EMBEDDINGS OF k-ORIENTABLE MANIFOLDS

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1. INTRODUCTION

Let M be a closed, k-connected, smooth, n-dimensional manifold, and let M_0 denote M minus a point $x_0 \in M$. In [2], J. C. Becker and H. Glover showed that for $j \leq 2k$ and $2j \leq n - 3$, the manifold M embeds in R^{2n-j} if and only if M_0 immerses in R^{2n-j-1}. We shall extend this result to $j = 2k + 1$ by placing an additional condition of orientability on M.

A vector bundle is called k-*orientable* if its restriction to the k-skeleton of its base is stably fibre-homotopy-trivial. A manifold is k-orientable if its tangent bundle is k-orientable.

Letting M be (k + 1)-orientable with $k \leq (n - 5)/4$, we state our main theorem.

THEOREM 1.1. M *embeds in* $R^{2n-2k-1}$ *if and only if* M_0 *immerses in* $R^{2n-2k-2}$.

This result reduces an embedding problem to one involving an immersion in which the top obstruction vanishes.

As applications we obtain the following.

THEOREM 1.2. *Let* M *be an* n-*dimensional, simply-connected spin manifold with* $n \equiv 3$ (*mod* 4) *and* $n \geq 11$. *Then* M *embeds in* R^{2n-3}.

Proof. It is sufficient to show that the associated bundle with fibre $V_{m,m-n+4}$ has a cross-section, for large m. The obstructions to such a cross-section lie in $H^{i+1}(M_0; \pi_i(V_{m,m-n+4}))$. If $i < n - 4$, then $\pi_i = 0$. For $i = n - 4$, the obstruction $\overline{w}_{n-3}$ is 0, by [7]. The homotopy group π_{n-3} is 0, by [6]. By connectedness, $H^{n-1}(M_0) = 0$, and finally, $H^n(M_0) = 0$.

COROLLARY 1.3. *If* M *is a closed, almost parallelizable,* k-*connected* n-*manifold and* $k \leq (n - 5)/4$, *then* M *can be embedded in* $R^{2n-2k-1}$.

The corollary follows from the fact that M is (n - 1)-orientable and that by [4] M_0 can be immersed in R^n. This corollary extends a result of R. de Sapio [8], for some values of k.

2. ORIENTABILITY

Let $\mathscr{E}$ be a spectrum as defined in [10]. Let $\mathscr{S}$ denote the sphere spectrum, and let $\mathscr{S}^k$ denote the k-stem spectrum. (We obtain $(S^n)^k$ from S^n by killing the homotopy group for $i \geq n + k$ with the inclusion map $\lambda: S^n \to (S^n)^k$.) As in [10], we have a generalized homology and cohomology theory defined by

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$$H_n(Y;\ \mathscr{E}) = \lim \pi_{n+q}(E_q \wedge Y),$$

$$H^n(Y;\ \mathscr{E}) = \lim\ [S^q Y,\ E_{n+q}]\ .$$

An *$\mathscr{E}$-fundamental class* for M is an element $z \in H_n(M;\ \mathscr{E})$ such that $(j_x)_*(z)$ is a generator of $\tilde{H}_n(S^n;\ \mathscr{E})$ as an $H^*(pt;\ \mathscr{E})$-module for each $x \in M$. Using an argument similar to that found in [9, p. 304], we obtain the following.

LEMMA 2.1. *A vector bundle over* M *is* k*-orientable if and only if it has an* $\mathscr{S}^k$*-fundamental class.*

3. PROOF OF THEOREM 1.1

We can now prove the sufficiency, using some standard techniques of algebraic topology together with a result of Becker [1].

Since M embeds, M_0 immerses in $R^{2n-2k-1}$. To reduce the codimension of this immersion by one, it is sufficient to find a cross-section to $S(\alpha)$, the sphere bundle associated with the restricted normal bundle α to this immersion. Since M is $(k+1)$-orientable, α is $(k+1)$-orientable. Therefore, by [1], it is sufficient to show that M_0 is contractible in M_0^α, the Thom space. M_0^α is $(n-k-1)$-connected, by an argument using the Thom Isomorphism Theorem, the Van Kampen Theorem, and the Hurewicz Theorem. The groups $H^p(M_0)$ are 0 for $n-k \leq p \leq n-1$, and $H^n(M_0) = 0$. Therefore all obstructions vanish.

