HADAMARD'S INEQUALITY FOR MATRICES WITH POSITIVE-DEFINITE HERMITIAN COMPONENT

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Hadamard's inequality states that if $A = (a_{ij})$ is an n-by-n positive-definite Hermitian matrix, then

$$\det A \leq a_{11} a_{22} \cdots a_{nn}.$$

If we let $\Pi_n \equiv \{A \in M_n(C): A + A^* > 0\}$, then $A \in \Pi_n$ does not necessarily satisfy the analogous inequality

(1) $$|\det A| \leq |a_{11} \cdots a_{nn}|.$$

D. M. Koteljanskiĭ [3], F. R. Gantmacher and M. G. Krein [2], and K. Fan [1] have generalized the Hadamard inequality by isolating a class (which includes the positive-definite Hermitian matrices) throughout which (1) holds. In this note, we point out a different class (related to a convexly parametrized subclass of Π_n) in which (1) holds, and, in the process, also give a dual class in which

(2) $$|\det A| \geq |a_{11} \cdots a_{nn}|$$

holds. The former class also includes the positive-definite Hermitian matrices, so that an alternate proof of Hadamard's inequality is provided.

For $A \in M_n(C)$, we define $H(A) = \frac{A + A^*}{2}$ and $S(A) = \frac{A - A^*}{2}$, so that $A = H(A) + S(A)$. If $H = (h_{ij})$ is Hermitian, we define the *upper triangular part* of H by $T(H) \equiv (t_{ij})$ where

$$t_{ij} = \begin{cases} 2h_{ij} & \text{if } i < j, \\ h_{ij} & \text{if } i = j, \\ 0 & \text{if } i > j. \end{cases}$$

For $A \in M_n(C)$, we then define $T(A)$ by $T(A) \equiv T(H(A))$. It also follows from our definitions that $H(T(A)) = H(A)$. In this context, Hadamard's inequality states simply that

$$\det H \leq \det T(H)$$

when H is positive-definite Hermitian.

Received July 9, 1974.

This work comprises a portion of the author's doctoral thesis written under the advisement of Professor Olga Taussky Todd at the California Institute of Technology, Pasadena, California 91109.

Michigan Math. J. 22 (1975).

We shall be interested in the following subclass of Π_n:

$$\Pi_n^{(\alpha)} \equiv \{\alpha T(A) + (1-\alpha) T(A)^*: A \in \Pi_n\}.$$

THEOREM. *Suppose* $A \in \Pi_n^{(\alpha)}$. *Then*

(i) *if* $0 \le \alpha \le 1$, $|\det A| \le a_{11} \cdots a_{nn}$, *and*

(ii) *if* $\alpha < 0$ *or* $\alpha > 1$, $|\det A| \ge a_{11} \cdots a_{nn}$.

Proof. Suppose $A = \alpha T + (1-\alpha)T^*$, where $T = T(C)$ for $C \in \Pi_n$. We first note that $S(A) = (2\alpha - 1)S(T)$:

$$S(A) = \frac{1}{2}(A - A^*) = \frac{1}{2}([\alpha T + (1-\alpha)T^*] - [\alpha T^* + (1-\alpha)T])$$

$$= \frac{1}{2}[(2\alpha - 1)T - (2\alpha - 1)T^*] = (2\alpha - 1)S(T).$$

Let $H(T) = H$ and $S(T) = S$. Then, since $H(A) = H(T)$ and $A = H(A) + S(A)$, we have the relation $A = H + (2\alpha - 1)S \in \Pi_n^{(\alpha)}$.

Now consider the quotient TA^{-1}. The theorem follows if $|\det TA^{-1}| \ge 1$ for $\alpha \in [0, 1]$ and $|\det TA^{-1}| \le 1$ for $\alpha \notin [0, 1]$. But

$$\det TA^{-1} = \frac{\det(H+S)}{\det(H + (2\alpha - 1)S)} = \frac{\det(I + \hat{S})}{\det(I + (2\alpha - 1)\hat{S})},$$

where $\hat{S} = H^{-1/2} S H^{-1/2}$ is skew-Hermitian ($H^{-1/2}$ is the inverse of the positive definite square root of H). Thus the eigenvalues of $I + \hat{S}$ are of the form $1 + i\lambda_j$, while those of $I + (2\alpha - 1)\hat{S}$ are of the form $1 + i(2\alpha - 1)\lambda_j$ (λ_j real, $j = 1, \cdots, n$).

