ON FULLY INVARIANT SUBGROUPS OF PRIMARY ABELIAN GROUPS

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We study the relation between the structure of fully invariant subgroups of certain p-primary abelian groups and the structure of the containing groups. Our major theorems are: (1) If F is a fully invariant subgroup of the totally projective group G, then both F and G/F are totally projective groups. (2) If F is a fully invariant subgroup of a C_λ-group G of length λ, and if λ is a limit ordinal cofinal with ω (the first infinite ordinal), then G/F is also a C_λ-group. Moreover, if F is unbounded of length μ, then G/F is a totally projective group, and F is a C_μ-group. (3) If F is an unbounded, fully invariant subgroup of a C_λ-group G, and if λ is a limit ordinal cofinal with ω, then F is a totally projective group only if G is a totally projective group.

All our groups are additively written, abelian, p-primary groups for some prime p. Most of the terminology and notation we use can be found in [1]. Theorems 1 and 2 were announced by the author in [3], and they have been proved independently by L. Fuchs and E. A. Walker (see Exercise 7, page 101 in [1]).

THEOREM 1. *If* F *is a fully invariant subgroup of the totally projective group* K, *then* F *is a totally projective group.*

Proof. We first prove the theorem for the case where K is a generalized Prüfer group, say H_β, for some ordinal β. If β is not larger than ω, then the conclusion follows from well-known results. Thus we assume that the theorem is true whenever $K = H_\alpha$ and α is less than β. We let F denote an unbounded, fully invariant subgroup of H_β. Since H_β is a totally projective group, it is fully transitive, and we can write $F = H_\beta(u)$, where $u = (\sigma(0), \sigma(1), \cdots, \sigma(n), \cdots)$ is an increasing sequence of ordinals that satisfies the gap condition (see Theorem 67.1 in [1]). If β is a limit ordinal, then we set $H_\beta = \bigoplus\{H_\alpha\colon \alpha < \beta\}$, where H_α is the generalized Prüfer group of length α, and we note that $H_\beta(u) = \bigoplus\{H_\alpha(u)\colon \alpha < \beta\}$ and that the class of the totally projective groups is closed under the operation of taking direct sums. If, on the other hand, β is not a limit ordinal, then we set $\beta = \gamma + 1$. Now, unless $F = H_{\gamma+1}(u)$ is bounded, we have the relation $\sigma(n) < \gamma + 1$ for all n, and thus

$$H_{\gamma+1}(u)/p^\gamma H_{\gamma+1} = (H_{\gamma+1}/p^\gamma H_{\gamma+1})(u) .$$

But the right-hand side is isomorphic to $H_\gamma(u)$, a fully invariant subgroup of H_γ. Since γ is less than β, this subgroup is a totally projective group by assumption. Thus, we see that $H_{\gamma+1}(u)/p^\gamma H_{\gamma+1}$ is a totally projective group. Because $\sigma(n)$ is less than γ for all n, there is an ordinal δ satisfying the condition $p^\gamma H_{\gamma+1} = p^\delta(H_{\gamma+1}(u))$. It follows that $H_{\gamma+1}(u)/p^\delta(H_{\gamma+1}(u))$ is a totally projective group, as is $p^\gamma H_{\gamma+1}$. By appealing to familiar properties of totally projective groups, we can conclude that $H_{\gamma+1}(u)$ is a totally projective group. It follows that the theorem is true in the case where K is a generalized Prüfer group. For the general case, we recall that every totally projective group is a summand of a direct

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sum of generalized Prüfer groups and that the class of totally projective groups is closed under the operation of taking direct summands.

