EXTREMAL PROBLEMS IN ARBITRARY DOMAINS, II

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1. INTRODUCTION

This paper deals with the extremal functions of a certain class of extremal problems. We obtain uniqueness of the extremal functions associated with a class of extremal problems, including the problems treated by D. Hejhal in [7]. We also study the behavior of the extremal function near a free analytic boundary arc. Our techniques are adapted from [4]; they are the techniques of function algebras, and they offer a perspective that is "dual" in some sense to the classical approach of S. Ja. Havinson [6] and others.

In order to state explicitly the main results, we fix some notation.

Let D be a bounded domain in the complex plane $\mathbb{C}$, let K be a compact subset of D, and let η be a measure on K. Let u be a continuous real-valued function on D. The basic extremal problem is the following:

(1.1) To maximize $\left|\int f\, d\eta\right|$, among all analytic functions f on D such that $|f| \leq e^u$ on D.

The extremal problem (1.1) is *nontrivial* if there exists a competing function f for which $\int f\, d\eta \neq 0$.

A normal-families argument shows that there exists an extremal function F for (1.1). Upon multiplying F by a unimodular constant, we can arrange that $\int F\, d\eta \geq 0$. Such an extremal function is said to be *normalized.*

An example of Hejhal [7, p. 114] shows that the normalized extremal function for (1.1) need not be unique, even if u is harmonic. One of Hejhal's uniqueness theorems can be stated as follows.

THEOREM 1.1. *Suppose that* u *is harmonic on* D *and that every component of* $D \setminus K$ *includes in its boundary an essential boundary point of* D. *If the extremal problem* (1.1) *is nontrivial, then the problem has a unique normalized extremal function.*

Hejhal's proof of Theorem 1.1 depends on the methods developed by Havinson [6], who proved the uniqueness of the Ahlfors function of arbitrary domains. Now there is in [4] an economical proof of Havinson's theorem that depends on function-algebraic techniques (see also [3] and [5]). In Section 2, we show how this simple proof can be extended to include Theorem 1.1. Sections 3 and 4 include assorted extensions of the basic uniqueness theorem. In particular, a result obtained in Section 3 includes the various uniqueness assertions of [7].

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S. Fisher proved in [2] that the Ahlfors function of an arbitrary planar domain extends analytically across free analytic boundary arcs and has unit modulus on these arcs. Proofs of this theorem, covering a broader class of extremal problems, have been given in [3] and in [4]. In Section 5 we show that the methods of [4] can be extended to yield the following result.

THEOREM 1.2. *Suppose that* u *is harmonic on* D, *that the extremal problem* (1.1) *is nontrivial, and that* $D \setminus K$ *is connected. Let* Γ *be a free analytic boundary arc for* D, *and let* v *be a conjugate harmonic function for* u, *defined on an open subset of* D *adjacent to* Γ. *If* F *is the extremal function for* (1.1), *then* $Fe^{-(u+iv)}$ *extends analytically across* Γ *and has unit modulus on* Γ.

We shall adhere to the usual conventions and notational usages. By "measure" we always mean "finite regular Borel measure." The closed support of a measure ν is denoted by supp ν. The space of continuous complex-valued functions on a topological space E is denoted by C(E). Associated with it is the norm of uniform convergence on E, defined by

$$\|f\|_E = \sup\{|f(x)|: x \in E\}.$$

The extended complex plane is denoted by $\mathbf{C}^*$. For the precise definitions of "essential boundary point," "free analytic boundary arc," and so forth, see [2], [3], or [4].

2. A PROOF OF THEOREM 1.1

Let $\beta(D)$ denote the Stone-Čech compactification of D, and let

$$Z: \beta(D) \to \overline{D}$$

be the extension of the coordinate function z from D to $\beta(D)$. Let B be the subspace of $C(\beta(D))$ consisting of functions of the form

$$g = fe^{-u},$$

where f is analytic on D and fe^{-u} is bounded on D. Evidently, B is a closed subspace of $C(\beta(D))$.

