TWO THEOREMS ON KAEHLER MANIFOLDS

Bang-yen Chen and Koichi Ogiue

1. INTRODUCTION

Let N be an n-dimensional submanifold (in this paper, we consider only manifolds of real dimension) of a 2m-dimensional Kaehler manifold M with complex structure J and Riemannian metric g, and let $\tilde{\nabla}$ and ∇ be the covariant differentiations on M and N, respectively. Then the second fundamental form σ of the immersion is defined by the equation $\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$, where X and Y are vector fields tangent to N, and where σ is a normal-bundle-valued symmetric 2-form on N. For a vector field ξ normal to N, we write

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

where $-A_\xi X$ (respectively, $D_X \xi$) denotes the tangential component (respectively, the normal component) of $\tilde{\nabla}_X \xi$. A normal vector field ξ is said to be *parallel* if $D_X \xi = 0$ for each vector field X tangent to N. The submanifold N is said to be *totally umbilical* if $\sigma(X, Y) = g(X, Y)H$, for all vector fields X and Y tangent to N, where $H = (1/n)\operatorname{trace}\sigma$ is the mean-curvature vector of N in M. In particular, if the second fundamental form σ vanishes identically, N is called a *totally geodesic* submanifold of M. The submanifold N is called a *holomorphic submanifold* (respectively, a *totally real submanifold*) of M if each tangent space of N is mapped into itself (respectively, into the normal space) by the complex structure J.

A Kaehler manifold of constant holomorphic sectional curvature is called a *complex-space-form*, and a Riemannian manifold of constant sectional curvature is called a *real-space-form*.

In his book on Riemannian geometry, É. Cartan [1, p. 231] proved that an n-dimensional, totally umbilical submanifold of a euclidean m-space is either an n-plane or an n-sphere (for more general cases, see [2, p. 50] for example). In Section 3, we shall prove the following result.

THEOREM 1. *Let* N *be an* n*-dimensional, totally umbilical submanifold* $(n \geq 2)$ *of a* 2m*-dimensional complex-space-form* M *of holomorphic sectional curvature* $c \neq 0$. *Then* N *is one of the following submanifolds*:

(a) *a complex-space-form immersed holomorphically in* M *as a totally geodesic submanifold, or*

(b) *a real-space-form immersed in* M *as a totally real and totally geodesic submanifold, or*

(c) *a real-space-form immersed in* M *as a totally real submanifold with nonzero parallel mean-curvature vector.*

Case (b) *occurs only when* $m \geq n$, *and case* (c) *occurs only when* $m > n$.

Received February 11, 1974.
The authors acknowledge partial support from the National Science Foundation and the Matsunaga Science Foundation, respectively.

Michigan Math. J. 21 (1974).

By an r-*plane* we mean an r-dimensional linear subspace of a tangent space. A 2-plane ϕ is called *holomorphic* (respectively, *antiholomorphic*) if $J\phi = \phi$ (respectively, if $J\phi$ is perpendicular to ϕ). A 3-plane is called *coholomorphic* if it contains a holomorphic 2-plane. It is clear that a coholomorphic 3-plane also contains an antiholomorphic 2-plane.

A Kaehler manifold M is said to satisfy the *axiom of holomorphic* 2-*planes* (respectively, the *axiom of antiholomorphic* 2-*planes*) if for each $x \in M$ and each holomorphic (respectively, each antiholomorphic) 2-plane ϕ, there exists a 2-dimensional, totally geodesic submanifold N such that $x \in N$ and $T_x(N) = \phi$.

K. Yano and I. Mogi [7] (respectively, B.-Y. Chen and K. Ogiue [3] and K. Nomizu [5]) proved that a Kaehler manifold with the axiom of holomorphic 2-planes (respectively, the axiom of antiholomorphic 2-planes) is a complex-space-form.

We now propose a new axiom.

Axiom of coholomorphic 3-*spheres.* For each point $x \in M$ and each coholomorphic 3-plane π, there exists a 3-dimensional, totally umbilical submanifold N such that $x \in N$ and $T_x(N) = \pi$.

In Section 4, we shall prove the following theorem.

THEOREM 2. *Let* M *be a* 2m-*dimensional Kaehler manifold* $(m \geq 3)$. *If* M *satisfies the axiom of coholomorphic* 3-*spheres, then* M *is flat.*

2. BASIC FORMULAS

Let N be a submanifold of a Kaehler manifold M with complex structure J and Riemannian metric g, and let R, R', and $R^{\perp}$ be the curvature tensors associated with $\tilde{\nabla}$, ∇, and D, respectively. Then

$$R(JX, JY) = R(X, Y), \tag{2.1}$$

$$R(X, Y)JZ = JR(X, Y)Z. \tag{2.2}$$

Let K(X, Y) be the sectional curvature of M determined by orthonormal vectors X and Y. Then

$$K(JX, JY) = K(X, Y), \tag{2.3}$$

$$K(X, JY) = K(JX, Y). \tag{2.4}$$

It is easy to see that

(2.5) *orthonormal vectors* X *and* Y *span an antiholomorphic* 2-*plane if and only if* X, Y, *and* JX *are orthonormal.*

By H(X) we denote the holomorphic sectional curvature determined by X; that is, H(X) = K(X, JX).