LEMMA 1. *If* K *is a* p*-primary group of length* $\beta + 1$ *such that* $p^\beta K$ *is finite, and if*

$$u = (\sigma(0), \sigma(1), \cdots, \sigma(n-1), \infty, \infty, \cdots) \text{ and } u' = (\sigma(0), \sigma(1), \cdots, \sigma(n-1), \beta, \infty, \infty, \cdots)$$

satisfy the gap condition, then $K(u')/K(u)$ *is finite.*

Proof. If $z_i \in p^\beta K = \{0, z_1, \cdots, z_m\}$, then Lemma 24 in [2] guarantees the existence of an element x_i in $K(u')$ such that $p^n x_i = z_i$. Now suppose that $x \in K(u')$. Then $p^n x \in p^\beta K$, and thus $p^n x = z_j = p^n x_j$ for some j. Hence $p^n(x - x_j) = 0$, and $p^t(x - x_j) \in p^{\sigma(t)}K$ for t less than n; that is, $x - x_j \in K(u)$. Thus $x + K(u) = x_j + K(u)$, and $K(u')/K(u)$ is finite.

THEOREM 2. *If* F *is a fully invariant subgroup of the totally projective group* K, *then* K/F *is a totally projective group, and the length of* K/F *does not exceed the length of* K.

Proof. We prove the theorem only for the case where $K = H_\beta$ for some ordinal β, since the proof for the general case is similar to the one offered in Theorem 1. We begin by assuming that the conclusion holds if $K = H_\alpha$ and α is less than β. The case where β is a limit ordinal presents no difficulties; suppose therefore that F is a fully invariant subgroup of H_β, where $\beta = \gamma + 1$. If we set $F = H_{\gamma+1}(u)$, where $u = (\sigma(0), \sigma(1), \cdots, \sigma(n), \cdots)$ satisfies the gap condition, then it follows from the definition of u that, for each n, $\sigma(n) \leq \gamma$ or $\sigma(n) = \infty$. We have three cases.

Case 1: $\sigma(n) < \gamma$ for all n. With this restriction on u, we can show that $H_{\gamma+1}(u)/p^\gamma H_{\gamma+1} = (H_{\gamma+1}/p^\gamma H_{\gamma+1})(u)$. Thus $H_{\gamma+1}/H_{\gamma+1}(u)$ is isomorphic to $(H_{\gamma+1}/p^\gamma H_{\gamma+1})/(H_{\gamma+1}/p^\gamma H_{\gamma+1})(u)$, where $(H_{\gamma+1}/p^\gamma H_{\gamma+1})(u)$ is a fully invariant subgroup of the totally projective group $H_{\gamma+1}/p^\gamma H_{\gamma+1}$. It follows from the induction hypothesis that $H_{\gamma+1}/H_{\gamma+1}(u)$ is a totally projective group. A straightforward argument shows that the length of $H_{\gamma+1}/H_{\gamma+1}(u)$ does not exceed the length of $H_{\gamma+1}$.

Case 2: $\sigma(n) = \gamma$ for some n. The proof for the sequence $u = (\gamma, \infty, \infty, \cdots)$ is not difficult, since it follows that $H_{\gamma+1}(u) = p^\gamma H_{\gamma+1}$. Suppose now that $u = (\sigma(0), \sigma(1), \cdots, \sigma(n-1), \gamma, \infty, \infty, \cdots)$ for some n larger than 1. If we set $v = (\sigma(0), \sigma(1), \cdots, \sigma(n-1), \infty, \infty, \infty, \cdots)$, then we see that $H_{\gamma+1}/H_{\gamma+1}(u)$ is isomorphic to $(H_{\gamma+1}/p^\gamma H_{\gamma+1})/(H_{\gamma+1}/p^\gamma H_{\gamma+1})(v)$, which is in turn isomorphic to $H_\gamma/H_\gamma(v)$; this latter quotient is a totally projective group, by assumption. The argument for the conclusion regarding the length of $H_{\gamma+1}/H_{\gamma+1}(u)$ is straightforward.