If $g \in B$, then $\log|g| - \log|f| - u$ is subharmonic on D. Consequently g cannot assume its maximum modulus on D unless g is constant. Every $g \in B$ then assumes its maximum modulus on $\beta(D) \setminus D$:

$$\|g\|_D = \|g\|_{\beta(D)\setminus D} \quad (g \in B). \tag{2.1}$$

Now define a continuous linear functional Λ on B by

$$\Lambda(g) = \int g e^u d\eta = \int f d\eta \quad (g \in B), \tag{2.2}$$

where $g = fe^{-u}$. Since $|f| \leq e^u$ if and only if $|g| \leq 1$, the extremal value for (1.1) coincides with the norm of Λ on B. An analytic function F on D is a normalized extremal function for (1.1) if and only if the corresponding function $G = Fe^{-u} \in B$ satisfies the conditions

$$|G| \leq 1, \qquad \Lambda(G) = \|\Lambda\| . \tag{2.3}$$

The problem then is to show that there exists a unique $G \in B$ that satisfies (2.3).

In view of (2.1), there is a measure ν on $\beta(D) \setminus D$ satisfying the conditions

$$\Lambda(g) = \int g\,d\nu \qquad (g \in B), \tag{2.4}$$

$$\|\Lambda\| = \|\nu\| . \tag{2.5}$$

The hypotheses of Theorem 1.1 guarantee that $\Lambda \neq 0$, so that $\nu \neq 0$. The crucial properties of ν are given in the form of two lemmas.

LEMMA 2.1. *Let* V *be a component of* $D \setminus K$. *Suppose there exists* $z_0 \in V$ *such that the evaluation functional* $g \to g(z_0)$ $(g \in B)$ *is <u>not</u> continuous in the norm of uniform convergence on* $K \cup \operatorname{supp} \nu$. *Then*

$$|\nu|(Z^{-1}(\partial V)) = 0 .$$

Proof. Note that Z maps $\beta(D) \setminus D$ onto ∂D. Since $\partial V \cap \partial D$ is a closed and open subset of ∂D, $Z^{-1}(\partial V \cap \partial D)$ is a closed and open subset of $\beta(D) \setminus D$. The lemma asserts that ν has no mass on this subset.

Define $E = K \cup \operatorname{supp} \nu$. Then the measure $e^u \eta - \nu \in B^{\perp}$ is supported on E.

Let $g \in B$. Since the evaluation functional is not continuous, its kernel is dense in B in the norm $\| \cdot \|_E$. Consequently there exists a sequence $\{g_n\}$ in B such that $g_n(z_0) = 0$, while $\{g_n\}$ converges uniformly to g on E. By the definition of B, the functions $g_n/(Z - z_0)$ also belong to B, and moreover $\{g_n/(Z - z_0)\}$ converges uniformly to $g/(Z - z_0)$ on E. Using $\overline{B|_E}$ to denote the uniform closure of $B|_E$ in $C(E)$, we conclude that $g \in \overline{B|_E}$ implies $g/(Z - z_0) \in \overline{B|_E}$. In fact, if $g \in \overline{B|_E}$, then $g/(Z - z_0)^m \in \overline{B|_E}$ for all integers $m \geq 0$.

Now V separates $\partial D \cap \partial V$ from K and from $\partial D \setminus \partial V$. By Runge's Theorem, there is then a sequence $\{h_n\}$ of polynomials, such that $h_n(1/(Z - z_0))$ converges uniformly to 0 on $\partial D \cap \partial V$, and to 1 on K and on $\partial D \setminus \partial V$. Now $e^u \eta - \nu$ is orthogonal to $\overline{B|_E}$, and $h_n(1/(Z - z_0))\,g \in \overline{B|_E}$ for all $g \in B$. Consequently

$$\int h_n\left(\frac{1}{Z - z_0}\right) g\, e^u\, d\eta = \int h_n\left(\frac{1}{Z - z_0}\right) g\, d\nu \qquad (g \in B).$$

Passing to the limit, we obtain the relation

$$\Lambda(g) = \int_K g\, e^u\, d\eta = \int_{Z^{-1}(\partial D \setminus \partial V)} g\, d\nu \qquad (g \in B).$$

Hence the restriction of ν to $Z^{-1}(\partial D \setminus \partial V)$ also represents the functional Λ. Since ν has minimal norm among the measures representing Λ, the measure ν must coincide with its restriction to $Z^{-1}(\partial D \setminus \partial V)$. Consequently, ν has no mass on $Z^{-1}(\partial D \cap \partial V)$. ■

LEMMA 2.2. *Let* V *be a component of* $D \setminus K$ *such that* $|\nu|(Z^{-1}(\partial V)) > 0$. *If* $g \in B$ *and* $g = 0$ *on* $Z^{-1}(\partial V) \cap \operatorname{supp} \nu$, *then* $g \equiv 0$.

