(x, y) & \text{for } (x, y) \in c(\partial M_1 \times I) \times I^q. \end{cases}$$

LEMMA 2. *Let* $F: \partial M_1 \times I^q \to \partial M_2 \times I^q$ *be a homeomorphism extending* f. *If* $h: I^q \to I^q$ *is a homeomorphism such that* $h(0) = 0$, *and if* $H: \partial M_1 \times I^q \to \partial M_1 \times I^q$ *is defined by the equation* $H(x, y) = (x, h(y))$, *then there exists a* 0-*preserving homeomorphism* $G: N_q(M; F) \to N_q(M; FH)$.

This lemma is obvious.

Definition. Let V, V_1, V_2 be manifolds.

(a) Homeomorphisms $F_0, F_1: V_1 \times I^q \to V_2 \times I^q$ will be called *equivalent* (notation: $F_0 \sim F_1$) if F_0 is concordant to F_1 relative to $V_1 \times \{0\}$.

(b) A homeomorphism $F: V \times I^q \to V \times I^q$ will be said to *reflect* $V \times I^q$ in V if the induced isomorphism $F_*: H_*(V \times I^q, V \times \partial I^q) \to H_*(V \times I^q, V \times \partial I^q)$ is not the identity.

(c) $C_q(V)$ will denote the group of $\sim$-equivalence classes of 0-preserving homeomorphisms of $V \times I^q$ onto itself that do not reflect $V \times I^q$ in V. An element of an equivalence class will be called a C_q-*homeomorphism.*

From Lemmas 1 and 2 it is apparent that equivalent extensions of $f: \partial M_1 \to \partial M_2$ define the same neighborhood of M. We shall see, however, that in some cases non-equivalent homeomorphisms define the same neighborhood. For example, if $F: \partial M_1 \times I^q \to \partial M_2 \times I^q$ is a homeomorphism extending f, and if

$$G: M_2 \times I^q \to M_2 \times I^q$$

is a 0-preserving homeomorphism, then $N_q(M; F) \cong N_q(M; GF)$, but there is no reason to suppose that F is equivalent to GF.

In any event, the first step is to understand better the groups $C_q(V)$ for $V = \partial M_1$ or ∂M_2 and to determine the conditions under which one and hence each representative of a particular equivalence class can be extended to a homeomorphism of $M_i \times I^q$.

3. C_q-HOMEOMORPHISMS

Let M be a manifold, and let $g: M \times I^q \to M \times I^q$ be a C_q-homeomorphism. Then we can regard M as a simplicial complex, and we can assume that g is simplicial. Moreover, by virtue of [6, Theorem 4.4], g is ~-equivalent to a block isomorphism, and if $g \mid \partial M \times I^q$ is already a block isomorphism, then the concordance can be taken relative to $\partial M \times I^q$.

Suppose then that K is a triangulation of M, and let $\underline{K}$ represent the Δ-set defined by K (with some ordering of the vertices). Since block isomorphisms of $K \times I^q$ onto itself correspond bijectively with homotopy classes of Δ-maps from $\underline{K}$ into $PL_q^{\sim}$, and since Δ-homotopies correspond bijectively with ~-equivalences, there exists a bijection (of sets) between $C_q(M)$ and the set of homotopy classes $[\underline{K}; PL_q^{\sim}]$. Likewise, if $L < K$ is a triangulation of ∂M, then a block isomorphism $g: L \times I^q \to L \times I^q$ extends to a 0-preserving homeomorphism of $K \times I^q$ onto itself if and only if the induced Δ-map $\underline{g}: \underline{L} \to PL_q^{\sim}$ extends to a map of $\underline{K}$ into $PL_q^{\sim}$.

LEMMA 3. *Let* $L < K$ *be simplicial complexes, let* $(\underline{K}, \underline{L})$ *be the associated* Δ*-sets, and let* $\underline{g}: (\underline{K}, \underline{L}) \to (PL_q^{\sim}, PL_q(I))$ $(q \geq 3)$.

(a) *If* $\dim(K) \leq 2$, *then* $\underline{g}$ *is homotopic relative to* $\underline{L}$ *to a* Δ*-map* $\underline{h}: \underline{K} \to PL_q(I)$.

(b) *If* $\dim(K) \leq 3$, *then there exists a* Δ*-map* $\underline{h}: \underline{K} \to PL_q(I)$ *that agrees with* $\underline{g}$ *on* $\underline{L}$.