Case 3: $\sigma(n-1) < \gamma$ and $\sigma(n) = \infty$ for some n. We set

$$u' = (\sigma(0), \sigma(1), \cdots, \sigma(n-1), \gamma, \infty, \infty, \cdots)$$

and note that u' satisfies the gap condition and that $H_{\gamma+1}(u)/H_{\gamma+1}(u')$ is finite, by Lemma 1. By the argument given in Case 2, $H_{\gamma+1}/H_{\gamma+1}(u')$ is a totally projective group, and thus $(H_{\gamma+1}/H_{\gamma+1}(u'))/(H_{\gamma+1}(u)/H_{\gamma+1}(u'))$ is a totally projective group. However, this latter quotient is an isomorphic copy of $H_{\gamma+1}/H_{\gamma+1}(u)$.

Note that if A is a finite subgroup of G, then the length of G/A does not exceed the length of G; thus the length of $(H_{\gamma+1}/H_{\gamma+1}(u'))/(H_{\gamma+1}(u)/H_{\gamma+1}(u'))$ is less than or equal to the length of $H_{\gamma+1}/H_{\gamma+1}(u')$. Since the former quotient is isomorphic to $H_{\gamma+1}/H_{\gamma+1}(u)$, we can complete the proof by appealing to Case 2.

In his generalization of C. Megibben's C_λ-theory [5], K. Wallace [9] calls a reduced p-primary group G a *C_λ-group* if λ is a limit ordinal and $G/p^\alpha G$ is a totally projective group for all α less than λ; the C_ω-groups are precisely the p-primary groups. Wallace generalizes the notion of a basic subgroup by defining B to be a *λ-basic subgroup* of the reduced p-primary group G if B is a totally projective group of length at most λ, B is p^λ-pure [7] in G, and G/B is divisible. In addition to other extensions of classical theorems, Wallace shows that a reduced p-primary group contains a proper λ-basic subgroup if and only if λ is cofinal with ω and the group is a C_λ-group. Since the remaining proofs require the existence of λ-basic subgroups, we restrict our attention to the case where λ is cofinal with ω.

An important tool in our previous proofs is the sequential characterization of fully invariant subgroups of fully transitive groups. It is known [4] that C_λ-groups having length λ are fully transitive; those having greater length do not in general possess this property, and thus characteristics of their fully invariant subgroups are relatively unknown. Thus we shall restrict our attention to C_λ-groups having length λ. (However, it should be observed that there are examples demonstrating the necessity for the restriction on the length in the hypotheses of Theorems 4 and 5.) Note that, by applying results in [7], we can show that the subgroup B of the p-primary group G is a λ-basic subgroup of G if and only if B is a totally projective group of length at most λ, $pG \cap B = pB$, and $G[p] \subseteq p^\alpha G + B[p]$ for all $\alpha < \lambda$.

THEOREM 3. *If* F *is a bounded, fully invariant subgroup of the* C_λ-*group* G, *and if* G *has length* λ, *then* G/F *is a* C_λ-*group*.

Proof. We can assume that F does not have the form $G[p^m]$ for any positive integer m. We shall show that if B is a λ-basic subgroup of G, then $(B + F)/F$ is a λ-basic subgroup of G/F. Since G is fully transitive, we can write $F = G(u)$, where $u = (\sigma(0), \sigma(1), \cdots, \sigma(k), \infty, \infty, \cdots)$ satisfies the gap condition, k is greater than 0, and $p^{k+1} F = 0$.

First we note that $(B + F)/F$ is isomorphic to $B/B(u)$, where $B(u)$ is a fully invariant subgroup of B, and where B is a totally projective group. Thus, we can apply Theorem 2 and show that $(B + F)/F$ is a totally projective group of length at most λ.