Proof. Since ∂V includes an essential boundary point of D, there exists an analytic function h on D such that

$$|h(z)| < 1 \qquad (z \in D), \tag{2.6}$$

$$\limsup_{D \ni z \to \partial D \cap \partial V} |h(z)| = 1, \tag{2.7}$$

$$\limsup_{D \ni z \to \partial D \setminus \partial V} |h(z)| < 1. \tag{2.8}$$

Now h extends continuously to $\beta(D)$, and (2.8) shows that the maximum modulus of h on $Z^{-1}(\partial D \setminus \partial V)$ is strictly less than 1. Since also $\|h\|_K < 1$, we can by (2.7) find an open disc Δ_0 in V such that

$$|h(z)| > \|h\|_{K \cup Z^{-1}(\partial D \setminus \partial V)} \qquad (z \in \Delta_0). \tag{2.9}$$

Now fix $z_0 \in \Delta_0$. The functions

$$g_n = h^n g / h(z_0)^n \qquad (n \geq 1),$$

belong to B. On account of (2.9), $\{g_n\}$ converges uniformly to zero on K and on $Z^{-1}(\partial D \setminus \partial V)$. By Lemma 2.1, $|g_n(z_0)| \to 0$. Since $g_n(z_0) = g(z_0)$, we see that $g(z_0) = 0$. Since $z_0 \in \Delta_0$ is arbitrary, g vanishes on Δ_0. In view of the definition of B, the function g is identically zero on D. ■

Now we complete the proof of Theorem 1.1 as follows. Suppose $G \in B$ satisfies (2.3). Then the inequalities in the chain

$$\|\Lambda\| = \Lambda(G) = \int g\, d\nu \leq \int |G|\, d|\nu| \leq \|G\|\, \|\nu\| = \|\Lambda\|$$

become equalities. We conclude that $G\nu \geq 0$ and

$$|G| = 1 \quad \text{on supp } \nu. \tag{2.10}$$

If $G_0, G_1 \in B$ both satisfy (2.3), then each convex combination of G_0 and G_1 also satisfies (2.3). By (2.10), every convex combination of G_0 and G_1 is unimodular on supp ν. This can occur only if $G_0 = G_1$ on supp ν. Since $\nu \neq 0$, there is at least one component V of $D \setminus K$ such that $|\nu|(\partial V) > 0$. Applying Lemma 2.2 to this component V, we see that $G_0 - G_1 = 0$, and therefore $G_0 = G_1$. The function satisfying (2.3) is therefore unique, and the proof is concluded.

3. RELAXING THE HYPOTHESES

There are various ways in which the hypotheses of Theorem 1.1 can be relaxed.

First, u can be any continuous superharmonic function on D. The proof goes over verbatim for such u. We can also easily reduce this case to the harmonic case, by replacing u by its greatest harmonic minorant.

Second, D can be any domain in the extended complex plane. In this case, it is convenient to perform a rotation so that $\infty \in D \setminus K$. Again the coordinate function extends to a mapping Z of $\beta(D)$ onto $\overline{D}$, and Z maps $\beta(D) \setminus D$ onto ∂D. If the point at ∞ is avoided, then the remainder of the proof of Theorem 1.1 goes over unchanged.

Third, one can consider a more restricted class of admissible functions. For instance, let $\{z_k\}_{k=1}^{\infty}$ be a sequence of points in D, and let $\{m_k\}_{k=1}^{\infty}$ be a sequence of positive integers. Suppose we modify the extremal problem (1.1) by considering only analytic functions f on D that have zeros of order m_k at each z_k $(k \geq 1)$. Then Theorem 1.1 remains valid, and the proof given in Section 2 goes over as it is, once B is defined appropriately.

Fourth, the condition that each component B of $D \setminus K$ contain an essential boundary point of D in its closure can be replaced by the following condition.

(3.1) If V is a component of $D \setminus K$ such that ∂V does not include an essential boundary point of D, then there is a closed subset T of $\partial D \cap \partial V$ that has logarithmic capacity 0, and such that u extends continuously to a superharmonic function on a neighborhood of $(\partial D \cap \partial V) \setminus T$.

Because in this case the generalization is not completely trivial, we state the result explicitly and give a proof. For convenience, we omit the conditions on the zeros of the admissible functions.

