TOPOLOGICAL ENTROPY FOR NONCOMPACT SPACES

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1. INTRODUCTION

For compact spaces, topological entropy was defined in [1]. In this paper we extend the concept to noncompact spaces. Let T be a continuous mapping from a topological space X into itself. We define the topological entropy of T for noncompact spaces in three different ways: $h^2(T)$, $h^3(T)$, and $h^*(T)$ (see [6] for more ways). Then we establish some properties.

We shall use the following notation: $\mathscr{A}(X)$, or simply $\mathscr{A}$ when the meaning is clear, will denote the class of all open covers of X, while $\mathscr{A}_f(X)$, or simply $\mathscr{A}_f$, will denote the class of all finite open covers of X. Suppose that X is a compact topological space and ϕ: X → X is a continuous mapping. Let $\alpha_i \in \mathscr{A}$ for $i = 1, 2, \cdots, n$. We define the join $\bigvee_{i=1}^n \alpha$ of the covers α_i by the formula

$$\bigvee_{i=1}^{n} \alpha_i = \alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n = \{U_1 \cap \cdots \cap U_n : U_i \in \alpha_i,\ i = 1, \cdots, n\}.$$

We define $N_X(\alpha)$ (or simply $N(\alpha)$ when the space X is understood) as the number of sets in a subcover of α of minimal cardinality. Set

$$h(\alpha, \phi) = \lim_{n \to \infty} \frac{1}{n} \log N\left(\bigvee_{i=0}^{n-1} \phi^{-i}\alpha\right) \quad \text{and} \quad h(\phi) = \sup_{\alpha \in \mathscr{A}} h(\alpha, \phi).$$

The quantity $h(\phi)$ is called the topological entropy of ϕ (see [1]).

Now let X be a noncompact Hausdorff space, and let T: X → X be a continuous mapping. In defining topological entropy for noncompact spaces, at least two approaches appear natural: one is to compactify the space and to consider the extension T^* of T to the compactification X^* of X; the second approach is to consider only finite open covers of X.

Another approach that involves no compactification is based on the notion of uniform spaces (see Section 4).

2. BASIC DEFINITIONS AND PROPERTIES

Although many of our results are valid when T is merely assumed to be continuous, we shall assume, unless we specify otherwise, that T is in fact a homeomorphism.

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Let $(X, \mathscr{T})$ be a *noncompact* Hausdorff space, and let $T: X \to X$ be a *homeomorphism.*

Definitions. (a) The entropy h^2 of the mapping T is defined by the formula $h^2(T) = h(T^*)$, where T^* is the unique continuous extension of T to the Stone-Čech compactification X^* of X. Here $(X, \mathscr{T})$ is a completely regular topological T_1-space.

(b) The entropy h^3 of the mapping T is defined by the formula

$$h^3(T) = \sup_{\alpha \in \mathscr{A}_f} h(\alpha, T).$$

Remark. If X is compact, it is clear that $h^i(T) = h(T)$ for $i = 2$ and $i = 3$.

PROPOSITION 1. *Let* X *be a zero-dimensional Hausdorff space. Then*

$$h^2(T) \leq h^3(T).$$

For the proof, we refer the reader to [6].

Definition. A *flow* is a pair (X, T), where X is a compact Hausdorff space and $T: X \to X$ is a continuous mapping. If T is a homeomorphism, then (X, T) is a *cascade.* That is, a cascade is a transformation group in which the acting group is the set of integers.

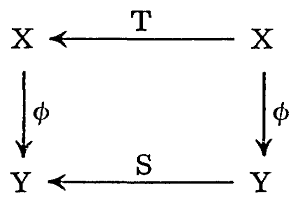
Example. Let Z be the set of integers, and let $T: Z \to Z$ be the shift defined by $T(x) = x + 1$. In [4] it is shown that (X^*, T^*) (again, X^* is the Stone-Čech compactification of X) is the universal point-transitive cascade, and hence $h^2(T) = h^3(T) = \infty$.

Proof. Because every point-transitive cascade is the homomorphic image of the universal transitive cascade, the universal transitive cascade has entropy greater than any homomorphic image. But there exist point-transitive cascades of arbitrarily large entropy. Hence $h^2(T) = h^3(T) = \infty$.

Definitions. (a) We say that h satisfies the *power formula* if $h(T^k) = kh(T)$ for each positive integer k (see [1]).

(b) Let Y be a closed T-invariant subset of X. We say that h has the *monotonic property* if $h(T \mid Y) \leq h(T)$.

(c) Let X and Y be topological spaces. We say that h has the *continuous-image property* if, whenever the diagram



commutes (that is, $\phi \circ T = S \circ \phi$), then $h(S) \leq h(T)$, where S and T are continuous mappings and ϕ is a continuous and surjective mapping.