Now,

$$|\det TA^{-1}| = \frac{|(1 + i\lambda_1)\cdots(1 + i\lambda_n)|}{|(1 + i(2\alpha - 1)\lambda_1)\cdots(1 + i(2\alpha - 1)\lambda_n)|}$$

$$= \left|\frac{1 + i\lambda_1}{1 + i(2\alpha - 1)\lambda_1}\right| \cdots \left|\frac{1 + i\lambda_n}{1 + i(2\alpha - 1)\lambda_n}\right|.$$

Since $|2\alpha - 1| \le 1$ if $\alpha \in [0, 1]$ and $|2\alpha - 1| \ge 1$ if $\alpha \notin [0, 1]$, this means that

$$\left|\frac{1 + i\lambda_j}{1 + i(2\alpha - 1)\lambda_j}\right| \ge 1 \quad \text{if } \alpha \in [0, 1] \text{ and}$$

$$\left|\frac{1 + i\lambda_j}{1 + i(2\alpha - 1)\lambda_j}\right| \le 1 \quad \text{if } \alpha \notin [0, 1].$$

Thus we conclude that

$$|\det TA^{-1}| \ge 1 \text{ if } \alpha \in [0, 1] \quad \text{and} \quad |\det TA^{-1}| \le 1 \text{ if } \alpha \notin [0, 1],$$

which is equivalent to the statement of the theorem. The special case $\alpha = 1/2$ is Hadamard's result.

Remark 1. The cases of equality in the theorem are easily analyzed from the preceding discussion. Equality is attained if and only if either $\alpha = 0$ or $\alpha = 1$ or A is diagonal.

Remark 2. It is also clear from the proof of the theorem that the function $f(\alpha) \equiv \det(\alpha T + (1-\alpha)T^*)$, where $T = T(C)$, $C \in \Pi_n$, and α is real, attains a minimum for $\alpha = 1/2$ and is decreasing everywhere to the left and increasing everywhere to the right.

COROLLARY 1. *Suppose* $A \in \Pi_n$. *Then*

$$\det H(A) \leq |\det(\alpha T(A) + (1-\alpha)T(A)^*)| \leq \det T(A) \leq |\det(\beta T(A) + (1-\beta)T(A)^*)|$$

($\alpha \in [0, 1]$, $\beta \notin [0, 1]$, β *real*).

Since right or left multiplication by a positive diagonal matrix has the same relative impact on the determinant and the product of the diagonal entries of a matrix, we also have a slight generalization of the main result.

COROLLARY 2. *Suppose* $A = DBE$, *where* D *and* E *are positive diagonal matrices and* $B \in \Pi_n^{(\alpha)}$. *Then* $|\det A| \leq a_{11} \cdots a_{nn}$ *for* $0 \leq \alpha \leq 1$ *and* $|\det A| \geq a_{11} \cdots a_{nn}$ *for* $\alpha > 1$ *or* $\alpha < 0$.

Example. The classes $\Pi_n^{(\alpha)}$ are not contained in the GKK class as defined by Fan [1]. Let

$$A = \begin{bmatrix} 3 & 3 & -3 \\ 1 & 4 & -6 \\ -1 & -2 & 5 \end{bmatrix}.$$

Then $A \in \Pi_n^{(3/4)}$, but A is not a GKK matrix, since

$$\det \begin{bmatrix} 3 & -3 \\ 4 & -6 \end{bmatrix} \det \begin{bmatrix} 1 & 4 \\ -1 & -2 \end{bmatrix} = (-6)(2) = -12 \not\geq 0.$$

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