# ON THE OVALS OF EVEN-DEGREE PLANE CURVES

R. W. Sharpe

## 1. INTRODUCTION

Let $P(x_0, x_1, x_2)$ be a real homogeneous polynomial of degree d. In 1876, Axel Harnack showed that the real locus $A \subset \mathbb{R}P^2$ of the equation $P = 0$ has at most $g + 1$ components, where $g = \text{genus} = (d - 1)(d - 2)/2$. In the same paper [3], he gave a construction in every degree for a curve with the maximal number of components (an M-*curve*).

If P is nonsingular on $\mathbb{R}P^2$, then A consists of disjoint circles. In even degrees, the *sign* of P is well defined on $\mathbb{R}P^2$, so that each of these circles is two-sided. In odd degrees, exactly one of these circles fails to be two-sided.

Two-sided circles are called *ovals*, and each oval divides $\mathbb{R}P^2$ into a disc and a Möbius band. Thus we can say that one oval lies in another if the first lies in the disc component of the complement of the second.

Our aim here is to publicize some recent Russian work describing the mutual disposition of the ovals of a plane curve, and also to provide an independent proof of an inequality due to V. A. Rohlin.

Let P (respectively, N) be the number of ovals properly contained in an even (respectively, odd) number of ovals. In 1938, I. Petrovsky [4] showed that for a nonsingular curve of even degree $d = 2k$, $|2(P - N) - 1| \leq 3k^2 - 3k + 1$. In particular, this implies that $N \geq (k - 1)(k - 2)/4$.

V. I. Arnol'd [1] has suggested that for M-curves $|P - N - 1| \leq k^2 - 1$, which implies that $N \geq (k - 1)(k - 2)/2$. We give here an inequality, due to Rohlin [8], that is independent of Petrovsky's theorem, but would follow from Arnol'd's inequality.

Let the *complexity* of a curve be the number of proper inclusions among its ovals.

THEOREM. *A nonsingular* M-*curve of even degree* 2k *has complexity at least* $(k - 1)(k - 2)/2$, *and these numbers are equal* mod 2.

In the series of M-curves constructed by Harnack, in odd degrees no oval contains any other oval, while in even degree 2k, there are $g + 1 - (k - 1)(k - 2)/2$ ovals all exterior to each other, and one of these contains the remaining $(k - 1)(k - 2)/2$ ovals, which themselves are mutually exterior to each other. Thus not only is the bound on complexity exact in every even degree, but also there can be no direct analogue of the theorem in odd degrees.

Recently, Arnol'd [1] and Rohlin ([5], [6], and [7]) have proved Gudkov's conjecture [2] that a nonsingular M-curve of even degree 2k satisfies the condition $P - N \equiv k^2 \pmod 8$. Rohlin [8] gives a new proof of the Gudkov conjecture, but only mod 4. The novelty in our proof lies in the fact that it invokes only the geometry of

Received June 19, 1975.
This research was supported in part by NSF Grant GP 33019X.

Michigan Math. J. 22 (1975).

$CP^2$ itself, whereas all other proofs use a certain twofold cover of $CP^2$ ramified along the complex locus of $P = 0$.

## 2. PROOF OF THE THEOREM

Let $\sigma: \mathbb{C}P^2 \to \mathbb{C}P^2$ denote complex conjugation, and let $\mathbb{C}A$ denote the complex locus $P = 0$. We may assume $\mathbb{C}A$ is nonsingular (after a slight shift of the coefficients of P, which will not alter the disposition of the ovals). The proof of Harnack's theorem [3] shows that A divides $\mathbb{C}A$ into two components $\mathbb{C}A_{\pm}$ that are interchanged by complex conjugation. Choose one of these, say $\mathbb{C}A_+$. The orientation of $\mathbb{C}A_+$ orients the ovals (which form its boundary), so that we can compare the orientations of two related ovals. Now we number the ovals $0, 1, \cdots, g$, and the inclusion relation among the ovals gives a partial ordering (written $i < j$) on these indices. Let

$$\varepsilon_{ij} = \begin{cases} \pm 1 \\ 0 \end{cases} \text{according as the orientations of the } i^{th} \text{ and } j^{th} \text{ ovals agree, disagree, or cannot be compared.}$$

The theorem will follow from the following result.

PROPOSITION 1. $\sum_{i<j} \varepsilon_{ij} = -(k-1)(k-2)/2$.

*Proof.* Let $D = \sum_{i=0}^{g} D_i$ be the integral chain on $\mathbb{R}P^2$ made up of the discs bounded by the $g + 1$ components of A. We orient these discs so that $z = \mathbb{C}A_+ + D$ is an integral cycle. We obtain the proposition by computing $z \cdot z$ in two ways. First note that $\sigma_* = -1$ on $H_2(\mathbb{C}P^2; Z)$, since it reverses the orientation of the generator $[\mathbb{C}P^1]$. Thus

$$2z \sim z - \sigma z \sim \mathbb{C}A_+ + D - \sigma\mathbb{C}A_+ - \sigma D = \mathbb{C}A_+ + \mathbb{C}A_- = \mathbb{C}A \sim 2k\mathbb{C}P^1 .$$

Therefore $[z] = k[\mathbb{C}P]$, so that $z \cdot z = k^2$.