Next we prove that $p(G/F) \cap (B + F)/F \subseteq p((B + F)/F)$. Let $p(g + F) = b + F$ denote an element in $(B + F)/F$, where $g \in G$ and $b \in B$. If $\sigma(0) \neq 0$, then $F \subseteq p^{\sigma(0)} G \subseteq pG$, and it follows that $b \in pB$. If, on the other hand, $\sigma(0) = 0$, then $u \neq (0, 1, 2, \cdots, k, \infty, \infty, \cdots)$, because $F \neq G[p^{k+1}]$. Thus there is a first t such that $\sigma(t) > t$. The assumption that $pg - b \in F$ implies that $p^t(pg - b) \in p^{\sigma(t)} G$, where $p^{\sigma(t)} G \subseteq p^{t+1} G$. If we set $p^t b = p^{t+1} b'$, where $b' \in B$, then it follows that $pg - pb' \in G(u) = F$.

Finally, we show that $(G/F)[p] \subseteq p^\alpha(G/F) + ((B + F)/F)[p]$ for all α less than λ. Suppose that $g + F \in (G/F)[p]$ for some $g \in G$ and that α is less than λ. If t is the first positive integer such that $p^t g = 0$, then t is not larger than $k + 2$. Since B is a λ-basic subgroup of G, it follows that $G[p^t] = p^\beta G[p^t] + B[p^t]$, where β denotes the larger of α and $\sigma(k)$. Thus $g = g' + b$, where $g' \in p^\beta G[p^t]$ and $b \in B[p^t]$; furthermore, $g + F$ is in $(p^\beta G + F)/F + (B + F)/F$. In order to show that $pb \in F$, we note that the assumption $p(g + F) = 0$ requires that $pg' - pb \in F$. However, $pg' \in p^{\beta+1} G[p^{t-1}] \subseteq p^{\sigma(k)} G[p^{k+1}] \subseteq F$. This completes the proof.

If F is a fully invariant subgroup of the C_λ-group G, then F is called a *λ-large subgroup* of G if $G = B + F$ for all λ-basic subgroups B of G. This generalization of R. Pierce's [8] concept of a large subgroup is studied in [4], where it is shown that a fully invariant subgroup of a C_λ-group of length λ is a λ-large subgroup if and only if it is unbounded. Thus we can restate two of the results in [4] in the following form.

THEOREM 4. *Let* F *denote an unbounded, fully invariant subgroup of the* C_λ-*group* G, *where* G *has length* λ. *If* μ *denotes the length of* F, *then* F *is a* C_μ-*group, and* G/F *is a totally projective group.*

Proof. We only sketch the proof here. If B is a λ-basic subgroup of G, then we can show that $F \cap B$ is a fully invariant subgroup of B. Since B is a totally projective group, it follows from Theorem 1 that $F \cap B$ is also a totally projective group; furthermore, $F \cap B$ is a μ-basic subgroup of F, and thus F is a C_μ-group. Since G/F is isomorphic to $B/(F \cap B)$, Theorem 2 guarantees that G/F is a totally projective group.

THEOREM 5. *Let* F *denote an unbounded, fully invariant subgroup of the* C_λ-*group* G, *where* G *has length* λ. *Then* F *is a totally projective group only if* G *is a totally projective group.*

Proof. We sketch the proof, which is by induction on the length of G. Assume that the theorem is true for all cases where G is a C_β-group, β is less than λ, and β is cofinal with ω. If $F = G(u)$, where $u = (\sigma(0), \sigma(1), \cdots, \sigma(n), \cdots)$, then we set $\beta = \sup\{\sigma(n): n < \omega\}$. If β is less than λ, then $p^\omega F = p^\beta G$, and thus $p^\beta G$ is totally projective. The desired result now follows from the fact that G is a C_λ-group. On the other hand, if $\beta = \lambda$, then $p^\omega F = 0$, and F is a direct sum of cyclic groups; furthermore, $p^{\sigma(0)}G[p] = F[p] = \bigcup\{S(n): n < \omega\}$, where $S(n) \cap p^n F = 0$ for each n. It follows that $p^{\sigma(0)}G$ is a σ-summable [6] C_μ-group, where μ is cofinal with ω and less than λ. An application of the generalized Kulikov criterion [6] enables us to conclude that $p^{\sigma(0)}G$ is a totally projective group, as is $G/p^{\sigma(0)}G$.

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