THEOREM 3.1. *Let* D *be a domain in the extended complex plane, and let* u *be a continuous superharmonic function on* D. *Let* K *be a compact subset of* D, *and let* η *be a measure on* K *such that the extremal problem* (1.1) *is nontrivial. Suppose that* (3.1) *is valid. Then there exists a unique normalized extremal function for the extremal problem* (1.1).

Proof. Let $V_1, \cdots, V_r$ be the components of $D \setminus K$ with the property that $\partial V_j \cap \partial D$ is a nonempty set containing no essential boundary points of D. Then $\partial V_j \cap \partial D$ is a compact, totally disconnected subset of $\mathbb{C}^*$ of analytic capacity 0. Let T_j be the compact subset of $\partial V_j \cap \partial D$ given by (3.1), so that $\operatorname{cap}(T_j) = 0$. Then the union $\tilde{D}$ of D and the sets $(\partial V_j) \setminus T_j$ $(1 \leq j \leq r)$ is a domain. By hypothesis, u extends to a continuous and superharmonic function on $\tilde{D}$. If f is an analytic function on D such that $|f| \leq e^u$, then f is bounded near each point of $(\partial V_j) \setminus T_j$. Since the sets $(\partial V_j) \setminus T_j$ have zero analytic capacity, f extends analytically across $(\partial V_j) \setminus T_j$. Consequently, f extends analytically to $\tilde{D}$ and satisfies the inequality $|f| \leq e^u$ there. Replacing the extremal problem on D with the obvious equivalent extremal problem on $\tilde{D}$, and replacing V_j by $V_j \cup (\partial V_j \setminus T_j)$, we can make the following assumption:

(3.2) If $V_1, \cdots, V_r$ are the components of $D \setminus K$ such that $\partial V_j \cap \partial D$ is a nonempty set of analytic capacity 0, then each $\partial V_j \cap \partial D$ has logarithmic capacity 0.

Now let $T = \bigcup_{j=1}^{r} \partial V_j \cap \partial D$, so that $\operatorname{cap}(T) = 0$. Define B as in Section 2, and let $g \in B$. Then $\log|g|$ is subharmonic on D and bounded above. By a result in potential theory, which can be proved simply by means of the Evans function, $\log|g|$ extends across T to a subharmonic function on the domain $D \cup T$. Now there are two cases.

If $T = \partial D$, then $D \cup T$ coincides with the extended complex plane. In this case, $\log |g|$ is constant, and each function in B has constant modulus. It follows that B is a one-dimensional linear space, so that the uniqueness of the normalized extremal function is trivial.

On the other hand, suppose that $T \neq \partial D$. The subharmonicity then yields the estimate

$$\log |g(z)| \leq \limsup_{D \ni w \to (\partial D) \setminus T} \log |g(w)| \qquad (z \in D).$$

In other words, the following analogue of (2.1) is valid:

$$\|g\| = \|g\|_{\beta(D)\setminus(D \cup Z^{-1}(T))} \qquad (g \in B). \tag{3.3}$$

In this case, the norm-preserving extension ν of Λ can be chosen to be situated on the compact subset $\beta(D) \setminus (D \cup Z^{-1}(T)) = Z^{-1}((\partial D) \setminus T)$ of $\beta(D)$. In particular, if V is any component of $D \setminus K$ such that $|\nu|(Z^{-1}(\partial D \cap \partial V)) > 0$, then ∂V includes an essential boundary point of D. Since the proof of Lemma 2.2 is valid for such a V, the proof given in Section 2 serves to complete the proof of the theorem. ∎

Now consider the uniqueness theorem obtained by Hejhal [7, p. 94, Theorem 1]. The theorem breaks into four parts. Of these, part (b) is Theorem 1.1, while part (d) can be reduced easily to part (b). Parts (a) and (c) are included in Theorem 3.1, and one of the hypotheses of part (c) is found to be superfluous.

4. EXTREMAL PROBLEMS WITH SIDE CONDITIONS

We have already discussed side conditions requiring that admissible functions vanish on a given sequence. Another type of side conditions is related to Pick-Nevanlinna interpolation. Let D, K, u, and η be as before, let $\eta_1, \cdots, \eta_m$ be measures supported on K, and let $b_1, \cdots, b_m$ be complex numbers. The problem is the following:

(4.1) To maximize $\Re \int f\, d\eta$, among all analytic functions f on D satisfying the conditions $|f| \leq e^u$ and $\int f\, d\eta_j = b_j$ $(1 \leq j \leq m)$.