The necessity will follow after a series of lemmas and the application of a technique used in [3].

LEMMA 3.1. *If* M_0 *immerses in* $R^{2n-2k-1}$ *with a normal vector field, then* M_0 *embeds in* $R^{2n-2k-1}$ *with a normal vector field.*

The lemma follows from [5, Theorem 5.1].

Let M_1 denote M minus the interior of an n-disk E_1 of radius 1 and center at x_0, and let M_2 denote M minus the interior of an n-disk E_2 of radius 2 and center at x_0.

Assume $f\colon M_0 \to R^{2n-2k-1}$ is an embedding with a normal vector field. Let

$$X = R^{2n-2k-1} \setminus f(M_2).$$

LEMMA 3.2. *If* $g\colon E_2 \to X$ *is a proper map whose restriction to the complement of some compact set is an embedding, then there exists a homotopy, fixed outside some compact set, that deforms* g *into an embedding.*

Proof. Using Alexander and Poincaré dualities, we see that $H_i(X) = 0$ for all $i \leq n-k-2$. The set X is simply connected, by a general-position argument. Therefore, by the Hurewicz Theorem, X is $(n-k-2)$-connected. The result now follows from [3].

LEMMA 3.3. $H_{n-1}(X;\ \mathscr{S}) \simeq H_{n-1}(X;\ \mathscr{S}^{k+1})$.

Proof. It is sufficient to show

$$\pi_{n-1+q}(X \wedge S^q) \simeq \pi_{n-1+q}(X \wedge (S^q)^{k+1})\ .$$

The Kunneth formula gives us a commutative diagram of short exact sequences associated with the maps id: $X \to X$ and $\lambda: S^q \to (S^q)^{k+1}$. The homomorphism

$$(\mathrm{id} \wedge \lambda)_*: H_i(X \wedge S^q) \to H_i(X \wedge (S^q)^{k+1})$$

is an isomorphism for $i \leq n - 1 + q$, and it is surjective for $i = n - 1$, by the five-lemma. The result follows from Whitehead's Theorem.

LEMMA 3.4. *The homomorphism* $i_*: H_{n-1}(\partial M_1; \mathscr{S}^{k+1}) \to H_{n-1}(M_1; \mathscr{S}^{k+1})$ *is constant.*

Proof. The groups $H_i(E_1; \mathscr{S}^{k+1})$ are 0 for $i \geq k+1$. Therefore, the mappings

$$\partial: H_n(\overline{E}_1, \partial\overline{E}_1; \mathscr{S}^{k+1}) \to H_{n-1}(\partial\overline{E}_1; \mathscr{S}^{k+1})$$

and

$$i_*: H_n(\overline{E}_1, \partial\overline{E}_1; \mathscr{S}^{k+1}) \to H_n(M, M_1; \mathscr{S}^{k+1})$$

are isomorphisms.

Since $j_*: H_n(M; \mathscr{S}^{k+1}) \to H_n(M, M_1; \mathscr{S}^{k+1})$ is surjective, by the existence of a fundamental class, it follows that $\partial: H_n(M, M_1; \mathscr{S}^{k+1}) \to H_{n-1}(M_1; \mathscr{S}^{k+1})$ is zero. The sufficiency now follows from the commutative diagram

$$\begin{array}{ccc}
H_{n-1}(\partial M_1; \mathscr{S}^{k+1}) & \xrightarrow{i_*} & H_{n-1}(M_1; \mathscr{S}^{k+1}) \\
\downarrow \partial\ (\simeq) & & \downarrow \partial\ (\text{zero}) \\
H_n(\overline{E}_1, \partial\overline{E}_1; \mathscr{S}^{k+1}) & \xrightarrow{i_*(\simeq)} & H_n(M, M_1; \mathscr{S}^{k+1})
\end{array}$$

We now prove the necessity.