In other words, if K is a 1- or 2-manifold, then the natural map $[\underline{K}, PL_q(I)] \to [\underline{K}, PL_q^{\sim}]$ is a bijection.

Proof. The proof is by induction on the dimension of K, and it follows immediately from the triviality of the groups $\pi_i(PL_q^{\sim}, PL_q(I))$ and $\pi_2(PL_q(I))$ for $i \leq 2$ and $q \geq 3$ [3].

It follows that if V is a 1- or 2-manifold, then each equivalence class in $C_q(V)$ contains a fibre-preserving block isomorphism. In particular, if $|K| = S^1$, then

$$C_q(S^1) \approx [\underline{K}, PL_q(I)] \approx \pi_1(PL_q(I)) \approx Z_2 .$$

Likewise, since $\pi_2(PL_q(I)) \approx 0$,

$$C_q(S^1 \times S^1) \approx \pi_1(PL_q(I)) \oplus \pi_1(PL_q(I)) \approx Z_2 \oplus Z_2 .$$

If I^q is identified with the unit q-ball B^q, then, since $\pi_1(PL_q(I)) \approx \pi_1(SO(q))$, the induced homeomorphism $f: S^1 \times B^q \to S^1 \times B^q$ can be realized as a fibre-preserving SO(q) bundle map of the form $G(x, y) = (x, g_x(y))$, where $g_x = g(x)$, and $g: S^1 \to SO(q)$ represents a generator of $\pi_1(SO(q))$. The map $g: S^1 \to SO(q)$ can in turn be defined at each point $e^{i\theta} \in S^1$ as the (q - 2)-fold suspension of the map $g_2: S^1 \to SO(2)$, where $g_2(e^{i\theta})$ is the rotation of B^2 through θ radians.

A nice geometric analysis of maps $f: S^1 \times I^3 \to S^1 \times I^3$ is presented in [2].

COROLLARY 1. *Let* $(M, \partial M)$ *be a* 2- *or* 3-*manifold with triangulation* (K, L), *and let* $g: M \times I^q \to M \times I^q$ *be a fibre-preserving block isomorphism induced by* $\underline{g}: \underline{L} \to PL_q(I)$. *Then* g *extends to a* 0-*preserving homeomorphism*

$$G: M \times I^q \to M \times I^q$$

if and only if $\underline{g}$ *extends to map* $\underline{K}$ *into* $PL_q(I)$.

Proof. If $\underline{g}$ extends, it is evident that g extends. Conversely, as we observed earlier, every 0-preserving extension of g is ~-equivalent relative to $\partial M \times I^q$ to a block isomorphism. Therefore the desired extension exists, by Lemma 3(b).

COROLLARY 2. *Let* $(M, \partial M)$ *be a* 2*-manifold with triangulation* (K, L), *and let* $\underline{g}: \underline{L} \to PL_q(I)$ *be a map. Then* $\underline{g}$ *extends to map* $\underline{K}$ *into* $PL_q(I)$ *if and only if* $\underline{g} \simeq 0$.

Proof. If $\underline{g} \simeq 0$, then $\underline{g}$ clearly extends, and conversely, since $\pi_1(PL_q(I)) \approx \mathbb{Z}_2$.

COROLLARY 3. *Let* U_n *be a solid orientable* 3*-dimensional handlebody of genus* n *with triangulation* K, *and let* $\underline{g}: \partial\underline{K} \to PL_q(I)$.

(a) *There exist a solid torus* $T \subseteq \operatorname{int}(U_n)$ *with triangulation* $K_0 < K''$ (rel ∂K) *and an extension* $\underline{G}: \underline{L} \to PL_q(I)$, *where* $L = cx(K''(\text{rel } \partial K) - K_0)$.

(b) *If* $n = 1$ *and* $J < \partial K$ *represents a meridian, then* $\underline{g}: \partial\underline{K} \to PL_q(I)$ *extends if and only if* $\underline{g} \mid \underline{J} \simeq 0$.

Proof. If $n = 0$, then $U_n = I^3$ and $\underline{g}$ extends to all of $\underline{K}$, since $\pi_2(PL_q(I)) \approx 0$.

Otherwise, let $A_1, \cdots, A_n < \partial K$ be subcomplexes representing meridians of U_n, and let $\Delta_1, \cdots, \Delta_n < K$ be disjoint discs with $\partial\Delta_i = A_i$. Choose discs $D_1, \cdots, D_{n-1} < K$ disjoint from each other and from the Δ_i, with $(D_i, \partial D_i) < (K, \partial K)$ and such that $A_i \cup \partial D_i \cup A_{i+1}$ bound on ∂K.