Next we compute $z \cdot z = -z \cdot \sigma z$ by pushing $z$ into a new position $z'$ where it meets $\sigma z$ in isolated points, and counting geometric intersections.

We choose a vector field $\xi$ along $\mathbb{R}P^2$, and tangent to it, so that $\xi \mid A$ is tangent to $\mathbb{C}A$ (the same is then true of $i\xi \mid A$) and oriented so that $i\xi \mid A$ points into $\mathbb{C}A_+$ (rather than $\mathbb{C}A_-$). Thus $\xi \mid A$ is the opposite of the orientation on A induced from the chain D. Now, using $i\xi$, we deform $z$ to a new position $z'$, where $\mathbb{C}A_+$ has shrunk back from its boundary and the discs D have tilted into the complex direction to a new position $D' = \sum_{i=0}^{g} D_i'$, so that they now meet $\mathbb{R}P^2$ only at the zeros of the vector field $\xi$. Indeed, the cycles $\sigma z$ and $z'$ now meet only at these zeros. Thus

$$-z' \cdot \sigma z = -\sum D_i' \cdot D_j = \sum \varepsilon_{ij} = 2 \sum_{i<j} \varepsilon_{ij} + g + 1 .$$

Since $g = (2k-1)(2k-2)/2$, this implies the proposition.

The theorem follows on the observation that

$$(k-1)(k-2)/2 = \sum_{i<j} -\varepsilon_{ij} \leq \sum_{i<j} |\varepsilon_{ij}| = \text{complexity},$$

and the inequality is clearly an equality mod 2.

We remark that in general it is difficult to obtain the numbers $\varepsilon_{ij}$. However, they do satisfy the relation $\varepsilon_{ij}\varepsilon_{jk} = \varepsilon_{ik}$ if $i \leq j \leq k$. Can one use this property to obtain new information about M-curves?

## 3. THE GUDKOV CONJECTURE mod 4

PROPOSITION 2. *Let* $n_i = \sum_{i \leq j} \varepsilon_{ij}$, *and let* $B_i$ *be the component of* $\mathbb{R}P^2 - A$ *whose outer boundary is the* $i^{th}$ *oval. Then* $\sum n_i^2 \chi(B_i) = k^2$.

*Proof.* Let $J_i = \{j \mid j < i,\ j \text{ maximal}\}$. Then $\chi(B_i) = 1 - |J_i|$, and if $j \in J_i$, then $n_j = 1 + n_i \varepsilon_{ij}$, so that $n_i^2 = n_j^2 - 2n_j + 1$. Thus

$$\sum n_i^2 \chi(B_i) = \sum \Big( n_i^2 - \sum_{j \in J_i} (n_j^2 - 2n_j + 1) \Big) = \sum n_i^2 - \sum_{j < \max} (n_j^2 - 2n_j + 1)$$

$$= \sum_{i = \max} n_i^2 + 2 \sum_{j < \max} n_j - \sum_{j < \max} 1 = 2 \sum n_j - \sum 1$$

$$= 2 \sum_{i \leq j} \varepsilon_{ij} - \sum 1 = 2 \sum_{i < j} \varepsilon_{ij} + \sum 1 = -(k-1)(k-2) + g + 1 = k^2 .$$

COROLLARY. $P - N = k^2 \pmod 8$.

*Proof.* We call $B_i$ positive or negative according as its outer boundary is properly contained in an even or odd number of ovals. Thus $n_i$ is odd or even according as $B_i$ is positive or negative. Thus

$$k^2 = \sum n_i^2 \chi(B_i) \equiv \sum_{B_i \text{ positive}} \chi(B_i) = P - N \pmod 4 .$$

We note that we could also prove Proposition 2 directly by mimicking the proof of Proposition 1 in the twofold cover of $\mathbb{C}P^2$ ramified along $\mathbb{C}A$.

## REFERENCES


1. V. I. Arnol'd, *The situation of ovals of real plane algebraic curves, the involutions of four-dimensional smooth manifolds, and the arithmetic of integral quadratic forms.* (Russian) Funkcional. Anal. i Priložen. 5 (1971), no. 3, 1-9.
2. D. A. Gudkov, *Construction of a new series of* M-*curves.* (Russian) Dokl. Akad. Nauk SSSR 200 (1971), 1269-1272.
3. A. Harnack, *Über die Vieltheiligkeit der ebenen algebraischen Curven.* Math. Ann. 10 (1876), 189-199.
4. I. Petrovsky, *On the topology of real plane algebraic curves.* Ann. of Math. (2) 39 (1938), 189-209.
5. V. A. Rohlin, *Proof of a conjecture of Gudkov.* (Russian) Funkcional. Anal. i Priložen. 6 (1972), no. 2, 62-64.

6. V. A. Rohlin, *Congruences modulo* 16 *in Hilbert's sixteenth problem.* (Russian) Funkcional. Anal. i Priložen. 6 (1972), no. 4, 58-64.

7. ———, *Congruences modulo* 16 *in Hilbert's sixteenth problem. II.* (Russian) Funkcional. Anal. i Priložen. 7 (1973), no. 2, 91-92.

8. ———, *Complex orientations of real algebraic curves.* (Russian) Funkcional. Anal. i Priložen. 8 (1974), no. 4, 71-75.

Columbia University
New York, N.Y. 10027