One can deduce the uniqueness of the extremal function for (4.1) by imposing the hypotheses of Theorem 1.1 (or Theorem 3.1), together with the two trivial requirements

(4.2) there exists a competing function f for (4.1),

(4.3) η is not congruent to a linear combination of $\eta_1, \cdots, \eta_m$ modulo the annihilator of the space of analytic functions f such that $|f|\, e^{-u}$ is bounded.

The reduction of a uniqueness assertion for a problem of type (4.1) to the simpler problem treated in Theorem 1.1 can be handled as in [4, p. 9]. The reduction is a simple consequence of an assertion that is valid for all Banach spaces.

THEOREM 4.1. *Let* B *be a Banach space, and let* $L, L_1, \cdots, L_m$ *be continuous linear functionals on* B *such that* L *does not depend linearly on* $L_1, \cdots, L_m$. *There exists a nonzero linear combination* Λ *of* $L, L_1, \cdots, L_m$ *such that each extremal function for the problem*

(*) *to maximize* $\Re L(f)$, *among all* $f \in B$ *satisfying the conditions* $\|f\| \leq 1$ *and* $L_j(f) = b_j$ $(1 \leq j \leq m)$,

is also an extremal function for the problem

(**) *to maximize* $|\Lambda(f)|$, *among all* $f \in B$ *satisfying the condition* $\|f\| \leq 1$.

The proof of Theorem 5.1 of [4] extends immediately to this generalization. We do not give details, except to mention that the proof involves a simple application of the separation theorem for convex sets.

5. PROOF OF THEOREM 1.2

The proof of Theorem 1.2 will be based on the methods used in [4, Section 4]. We shall omit the details when they overlap with [4].

Because Theorem 1.2 is local, we can restrict our attention to arbitrarily small neighborhoods of a fixed point on Γ. By applying a conformal map, we can assume that D is a bounded subset of the upper half-plane, that Γ is an open interval lying on the real axis, and that the point in question is the origin. Let W be the open upper half of an open disc centered at 0, such that $W \subset D$ and $W \cap K = \emptyset$. By shrinking Γ, we can assume that $\overline{\Gamma}$ lies on an open interval contained in ∂W. By shrinking W and Γ, we can also assume that there exists a continuous real-valued function v on D such that $u + iv$ is analytic on W.

Define B to be the set of all continuous complex-valued functions on D of the form $f e^{-(u+iv)}$, where f is analytic on D and $|f| \leq e^u$. In other words, we are multiplying the subspace used in Section 2 by the unimodular function e^{-iv}, so that the functions in B are now analytic on W.

The functions in B now have nontangential limits a.e. (dx) on Γ. The boundary-value function of $g \in B$ will be denoted by $\tilde{g} \in L^\infty(\Gamma, dx)$.

Define Λ analogously to (2.2), by the formula

$$\Lambda(g) = \int g e^{u+iv} d\eta \qquad (g \in B). \tag{5.1}$$

If F is the normalized extremal function for (1.1), then the corresponding function $G = F e^{-(u+iv)}$ in B is again characterized by (2.3). We must show that G extends analytically across Γ and has unit modulus on Γ. We begin with the following lemma.

LEMMA 5.1. *There exists* $g \in B$ *that extends analytically across* Γ *and has unit modulus on* Γ.

Proof. By hypothesis, there is a nonzero function $h \in B$. By the Nevanlinna factorization theory, there is an analytic function k in the Nevanlinna class of the upper half-plane such that k and $1/k$ are bounded outside W, k has unit modulus on

$\mathbb{R} \setminus \overline{\Gamma}$, and h/k is analytic on $W \cup \Gamma$ and has unit modulus on Γ. The function k can be expressed explicitly in the form

$$k = S_0 S_1 \exp(w + i^* w),$$

where S_0 is the Blaschke product (in the upper half-plane) of the zeros of h near Γ, S_1 is an appropriate singular inner function on the upper half-plane, and w is the Poisson integral of the function that is 0 on $\mathbb{R} \setminus \Gamma$ and $\log|\tilde{h}|$ on Γ. From the definition of B, one verifies that $g = h/k$ belongs to B. ■

Let $\lambda \in \overline{\Gamma}$, and consider the fiber $Z^{-1}(\{\lambda\}) \subset \beta(D)$ of $\beta(D)$ over λ. Since W coincides with D near λ, every bounded continuous function W also extends continuously to $Z^{-1}(\{\lambda\})$. In particular, the functions in $H^\infty(W)$ extend continuously to $Z^{-1}(\{\lambda\})$.