Suppose first that $\underline{g} \mid \underline{A}_i \not\simeq 0$ for all i $(1 \leq i \leq n)$. Then, since $\pi_1(PL_q(I)) \approx \mathbb{Z}_2$, $\underline{g} \mid \partial\underline{D}_i \simeq 0$ and hence $\underline{g}$ extends to a map of $\partial\underline{K} \cup \left(\bigcup_{i=1}^{n-1} \underline{D}_i\right)$ into $PL_q(I)$. Let $T \subseteq \operatorname{int}(U_n)$ be a solid torus that intersects each Δ_i once and misses each D_i and such that the region between T and $\partial U_n \cup \left(\bigcup_{i=1}^{n-1} |D_i|\right)$ is homeomorphic to $\partial T \times (0, 1)$. Since T can easily be chosen to coincide with a subcomplex of K''(rel ∂K), the map $\underline{g}$ will extend.

Now, if $\underline{g} \mid \underline{A}_i \not\simeq 0$ for $1 \leq i \leq k < n$, and $\underline{g} \mid \underline{A}_i \simeq 0$ for $k < i \leq n$, then $\underline{g}$ extends to map $\underline{\Delta}_i$ into $PL_q(I)$ for $k < i \leq n$. After we cut U_n along these Δ_i, the proof follows as above.

(b) follows immediately from (a), since $\pi_2(PL_q(I)) \approx 0$.

4. REGULAR NEIGHBORHOODS OF TWO-MANIFOLDS

As we have seen, if M is a closed orientable 2-manifold embedded in a $(q + 2)$-manifold Q $(q \geq 3)$, and if N is an orientable regular neighborhood of M in Q, then corresponding to each decomposition $M = M_1 \cup M_2$ with

$$M_1 \cap M_2 = \partial M_1 = \partial M_2 = \Sigma \cong S^1$$

there exist a C_q-homeomorphism $g: \Sigma \times I^q \to \Sigma \times I^q$ and a homeomorphism $H: N \to N_q(M_1, M_2; g)$ such that $H(x) = (x, 0)$ for all $x \in M$. Since $C_q(\Sigma) \approx \mathbb{Z}_2$, it remains only to observe that the two possible neighborhoods are distinct.

Let K be a triangulation of Σ, let $\underline{g}: \underline{K} \to PL_q(I)$ be a map onto a generator of $\pi_1(PL_q(I))$, and let $g: \partial M_1 \times I^q \to \partial M_2 \times I^q$ be the induced homeomorphism.

THEOREM 1 (T. M. Price [5]). *Let* M *be a closed orientable* 2*-manifold piecewise-linearly embedded in a* $(q + 2)$*-manifold* Q $(q \geq 3)$. *Let* $M_1, M_2 \subseteq M$ *be compact submanifolds such that*

$$M = M_1 \cup M_2 \qquad \textit{and} \qquad M_1 \cap M_2 = \partial M_1 \cap \partial M_2 = \Sigma \cong S^1 .$$

If N *is an orientable regular neighborhood of* M *in* Q, *then either* $N \cong M \times I^q$ *or* $N \cong (M_1 \times I^q) \cup_g (M_2 \times I^q) = N_q(M_1, M_2; g)$.

Furthermore, the two possibilities are distinct; that is, there exists no 0-*preserving homeomorphism* $M \times I^q \to N_q(M_1, M_2; g)$.

Proof. As we observed above, it remains only to prove that the possibilities are distinct.

Suppose there exists a 0-preserving homeomorphism

$$H: M \times I^q \to N_q(M_1, M_2; g).$$

By [6, Theorem 4.4] and Lemma 3(a), we can suppose that H is a fibre-preserving block isomorphism, and in particular that H restricts to fibre-preserving homeomorphisms $H_i: M_i \times I^q \to M_i \times I^q$ for $i = 1, 2$. The diagram

$$\begin{array}{ccc} \partial M_1 \times I^q & \xrightarrow{\;1\;} & \partial M_2 \times I^q \\ h_1 \big\downarrow & & \big\downarrow h_2 \\ \partial M_1 \times I^q & \xrightarrow{\;g\;} & \partial M_2 \times I^q \end{array}$$

where $h_i = H_i \mid \partial M_i \times I^q$, must then commute. Now h_1 and h_2: $\Sigma \times I^q \to \Sigma \times I^q$ are induced by Δ-maps $\underline{h}_1$, $\underline{h}_2$: $\underline{K} \to PL_q(I)$, and since the diagram commutes, $\underline{g}\,\underline{h}_1 = \underline{h}_2$, where the product is in $\pi_1(PL_q(I))$. But $\underline{h}_1 \simeq 0 \simeq \underline{h}_2$, by Corollary 2. Hence $\underline{g} \simeq 0$, a contradiction.