LEMMA 1. *Let* X *be a topological space, and let* Y *be a Hausdorff space. Let* $f, g: X \to Y$ *be continuous. If* $D \subseteq X$ *is dense and* $f \mid D = g \mid D$, *then* $f = g$ *on* X.

PROPOSITION 2. *The mappings* h^2 *and* h^3 *satisfy the power formula, and both have the monotonic property.*

Proof. We prove only that h^2 satisfies the power formula, and we refer the reader to [6] for the proof of the remaining assertions. It is clear that $T^{*k} \mid X = T^{k*} \mid X$. By Lemma 1, this implies that $T^{*k} = T^{k*}$ on X^*. Hence

$$h^2(T^k) = h(T^{k*}) = h(T^{*k}) = kh(T^*) = kh^2(T).$$

We shall need the following two well-known results from topology.

THEOREM 1. *If* X *and* Y *are topological spaces, then a function* f *from* X *onto* Y *is continuous if and only if* $\langle a_\nu \rangle \to a^*$ *in* X *implies that* $\langle f(a_\nu) \rangle \to f(a^*)$ *in* Y, *where* $\langle a_\nu \rangle$ *is a net in* X.

THEOREM 2. *A space* X *is a Hausdorff space if and only if every convergent net in* X *has a unique limit.*

PROPOSITION 3. *The mappings* h^2 *and* h^3 *have the continuous-image property.*

Proof. The proof for h^3 is easy; we give that for h^2. Let X and Y be completely regular T_1-spaces. Consider the diagram

$$\begin{array}{ccc} X & \xleftarrow{\ T\ } & X \\ \downarrow \phi & & \downarrow \phi \\ Y & \xleftarrow{\ S\ } & Y \end{array}$$

where T and S are continuous, ϕ is continuous and surjective, and $\phi \circ T = S \circ \phi$. The problem is to show that $h^2(S) \leq h^2(T)$, or equivalently, that $h(S^*) \leq h(T^*)$. We extend the diagram by forming the Stone-Čech compactification of each space and extending the corresponding maps. This gives the diagram

$$\begin{array}{ccc} X^* & \xleftarrow{\ T^*\ } & X^* \\ \downarrow \phi^* & & \downarrow \phi^* \\ Y^* & \xleftarrow{\ S^*\ } & Y^* \end{array}$$

From topology we know that ϕ^* exists and is unique and continuous. It remains to show that $\phi^* \circ T^* = S^* \circ \phi^*$. Let $x \in X^*$. If $x \in X$, then

$$\phi^* \circ T^*(x) = \phi \circ T(x) = S \circ \phi(x) = S^* \circ \phi^*(x).$$

Let $x \in X^* - X$. Then there exists a net $\langle x_\nu \rangle \in X$ such that $\langle x_\nu \rangle \to x$. By Theorem 1,

$$\phi \circ T(x_\nu) = \phi^* \circ T^*(x_\nu) \to \phi^* \circ T^*(x) \quad \text{and} \quad S \circ \phi(x_\nu) = S^* \circ \phi^*(x_\nu) \to S^* \circ \phi^*(x).$$

Hence $\phi^* \circ T^* = S^* \circ \phi^*$, by Theorem 2. This implies $h(S^*) \leq h(T^*)$. The proof is complete.

3. THE MAIN RESULT

A good reference for the material of this section is [7, p. 167, Problem 5R]. The next result is of considerable importance, since it shows that in calculating the topological entropy for normal T_1-spaces we obtain the same result using finite open covers of X as with the Stone-Čech compactification of X.

THEOREM 3. *Let* X *be a normal* T_1*-space. Then* $h^2(T) = h^3(T)$.

The proof requires a few lemmas, all easy to prove.

LEMMA 2. *Let* X *be a normal* T_1*-space, and let* $\phi: X \to w(X)$ *be defined as in* [7]. *Then, if* U *is open in* X,

$$U^* \cap \phi(X) = \phi(U).$$

LEMMA 3. *Let* U *and* V *be open subsets of* X. *Then the relation* $U \subseteq V$ *implies that* $U^* \subseteq V^*$.

LEMMA 4. *Let* X *be a normal* T_1*-space. Define* $\alpha^* = \{U^*: U \in \alpha\}$, *where the members of* α *are open sets in* X. *Then* α *is a finite open cover of* X *if and only if* α^* *is an open cover of* w(X).

LEMMA 5. *Consider the correspondence* $U \leftrightarrow U^*$, *where* U *is open in* X *and* U^* *is open in* w(X). *Let* α *be a finite open cover of* X. *Then*

$$N_X(\alpha) = N_{w(X)}(\alpha^*).$$

LEMMA 6. *Let* X *be a normal* T_1*-space, and let* $T: X \to X$ *be a homeomorphism. Define* $T^*: w(X) \to w(X)$ *by* $T^*(\mathscr{A}) = \{T(A): A \in \mathscr{A}\}$. *Then* $T^{*-1}(A^*) = (T^{-1}(A))^*$.