Let Y be the compactification of D obtained from $\beta(D)$ by the identification of all points of $Z^{-1}(\overline{\Gamma})$ that are identified by $H^\infty(W)$. In other words, the fiber of Y over $\lambda \in \partial D$ coincides with the fiber of $\beta(D)$ over λ if $\lambda \notin \overline{\Gamma}$, while it coincides with the fiber of the maximal-ideal space of $H^\infty(W)$ over λ whenever $\lambda \in \overline{\Gamma}$. The functions in B all extend continuously to Y.

When it is convenient, we shall regard Z as the extension of the coordinate function to Y, so that Z maps Y onto $\overline{D}$. Let $\check{S}(W)$ be the Shilov boundary of $H^\infty(W)$, and set

$$Q = Z^{-1}(\partial D \setminus \Gamma) \cup [Z^{-1}(\Gamma) \cap \check{S}(W)];$$

then Q is a compact subset of $Y \setminus D$. The term $Z^{-1}(\Gamma) \cap \check{S}(W)$ has a simple interpretation. Let $\Sigma(dx)$ be the maximal-ideal space of $L^\infty(\Gamma, dx)$, so that

$$L^\infty(\Gamma, dx) \cong C(\Sigma(dx)). \tag{5.2}$$

Again the coordinate function z in $L^\infty(dx)$ determines a continuous function, also denoted by Z, that projects $\Sigma(dx)$ onto the closure $\overline{\Gamma}$ of Γ. The term $Z^{-1}(\Gamma) \cap \check{S}(W)$ can be identified with the piece of $\Sigma(dx)$ lying over Γ,

$$Z^{-1}(\Gamma) \cap \check{S}(W) \cong Z^{-1}(\Gamma) \cap \Sigma(dx).$$

The values of a function $f \in H^\infty(W)$ on $Z^{-1}(\Gamma) \cap \check{S}(W)$ are determined via Fatou's theorem, which assigns to f its nontangential-boundary-value function $\tilde{f} \in L^\infty(\Gamma, dx)$.

Now B is a closed subspace of C(Y), and the analogue of (2.1) is the following.

LEMMA 5.2. *For all* $g \in B$, $\|g\|_D = \|g\|_Q$.

Proof. Let $g \in B$. Since $\log|g|$ is subharmonic on D, the function g attains its maximum modulus on the fringe $Y \setminus D = Z^{-1}(\partial D)$, and hence on $Z^{-1}(\{\lambda\})$ for some $\lambda \in \partial D$. If $\lambda \in (\partial D) \setminus \Gamma$, then $Z^{-1}(\{\lambda\}) \subset Q$; therefore g attains its maximum modulus on Q. On the other hand, if $\lambda \in \Gamma$, then

$$\limsup_{W \ni z \to \lambda} |g(z)| = \|g\|.$$

The nontangential-boundary-value function $\tilde{g}$ of g then satisfies the condition

$$\operatorname*{ess\,lim\,sup}_{\Gamma \ni x \to \lambda} |\tilde{g}(x)| = \|g\| .$$

Under the isomorphism (5.2), the "essential lim sup" of $|\tilde{g}|$ at λ coincides with the maximum modulus of g on $\Sigma(dx) \cap Z^{-1}(\lambda)$. This latter set is included in Q, so that again g attains its maximum modulus on Q. ■

In view of Lemma 5.2, there is a measure ν on Q that satisfies (2.4) and (2.5), that is, $\|\nu\| = \|\Lambda\|$, and $\Lambda(g) = \int g\, d\nu$ $(g \in B)$. As in [4], let dX denote the measure on $\Sigma(dx)$ that is the natural lift of dx to $\Sigma(dx)$.

LEMMA 5.3. *The restriction of* ν *to* $Q \cap Z^{-1}(\Gamma)$ *is absolutely continuous with respect to* dX.

We omit the proof, since it is the same as that in [4, p. 8].

By Lemma 5.3, there is a function $h \in L^1(\Gamma, dx)$ such that on $Z^{-1}(\Gamma)$, ν coincides with the lift of $h\,dx$. In other words,

$$\Lambda(g) = \int_{Q \setminus Z^{-1}(\Gamma)} g\, d\nu + \int_\Gamma \tilde{g} h\, dx \qquad (g \in B).$$

LEMMA 5.4. *There is an analytic function on* W *of class* H^1 *near* Γ *that has nontangential boundary values equal to* h a.e. (dx) *on* Γ.