5. REGULAR NEIGHBORHOODS OF THREE-MANIFOLDS

Let $M = M_1 \cup_f M_2$ be a Heegard splitting of an orientable 3-manifold M, where $M_1 \cong U_n \cong M_2$, and suppose that $M \subseteq Q^{q+3}$ ($q \geq 3$). As we have seen, every orientable regular neighborhood of M in Q is of the form

$$N \cong (M_1 \times I^q) \cup_F (M_2 \times I^q) = N_q(M_1, M_2; F),$$

where $F: \partial M_1 \times I^q \to \partial M_2 \times I^q$ is a 0-preserving homeomorphism that extends f.

Now $C_q(\partial M_1)$ is apt to be large, and in any case it depends on the Heegard splitting. The next lemma will allow modifications of the decomposition of M. Let $M = M_1 \cup_f M_2$, where M_1 and M_2 are arbitrary submanifolds of M identified by f along a common boundary, and let $F: \partial M_1 \times I^q \to \partial M_2 \times I^q$ be an extension of f.

LEMMA 4. *Let* K_1 *be a triangulation of* M_1, *and let* $\underline{g}$: $\partial \underline{K}_1 \to PL_q(I)$. *Let* $M_0 \subseteq M_1$ *be a submanifold with triangulation* $K_0 < K$, *and define* $L = cx(K_1 - K_0)$. *If* $\underline{g}$ *extends to a map* $\underline{G}$: $\underline{L} \to PL_q(I)$, *then there exists a* 0-*preserving homeomorphism*

$$H: M_1 \times I^q \cup_F M_2 \times I^q \to M_0 \times I^q \cup_{G_0} cl(M_1 - M_0) \times I^q \cup_{FG_1^{-1}} M_2 \times I^q,$$

where for $i = 0, 1$, G_i: $\partial M_i \times I^q \to \partial M_i \times I^q$ *is the homeomorphism induced by the restriction of* $\underline{G}$ *to* $\partial \underline{K}_i$.

In particular,

(a) *if* $FG_1^{-1} = f \times 1\colon \partial M_1 \times I^q \to \partial M_2 \times I^q$, *then*

$$N_q(M_1, M_2; F) \cong N_q(M_0, \mathrm{cl}(M - M_0); G_0),$$

(b) *if* $M_0 = \emptyset$, *then* $N_q(M_1, M_2; F) \cong N_q(M_1, M_2; FG_1^{-1})$.

Proof. The proof is identical to that of Lemma 1. Define

$$H(x, y) = \begin{cases} (x, y) & \text{for } (x, y) \in (M_0 \cup M_2) \times I^q, \\ G(x, y) & \text{for } (x, y) \in \mathrm{cl}(M_1 - M_0) \times I^q, \end{cases}$$

where $G\colon \mathrm{cl}(M_1 - M_0) \times I^q \to \mathrm{cl}(M_1 - M_0) \times I^q$ is the fibre-preserving block isomorphism induced by $\underline{G}$.

COROLLARY 4. *Let* M *be a closed orientable* 3*-manifold in a* $(q+3)$*-manifold* Q $(q \geq 3)$. *If* N *is an orientable regular neighborhood of* M *in* Q, *then there exist a solid torus* $T \subseteq M$, *a fibre-preserving block isomorphism*

$$G_0\colon \partial T \times I^q \to \partial(\mathrm{cl}(M - T)) \times I^q,$$

and a homeomorphism $H\colon N \to N_q(T, M; G_0)$ *such that* $H(x) = (x, 0)$ *for all* $x \in M$.