LEMMA 7. *The mapping* T^* *is a continuous mapping.*

Proof of Theorem 3. Since X is normal, w(X) is a Hausdorff space topologically equivalent to the Stone-Čech compactification of X. Let α be any finite open cover of X. Then

$$N_X\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right) = N_{w(X)}\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right)^* = N_{w(X)}\left(\bigvee_{i=0}^{n-1} (T^{-i}\alpha)^*\right)$$
$$= N_{w(X)}\left(\bigvee_{i=0}^{n-1} T^{*-i}\alpha^*\right).$$

Hence

$$\frac{1}{n}\log N_X\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right) = \frac{1}{n}\log N_{w(X)}\left(\bigvee_{i=0}^{n-1} T^{*-i}\alpha^*\right).$$

Letting $n \to \infty$, we get the relation

$$h(\alpha, T) = h(\alpha^*, T^*).$$

Hence $h(\alpha, T) \leq h(T^*) = h^2(T)$, and therefore $h^3(T) \leq h^2(T)$. Now let β be any open cover of w(X). Since $\{U^*: U \text{ is open in } X\}$ forms a base for the topology of w(X) and since w(X) is compact, we can refine β by a finite open cover of the form

$$\alpha^* = \{U^*: U \text{ is open in } X\}.$$

Hence $\beta \prec \alpha^*$ (see [1]), and therefore

$$h(\beta, T^*) \leq h(\alpha^*, T^*) = h(\alpha, T) \leq h^3(T).$$

But β is arbitrary, and consequently $h^2(T) = h(T^*) \leq h^3(T)$. The proof is complete.

Remark. We conjecture that $h^2(T) = h^3(T)$ for all completely regular spaces. This would validate the theorem to a larger class. Whether there exists a space on which h^2 and h^3 disagree appears to be an open question.

4. UNIFORM TOPOLOGICAL ENTROPY

In this section we consider the calculation of topological entropy on noncompact spaces, using a method that, unlike the treatment in Sections 2 and 3, does not depend upon first compactifying the space. We accomplish this by using the notion of uniform spaces. Our definition of $h^*(T)$ is motivated by the work done in [3]. A good reference for the material of this section is [7, pp. 174-199].

We shall reserve the letters H, K, K_1, K_2, $\cdots$ for compact subsets of X.

Definition. Let $(X, \mathscr{U})$ be a uniform space, and let α be a uniform cover of X. Let $T: X \to X$ be uniformly continuous. We define

$$h_K(\alpha, T) = \lim_{n\to\infty} \frac{1}{n} \log N_K\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right) \quad \text{and} \quad h^*(T) = \sup_{\alpha \text{ uniform cover}} \; \sup_{K\subseteq X} h_K(\alpha, T),$$

where K is any compact subset of X. We shall call $h^*(T)$ the *uniform topological entropy* of T.

Remark. A compact Hausdorff space is completely regular. It is easy to show that in the case of a compact Hausdorff space, $h^*(T) = h(T)$.

Example. Let R denote the space of real numbers, let $T: R \to R$ be defined by $T(x) = 2x$, and suppose R has the usual uniformity $\mathscr{U}$. Then $h^*(T) = \log 2$.

Proof. Clearly, $\mathscr{U}$ contains all subsets $U \subseteq R \times R$ such that

$$U_\varepsilon = \{(x, y): |x - y| < \varepsilon\} \subseteq U \qquad \text{for some } \varepsilon > 0.$$

Therefore each $U_\varepsilon \in \mathscr{U}$. The topology generated by U_ε is $\alpha_\varepsilon = \{B(x, \varepsilon): x \in R\}$, where $B(x, \varepsilon) = (x - \varepsilon, x + \varepsilon)$. Now it clearly suffices to consider the uniform covers α_ε. Also, since

$$H \subseteq K \Rightarrow N_H\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right) \leq N_K\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right)$$

(hence $h_H(\alpha_\varepsilon, T) \leq h_K(\alpha_\varepsilon, T)$, and since each compact subset of R is contained in a compact set of the form $K_m = [-m\varepsilon, m\varepsilon]$, where m is a positive integer, it suffices to consider compact subsets of the form K_m, where m is a positive integer. Hence, for each positive integer n,

$$N_{K_m}\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha_\varepsilon\right) = 2^{n-1} N_{K_m}(\alpha_\varepsilon).$$

Therefore

$$\frac{1}{n}\log N_{K_m}\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha_\varepsilon\right) = \frac{n-1}{n}\log 2 + \frac{1}{n}\log N_{K_m}(\alpha_\varepsilon).$$

Let $n \to \infty$. Then $h_{K_m}(\alpha_\varepsilon, T) = \log 2$ for each positive integer m and each $\varepsilon > 0$. Hence $h^*(T) = \log 2$.