Because M_0 immerses in $R^{2n-2k-2}$, it immerses in $R^{2n-2k-1}$ with a normal vector field. By Lemma 3.1, M_0 embeds in $R^{2n-2k-1}$ by some map f and with a normal vector field $\nu: M_0 \to R^{2n-2k-1}$ such that $\nu(x)$ is a unit vector orthogonal to the image under df of the tangent plane to M_0 at x. Let ε be a positive real number small enough to be the radius of a tubular neighborhood of $f(M_1)$. Let $\lambda: M_1 \to [0, \varepsilon]$ be a differentiable map equal to ε on M_0 and equal to 0 on ∂M_1. Define $g: M_1 \to X$ by $g(x) = f(x) + \lambda(x)\,\nu(x)$.

By Freudenthal's Suspension Theorem, the homomorphism

$$S: [S^{n-1+q}, S^q X] \to [S^{n+q}, S^{q+1} X]$$

is an isomorphism for all q, since X is $(n - k - 2)$-connected. Therefore, $i_0: \pi_{n-1}(X) \to H_{n-1}(X; \mathscr{S})$ is an isomorphism. From the commutative diagram

$$\begin{array}{ccccc}
\pi_{n-1}(\partial M_1) & \xrightarrow{i_0} & H_{n-1}(\partial M_1; \mathscr{S}) & \xrightarrow{\lambda_*} & H_{n-1}(\partial M_1; \mathscr{S}^{k+1}) \\
\downarrow i_\# & & \downarrow i_* & & \downarrow i_* \text{ (zero)} \\
\pi_{n-1}(M_1) & \xrightarrow{i_0} & H_{n-1}(M_1; \mathscr{S}) & \xrightarrow{\lambda_*} & H_{n-1}(M_1; \mathscr{S}^{k+1}) \\
\downarrow g_\# & & \downarrow g_* & & \downarrow g_* \\
\pi_{n-1}(X) & \xrightarrow{i_0(\simeq)} & H_{n-1}(X; \mathscr{S}) & \xrightarrow{\lambda_*(\simeq)} & H_{n-1}(X; \mathscr{S}^{k+1})
\end{array}$$

with the appropriate maps now identified, we see that $[f(\partial M_1)] = 0$ in $\pi_{n-1}(X)$.

Therefore $f \mid M_1 \colon M_1 \to R^{2n-2k-1}$ can be extended to a map $\tilde{f} \colon M \to R^{2n-2k-1}$ such that $\tilde{f}(M_2) \cap \tilde{f}(E_2) = \emptyset$. The map $\tilde{f} \mid E_2$ is a proper map whose restriction to $E_2 \setminus \overline{E}_1$ is an embedding. By Lemma 3.2, $\tilde{f} \mid E_2$ is homotopic to an embedding of E_2 in X that agrees with f in $E_2 \setminus \overline{E}_1$. This embedding and f now fit together to give an embedding of M in $R^{2n-2k-1}$.

REFERENCES

1. J. C. Becker, *Immersions of* k-*orientable manifolds*. Michigan Math. J. 17 (1970), 161-164.
2. J. C. Becker and H. H. Glover, *Note on the embedding of manifolds in Euclidean space*. Proc. Amer. Math. Soc. 27 (1971), 405-410.
3. A. Haefliger and M. W. Hirsch, *On the existence and classification of differentiable embeddings*. Topology 2 (1963), 129-135.
4. M. W. Hirsch, *On imbedding differentiable manifolds in euclidean space*. Ann. of Math. (2) 73 (1961), 566-571.
5. ———, *Embeddings and compressions of polyhedra and smooth manifolds*. Topology 4 (1966), 361-369.
6. C. S. Hoo and M. E. Mahowald, *Some homotopy groups of Stiefel manifolds*. Bull. Amer. Math. Soc. 71 (1965), 661-667.
7. W. S. Massey and F. P. Peterson, *On the dual Stiefel-Whitney classes of a manifold*. Bol. Soc. Mat. Mexicana (2) 8 (1963), 1-13.
8. R. de Sapio, *Embedding* π-*manifolds*. Ann. of Math. (2) 82 (1965), 213-224.
9. E. H. Spanier, *Algebraic topology*. McGraw-Hill, New York, 1966.
10. G. W. Whitehead, *Generalized homology theories*. Trans. Amer. Math. Soc. 102 (1962), 227-283.

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