Proof. Let $M = M_1 \cup_f M_2$ be a Heegard splitting of M with triangulation $K = K_1 \cup_f K_2$, and let $F\colon \partial M_1 \times I^q \to \partial M_2 \times I^q$ be the product extension $F(x, y) = (f(x), y)$. By Lemma 3 and the remarks prior to it, there exists a Δ-map $\underline{G}_1\colon \partial\underline{K}_1 \to PL_q(I)$ such that $N \cong N_q(M_1, M_2; FG_1)$. By Corollary 3, there exist a solid torus $T \subseteq M_1$ with triangulation $K_0 < K_1''(\mathrm{rel}\ \partial K_1)$ and an extension $\underline{G}\colon \underline{L} \to PL_q(I)$, where $L = \mathrm{cx}(K_1''(\mathrm{rel}\ \partial K_1) - K_0)$ is a complex triangulating $\mathrm{cl}(M_1 - T)$. Let

$$G_0\colon \partial T \times I^q \to \partial(\mathrm{cl}(M - T)) \times I^q$$

be the map induced by $\underline{G}_0 = \underline{G} \mid \partial\underline{K}_0$. Then, because $(FG_1)G_1^{-1} = F = f \times 1$, the result follows from Lemma 4(a).

Now, for each torus $T \subseteq M$ with triangulation K_0, there are four possible homotopy classes of maps $\underline{g}\colon \partial\underline{K}_0 \to PL_q(I)$, since $C_q(S^1 \times S^1) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2$. However, we shall see in the next lemma that at most two different neighborhoods of the form $N_q(T, M; G)$ can occur, for any given T.

Let T be a solid torus with triangulation K_0, let $a, b\colon \partial\Delta^2 \to \partial K_0$ be homeomorphisms onto a meridian and longitude, respectively, and for $i, j = 0$ or 1, define maps $\underline{g}_{i,j}\colon \partial\underline{K}_0 \to PL_q(I)$ by

$$\underline{g}_{i,j} \circ \underline{a} \simeq 0 \quad \text{if and only if } i = 0,$$

$$\underline{g}_{i,j} \circ \underline{b} \simeq 0 \quad \text{if and only if } j = 0.$$

As usual, $g_{i,j}\colon \partial T \times I^q \to \partial T \times I^q$ will be the induced homeomorphism.

LEMMA 5. *Let* M *be an orientable* 3*-manifold, and let* $T \subseteq M$ *be a solid torus. Then*

(a) $N_q(T, M; g_{0,j}) \cong M \times I^q$ *for* $j = 0, 1$,

(b) $N_q(T, M; g_{1,0}) \cong N_q(T, M; g_{1,1})$.

Proof. By Corollary 3(b), the map $g_{0,j}$: $\partial K_0 \to PL_q(I)$ extends to a map of $\underline{K}_0$ into $PL_q(I)$.

(a) By Lemma 4(b) (with $M_1 = T$ and $M_0 = \emptyset$), there exists a 0-preserving homeomorphism $N_q(T, M; g_{0,j}) \cong M \times I^q$.

(b) Let $h = g_{1,0} \circ g_{0,1}^{-1}$. By Lemma 4 (with $M_0 = \emptyset$), there exists a 0-preserving homeomorphism

$$T \times I^q \cup_{g_{1,0}} cl(M - T) \times I^q \to T \times I^q \cup_h cl(M - T) \times I^q.$$

But $h = g_{1,0} \circ g_{0,1}^{-1} \sim g_{1,0} \circ g_{0,1} \sim g_{1,1}$. Therefore, by Lemma 1, $N_q(T, M; g_{1,0}) \cong N_q(T, M; g_{1,1})$.

THEOREM 2. *Let* M *be a closed orientable* 3-*manifold, piecewise-linearly embedded in a* $(q + 3)$-*manifold* Q $(q \geq 3)$, *and let* N *be an orientable regular neighborhood of* M *in* Q. *Then there exist a polyhedral simple closed curve* $J_0 \subseteq M$, *a regular neighborhood* $T_0 \cong J_0 \times I^2$ *of* J_0 *in* M, *and a homeomorphism* $F: N \to N_q(T_0, M; g_{1,0})$ *such that* $F(x) = (x, 0)$ *for all* $x \in M$. *Furthermore, if* $J_1 \subseteq M$ *is another polyhedral simple closed curve with regular neighborhood* $T_1 \subseteq M$, *then there exists a* 0-*preserving homeomorphism*

$$H: N_q(T_0, M; g_{1,0}) \cong N_q(T_1, M; g_{1,0})$$

if and only if $J_0 \sim J_1 \pmod{\mathbb{Z}_2}$. *In particular,* $N_q(T_0, M; g_{1,0}) \cong M \times I^q$ *if and only if* $J_0 \sim 0 \pmod{\mathbb{Z}_2}$.