Remark. If $\mathscr{U}$ is the uniformity of all neighborhoods of the diagonal in $R \times R$, then $h^*(T) = \infty$.

Example. Under the conditions of the example above, let $T(x) = x + 1$. Then $h^*(T) = 0$. If T is the identity, then $h^*(T) = 0$.

Proof. The proof is identical to that of the example above, except that now for each positive integer n we have the relation

$$N_{K_m}\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha_\varepsilon\right) = N_{K_m}(\alpha_\varepsilon).$$

Remark. Let Z be the set of integers, and let $T: Z \to Z$ be defined by $T(n) = n + 1$. Then $h^*(T) = 0$.

PROPOSITION 5. *The mapping* h^* *satisfies the power formula and has the monotonic property.*

The proof that $h^*(T^m) = mh^*(T)$ for each positive integer m is similar to that for h given in [1], and we refer the reader to [6] for the proof that h^* has the monotonic property.

Definition. We say that h *distinguishes* between the transformations S and T if $S \neq T$ implies $h(S) \neq h(T)$. We want to test the ability of h^1, h^2, and h^* to distinguish between the transformations $T, S: R \to R$ defined by $T(x) = x + 1$ and $S(x) = 2x$.

Recall that $h^*(T) = 0$ and $h^*(S) = \log 2$. Now R with the usual topology is a normal T_1-space, and hence $h^2(T) = h^3(T)$ and $h^2(S) = h^3(S)$. We shall show that

$$h^3(S) = h^3(T) = h^2(S) = h^2(T) = \infty.$$

Hence h^* distinguishes between the transformations T and S, but h^2 and h^3 do not.

Example. Let $T: R \to R$ be defined by $T(x) = x + 1$. Then $h^2(T) = h^3(T) = \infty$.

Proof. From the preceding paragraph we know that $h^2(T) = h^3(T)$. Let R^* be the Stone-Čech compactification of R, and let $T^*: R^* \to R^*$ be the unique continuous extension of T to R^*. Note (see [4] and [5, p. 167]) that (R^*, T^*) is the universal point-transitive cascade and hence $h^2(T) = h(T^*) = \infty$.

Definition. Let K be the circle group $K = \{z = C: |z| = 1\}$, where C is the set of complex numbers. Then $K^n = K \times K \times \cdots \times K$ (n terms) is called the n-torus.

Choose a line L in R^n passing through the origin such that L is orthogonal to no lattice lines (lines joining points of Z^n). Clearly, L is isomorphic to R. Project the line L onto K^n, using the map π defined by

$$\pi(x) = \pi(x_1, \cdots, x_n) = (e^{2\pi i x_1}, \cdots, e^{2\pi i x_n}) \qquad \text{for } x \text{ in } L.$$

LEMMA 8. *The set* πL *is a dense subgroup of* K^n.

We refer the reader to [6] for the proof.

Example. Let $S: R \to R$ be defined by $S(x) = 2x$. Then

$$h^2(S) = h^3(S) = \infty.$$

Proof. The n-torus has a dense subgroup isomorphic to R. Consider the diagram

$$\begin{array}{ccc} R & \xleftarrow{\;S\;} & R \\ \downarrow \pi & & \downarrow \pi \\ K^n & \xleftarrow{\;S'\;} & K^n \end{array}$$

where $S(x) = 2x$, where π is as defined previously, and where S' (not yet determined) satisfies the condition $\pi \circ S = S' \circ \pi$. From the relation $\pi \circ S = S' \circ \pi$ it follows easily that S' squares each component; that is, $S'(x_1, \cdots, x_n) = (x_1^2, \cdots, x_n^2)$. Now π induces a continuous mapping ϕ from R^* onto K^n (here R^* is the Stone-Čech compactification of R), and we can extend the diagram to get

$$\begin{array}{ccc} R^* & \xleftarrow{\;S^*\;} & R^* \\ \downarrow \phi & & \downarrow \phi \\ K^n & \xleftarrow{\;S'\;} & K^n \end{array}$$

where, clearly, $\phi \circ S^* = S' \circ \phi$. Now the map from the 1-torus onto the 1-torus defined by $x \to e^{2\pi i(2x)}$ has the associated matrix $A = (2)$. This implies (see [2, pp. 67-79]) that $h(A) = \log 2$. Hence $h(S') = n \log 2$, by the product theorem (see [1]). Hence, for each positive integer n, $h(S^*) \geq n \log 2$. Therefore

$$h^2(S) = h^3(S) = h(S^*) = \infty.$$

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