Proof. By Lemma 5.1, some function g_0 in B extends analytically across Γ and has unit modulus on Γ. Define the measure σ on $K \cup \partial D$ to be the projection of $g_0\,\nu - g_0\, e^{u+iv}\eta$:

$$\sigma = Z^*(g_0\,\nu - g_0\, e^{u+iv}\eta) = Z^*(g_0\,\nu) - g_0\, e^{u+iv}\eta .$$

On account of the definition of Λ, $\nu - e^{u+iv}\eta$ is orthogonal to B. Since $g_0/(Z - \zeta) \in B$ for all $\zeta \notin \overline{D}$,

$$0 = \int \frac{1}{Z - \zeta}\, g_0\, d(\nu - e^{u+iv}\eta) = \int \frac{1}{Z - \zeta}\, d\sigma(z) \qquad (\zeta \notin D).$$

By [4, Lemma 4.3], the analytic function

$$\hat{\sigma}(\zeta) = \int \frac{d\sigma(z)}{z - \zeta} \qquad (\zeta \in W), \tag{5.3}$$

is of class H^1 near Γ and has nontangential boundary values equal to $2\pi i g_0 h$ a.e. (dx) along Γ. It follows that $g_0 h$, and hence h, is the boundary-value function of an analytic function of class H^1 near Γ. ■

It is in the proof of the next lemma that we use the topological hypothesis on $D \setminus K$.

LEMMA 5.5. *The function* h *does not vanish identically on* Γ.

Proof. Suppose, on the contrary, that $h = 0$ on Γ, so that ν has no mass on $Z^{-1}(\Gamma)$.

Let α be a cycle in $D \setminus K$ that surrounds K in the usual sense of contour integration. Let $g \in B$. Since $\nu - e^{u+iv}\eta$ is orthogonal to B,

$$\int \frac{g\,d\nu}{z - \zeta} = \int \frac{g\,e^{u+iv}\,d\eta}{z - \zeta} \qquad (\zeta \in \mathbb{C} \setminus \overline{D}). \tag{5.4}$$

Now both sides of (5.4) are analytic on $D \setminus K$ and across Γ. Since $D \setminus K$ is connected, the identity (5.4) persists for all $\zeta \in D \setminus K$. Integrating both sides of (5.4) along the cycle α, and interchanging the orders of integration, we obtain the equation

$$0 = -2\pi i \int g\,e^{u+iv}\,d\eta = -2\pi i\,\Lambda(g).$$

Consequently $\Lambda = 0$. This contradiction establishes the lemma. ∎

Now the relations (2.4) and (2.5) show that the nontangential-boundary-value function $\tilde{G}$ of the extremal function G satisfies the conditions $\tilde{G}h \geq 0$ and $\tilde{G}h = |h|$ a.e. (dx) on Γ. The classical proof, which uses the H^1-version of the Schwarz reflection principle as in [4, p. 9], then shows that G is analytic across Γ, and that G is unimodular on Γ. That completes the proof of Theorem 1.2.

We close with some comments on the hypotheses of Theorem 1.2.

The hypothesis that $D \setminus K$ be connected is necessary, as the example of [4, p. 6] shows. Without this hypothesis, one can only conclude that there is a component V of $D \setminus K$ such that $Fe^{-(u+iv)}$ extends analytically across any free analytic boundary arc in $\partial D \cap \partial V$ and has unit modulus there.

The boundedness of D is irrelevant, since we can reduce the unbounded case to the bounded case by mapping D onto a bounded domain. Moreover, Theorem 1.2 remains valid if u is only continuous and superharmonic, providing in the conclusion u is replaced by its greatest harmonic minorant.

The imposition of side conditions as in (4.1) does not affect the behavior of the extremal function at a free analytic boundary curve, as Theorem 4.1 shows. However, if the admissible functions are required to vanish on a sequence $\{z_k\}$ in D, Then $Fe^{-(u+iv)}$ cannot extend analytically across any accumulation point on Γ of the points z_k. Nevertheless, in this case there is a certain "Blaschke product" S, bounded and analytic on D, such that $Fe^{-(u+iv)}/S$ extends analytically across Γ and has unit modulus on Γ.

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