LEMMA 6. *Let* M *be a* 3-*manifold with triangulation* L *such that* $\partial L = K = \bigcup_{n=1}^{N} K_n$ *is a collection of disjoint complexes with* $|K_n| \cong S^1 \times S^1$ *for all* n $(1 \leq n \leq N)$. *Let* a_n, b_n: $\partial\Delta^2 \to K_n$ *be simplicial maps representing generators of* $\pi_1(K_n)$, *and let* $\alpha_n, \beta_n \in H_1(K_n; \mathbb{Z}_2)$ *be the induced generators. Let* $\underline{g}$: $\underline{K} \to PL_q(I)$, *and suppose that the restriction of* $\underline{g}$ *to* $\underline{K}_n$ *is* $\underline{g}_{i_n, j_n}$, *where* i_n, $j_n = 0$ *or* 1. *Then* $\underline{g}$ *extends to map* $\underline{L}$ *into* $PL_q(I)$ *if and only if* $\sum_{n=1}^{N} i_*(j_n \alpha_n + i_n \beta_n) = 0$, *where* $i_*: H_1(K; \mathbb{Z}_2) \to H_1(L; \mathbb{Z}_2)$.

Proof. Let $\rho \in H_2(K; \mathbb{Z}_2)$ be the generator. Then the diagram (coefficients in $\pi_1(PL_q(I)) \approx \mathbb{Z}_2$)

$$\begin{array}{ccc} H^1(K) & \xrightarrow{\delta^*} & H^2(L, K) \\ \approx \big\downarrow \partial\rho\cap & & \approx \big\downarrow \rho\cap \\ H_1(K) & \xrightarrow{i_*} & H_1(L) \end{array}$$

commutes; here $\partial\rho\cap$ and $\rho\cap$ are the isomorphisms of Poincaré and Lefschetz duality, respectively.

Since $PL_q(I)$ represents the component that contains the identity map, the only obstruction to extending $\underline{g}$ over $\underline{L}$ is an element, $c(\underline{g})$, of $H^2(L, K; \pi_1(PL_q(I)))$. Moreover, it must be in the image of δ^*, since $j^*(c(\underline{g})) = 0$, where

$$j^*: H^2(L, K; \pi_1(PL_q(I))) \to H^2(L; \pi_1(PL_q(I))).$$

We shall show that the obstruction $c(\underline{g})$ is equal to $(\rho\cap)^{-1}\left(\sum_{n=1}^{N} i_*(j_n\alpha_n + i_n\beta_n)\right)$. The lemma then follows, since $\rho\cap$ is an isomorphism.

Let $\hat{\beta}_n = (\partial\rho\cap)^{-1}(\alpha_n)$, and let $\hat{\alpha}_n = (\partial\rho\cap)^{-1}(\beta_n)$. Then

$$(\rho\cap)^{-1}\left(\sum_{n=1}^{N} i_*(j_n\alpha_n + i_n\beta_n)\right) = \sum_{n=1}^{N} \delta^*(i_n\hat{\alpha}_n + j_n\hat{\beta}_n) .$$

Let $v*K$ be a cone over K, and let $p: L \to v*K$ be a projection that is the identity on K. Then if $\hat{c}(\underline{g})$ is the obstruction to extending $\underline{g}$ over $v*K$, $c(\underline{g}) = p^*(\hat{c}(\underline{g}))$.

Now, it is straightforward to verify that $\hat{c}(\underline{g}) = \sum_{n=1}^{N} \delta^*(i_n\hat{\alpha}_n + j_n\hat{\beta}_n)$, where $\delta^*: H^1(K; \mathbb{Z}_2) \to H^2(v*K, K; \mathbb{Z}_2)$. For if $z = v*a_n$ $(1 \leq n \leq N)$ is a generator of $H_2(v*K, K; \mathbb{Z}_2)$, then $[z, \hat{c}(\underline{g})] = 0$ if and only if $\underline{g}\circ\underline{a}_n \simeq 0$, by the definition of an obstruction. On the other hand, $\underline{g}\circ\underline{a}_n \simeq 0$ if and only if

$$i_n = \left[\partial z, \sum_{n=1}^{N} (i_n\hat{\alpha}_n + j_n\hat{\beta}_n)\right] = \left[z, \sum_{n=1}^{N} \delta^*(i_n\hat{\alpha}_n + j_n\hat{\beta}_n)\right] = 0 .$$

Likewise, if $z = v*b_n$ $(1 \leq n \leq N)$, then $[z, \hat{c}(\underline{g})] = 0$ if and only if

$$\left[z, \sum_{n=1}^{N} \delta^*(i_n\hat{\alpha}_n + j_n\hat{\beta}_n)\right] = 0 .$$

Since $[z, \hat{c}(\underline{g})]$ can only be 0 or 1, $\hat{c}(\underline{g})$ must be the indicated sum.

Therefore, $c(\underline{g}) = p^*(\hat{c}(\underline{g})) = \sum_{n=1}^{N} \delta^*(i_n\hat{\alpha}_n + j_n\hat{\beta}_n)$, since $p^*\delta^* = \delta^*p^*$, and since p^* is the identity on $H^1(K; \mathbb{Z}_2)$.

Proof of the theorem. In view of Corollary 4 and Lemma 5, it is only necessary to show that

(1) $N_q(T_0, M; g_{1,0}) \cong N_q(T_1, M; g_{1,0})$ if and only if $J_0 \sim J_1 \pmod{\mathbb{Z}_2}$,

(2) $N(T_0, M; g_{1,0}) \cong M \times I^q$ if $J \sim 0 \pmod{\mathbb{Z}_2}$.

By an isotopy, we may assume that T_0 and T_1 are disjoint.

Suppose then that $J_0 \sim J_1$, and let $\alpha_0, \alpha_1, \beta_0, \beta_1$ be meridians and longitudes on $\partial T_0 = \partial(\mathrm{cl}(M - T_0))$ and $\partial T_1 = \partial(\mathrm{cl}(M - T_1))$, respectively. From the Mayer-Vietoris sequence (coefficients in $\mathbb{Z}_2$)

$$H_1(\partial(\mathrm{cl}(M - (T_0 \cup T_1)))) \xrightarrow{\Psi} H_1(T_0 \cup T_1) \oplus H_1(\mathrm{cl}(M - (T_0 \cup T_1))) \xrightarrow{\Phi} H_1(M)$$

we see that $\Phi(0 \oplus (\beta_0 + \beta_1)) = 0$. Hence there exists

$$z = j_0\alpha_0 + j_1\alpha_1 + i_0\beta_0 + i_1\beta_1 \in H_1(\partial(\mathrm{cl}(\check{M} - (T_0 \cup T_1))))$$

with $j_0, j_1, i_0, i_1 = 0$ or 1 such that

$$\Psi(z) = (i_0\beta_0 + i_1\beta_1) \oplus (j_0\alpha_0 + j_1\alpha_1 + i_0\beta_0 + i_1\beta_1) = 0 \oplus (\beta_0 + \beta_1) .$$

Necessarily then, $i_0 = i_1 = 0$; that is,

$$(*)\qquad z = j_0\alpha_0 + j_1\alpha_1 = \beta_0 + \beta_1 .$$

Let L be a triangulation of $cl(M - (T_0 \cup T_1))$, and let $K_0, K_1 < L$ be subcomplexes that define triangulations of ∂T_0 and ∂T_1, respectively. Define

$$\underline{g}\colon \underline{K}_1 \cup \underline{K}_0 \to PL_q(I)$$

by the equations

$$\underline{g} \mid \underline{K}_0 = \underline{g}_{1,j_0}, \qquad \underline{g} \mid \underline{K}_1 = \underline{g}_{1,j_1} .$$

Then $\underline{g}$ will extend over $\underline{L}$ provided that $i_*(j_0\alpha_0 + \beta_0 + j_1\alpha_1 + \beta_1) = 0$, which it is, by (*). From Lemma 4 (with $M_0 = T_0$, $M_1 = cl(M - T_1)$, and $M_2 = T_1$), it follows that

$$cl(M - T_1) \times I^q \cup_{g_{1,j_1}} T_1 \times I^q \cong T_0 \times I^q \cup_{g_{1,j_0}} cl(M - T_0) \times I^q .$$

By Lemma 5(b),

$$cl(M - T_1) \times I^q \cup_{g_{1,j_1}} T_1 \times I^q \cong T_1 \times I^q \cup_{g_{1,0}} cl(M - T_1) \times I^q = N_q(T_1, M; g_{1,0})$$

and

$$T_0 \times I^q \cup_{g_{1,j_0}} cl(M - T_0) \times I^q \cong T_0 \times I^q \cup_{g_{1,0}} cl(M - T_0) \times I^q \cong N_q(T_0, M; g_{1,0}) .$$

Therefore, if $J_0 \sim J_1 \pmod{\mathbb{Z}_2}$, then

$$N_q(T_0, M; g_{1,0}) \cong N_q(T_1, M; g_{1,0}) .$$

Moreover, if we replace T_1 by the empty set in the proof above, case (2) (where $J \sim 0 \pmod{\mathbb{Z}_2}$) follows.

Conversely, then, if there exists a 0-preserving homeomorphism

$$H\colon T_0 \times I^q \cup_{g_{1,0}} cl(M - T_0) \times I^q \to T_1 \times I^q \cup_{g_{1,0}} cl(M - T_1) \times I^q ,$$

we can, by [6, Theorem 4.4] and Lemma 3, suppose that H is a fibre-preserving block isomorphism, and in particular that the restriction of H maps $T_i \times I^q$ onto itself for $i = 0, 1$. Hence the diagram

$$\begin{array}{ccccc}
\partial T_0 \times I^q & \xrightarrow{g_{1,0}} & \partial\, cl(M - (T_0 \cup T_1)) \times I^q & \xleftarrow{1} & \partial T_1 \times I^q \\
\downarrow h_0 & & \downarrow H_0 \mid H_1 & & \downarrow h_1 \\
\partial T_0 \times I^q & \xrightarrow{1} & \partial\, cl(M - (T_0 \cup T_1)) \times I^q & \xleftarrow{g_{1,0}} & \partial T_1 \times I^q
\end{array}$$

commutes; here h_i is the restriction of H to $\partial T_i \times I^q$, and H_i the restriction to $\partial\, cl(M - T_i) \times I^q$. Since h_i extends to map $T_i \times I^q$ onto itself, $h_0 \sim g_{0,j_0}$, where $j_0 = 0$ or 1; similarly, $h_1 \sim g_{0,j_1}$ ($j_1 = 0$ or 1). Therefore, by commutivity,

$H_0 \sim h_0 g_{1,0}^{-1} \sim g_{1,j_0}$. Likewise, $H_1 \sim g_{1,j_1}$. As before, let L be a triangulation of cl(M - $(T_0 \cup T_1)$), and let K_0, $K_1 < L$ be subcomplexes triangulating ∂ cl(M - T_0) and ∂ cl(M - T_1), respectively. Let $\underline{g}$: $\underline{K}_0 \cup \underline{K}_1 \to PL_q(I)$ be the Δ-map defined by the equations $\underline{g} \mid \underline{K}_0 = g_{1,j_0}$ and $\underline{g} \mid \underline{K}_1 = g_{1,j_1}$. Then $\underline{g}$ induces H_0 in ∂ cl(M - T_0) and H_1 in ∂ cl(M - T_1). Now, $\underline{g}$ extends to a map of $\underline{L}$ into $PL_q(I)$, by Corollary 1; therefore, $i_*(j_0 \alpha_0 + \beta_0 + j_1 \alpha_1 + \beta_1) = 0$, by Lemma 6. Equivalently, $i_*(\beta_0 + \beta_1) = i_*(j_0 \alpha_0 + j_1 \alpha_1)$ in $H_1(\mathrm{cl}(M - (T_0 \cup T_1)); \mathbb{Z}_2)$. Now let

$$i'_*: H_1(\mathrm{cl}(M - (T_0 \cup T_1)); \mathbb{Z}_2) \to H_1(M; \mathbb{Z}_2)$$

be the inclusion-induced morphism. Since $i'_* i_*(j_0 \alpha_0 + j_1 \alpha_1) = 0$, it follows that $i'_* i_*(\beta_0 + \beta_1) = 0$ also; hence $J_0 \sim J_1 \pmod{\mathbb{Z}_2}$ in M.

COROLLARY 5. *Let* M *be a closed* 3-*manifold with* $H_1(M; \mathbb{Z}_2) \approx 0$. *If* M *is piecewise-linearly embedded in* Q^{q+3} $(q \geq 3)$, *and if* N *is an orientable regular neighborhood of* M *in* Q, *then* $N \cong M \times I^q$.

Proof. It is sufficient to observe that M is orientable, since $H_1(M; \mathbb{Z}_2) \approx 0$.

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