For the second fundamental form σ of N in M we define the covariant derivative, denoted by $\overline{\nabla}_X \sigma$, to be

$$(\overline{\nabla}_X \sigma)(Y, Z) = D_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z). \tag{2.6}$$

Then, for all vector fields X, Y, Z, W tangent to N and all vector fields ξ and η normal to N, the equations of Gauss, Codazzi, and Ricci take the forms

$$g(R(X, Y)Z, W) = g(R'(X, Y)Z, W) + g(\sigma(X, Z), \sigma(Y, W)) - g(\sigma(X, W), \sigma(Y, Z)), \tag{2.7}$$

$$(R(X, Y)Z)^{\perp} = (\overline{\nabla}_X \sigma)(Y, Z) - (\overline{\nabla}_Y \sigma)(X, Z), \tag{2.8}$$

$$g(R(X, Y)\xi, \eta) = g(R^{\perp}(X, Y)\xi, \eta) - g([A_\xi, A_\eta](X), Y), \tag{2.9}$$

where $^{\perp}$ in (2.8) denotes the normal component.

The Kaehler manifold M is of constant holomorphic sectional curvature c if and only if

$$R(X, Y)Z = \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY + 2g(X, JY)JZ\}. \tag{2.10}$$

3. PROOF OF THEOREM 1

Let N be a totally umbilical submanifold of a complex-space-form M of constant holomorphic sectional curvature $c \neq 0$. Then

$$\sigma(X, Y) = g(X, Y) \cdot H \quad \text{or} \quad A_\xi = \frac{\operatorname{tr} A_\xi}{n} I, \tag{3.1}$$

where X and Y are vector fields tangent to N. By (2.6), we see that

$$(\overline{\nabla}_X \sigma)(Y, Z) = g(Y, Z) \cdot D_X H;$$

therefore equation (2.8) reduces to

$$(R(X, Y)Z)^{\perp} = g(Y, Z) \cdot D_X H - g(X, Z) \cdot D_Y H. \tag{3.2}$$

If $\dim N \geq 3$, then for each vector field X tangent to N we can choose a unit vector field Y tangent to N that is orthogonal to X and JX. For such a choice, it follows from (3.2) that

$$(R(X, Y)Y)^{\perp} = D_X H.$$

On the other hand, (2.10) implies $(R(X, Y)Y)^{\perp} = 0$, so that $D_X H = 0$ for each vector field X tangent to N. Therefore, by (3.2), we see that

$$(R(X, Y)Z)^{\perp} = 0 \tag{3.3}$$

for all vector fields X, Y, and Z tangent to N.

If $\dim N = 2$, put $N = N_1 \cup N_2$, where

$$N_1 = \{x \in N \mid JT_x(N) = T_x(N)\} \quad \text{and} \quad N_2 = \{x \in N \mid JT_x(N) \neq T_x(N)\}.$$

We can see that N_2 is an open submanifold of N, where the preceding argument is available, so that (3.3) holds. Let N_1' be the set of all interior points of N_1. Then N_1' is a complex analytic submanifold of M, so that $H \equiv 0$ on N_1', and hence (3.3) holds on N_1'. Since (3.3) is a tensorial equation, it holds on N.

Therefore (3.3) holds for N with $\dim N \geq 2$. This implies that N is an invariant submanifold of M (for the definition, see [6]). Thus, by Proposition 3.1 of Chen and Ogiue [4], we see that N is either a holomorphic or a totally real submanifold of M. It is well-known that if N is a holomorphic submanifold, then the mean-curvature vector H vanishes; that is, N is minimal, so that N is totally geodesic in M. Therefore, by equation (2.7) of Gauss, N is a complex-space-form of constant holomorphic sectional curvature c.

If N is a totally real submanifold, then it follows from (2.7) and (3.1) that N is a real-space-form of constant sectional curvature $c/4 + g(H, H)$. Since N is totally real, $m \geq n$. Moreover, if $H \neq 0$, then $m > n$. In fact, since $DH = 0$, it follows from (2.9) and (3.1) that

$$g(R(X, Y)H, JY) = 0$$

for all vector fields X and Y tangent to N. Hence, by (2.10) and the total reality of N, we see that

$$g(JY, H)\, g(JX, JY) = g(JX, H)\, g(JY, JY).$$

Choose Y in such a way that $g(JY, H) = 0$. Then

$$g(JX, H)\, g(Y, Y) = 0$$

for all vector fields X tangent to N; this implies that $g(JX, H) = 0$. From this we see that H is perpendicular to the n-dimensional normal subspace $JT_x(N)$, for each $x \in N$. Hence $m > n$. This completes the proof of Theorem 1.

Remark. A totally umbilical hypersurface N of a real projective n-space $RP^n(c/4)$ can be imbedded in a complex projective 2n-space $CP^{2n}(c)$ as a totally real and totally umbilical submanifold. In particular, if N is not totally geodesic in $RP^n(c/4)$, then N is not totally geodesic in $CP^{2n}(c)$.

4. PROOF OF THEOREM 2

Let x be any point in M, and let X and Y be any two orthonormal vectors in $T_x(M)$ such that X and Y span an antiholomorphic 2-plane. Then X, Y, and JX span a coholomorphic 3-plane π that is perpendicular to JY. By the axiom of coholomorphic 3-spheres, there exists a 3-dimensional totally umbilical submanifold N such that $x \in N$ and $T_x(N) = \pi$. By the argument in the proof of Theorem 1, we obtain the equations

$$(R(X, JX)Y)^{\perp} = g(JX, Y)\, D_X H - g(X, Y)\, D_{JX} H = 0.$$

Therefore

$$g(R(X, JX)Y, JY) = 0 \tag{4.1}$$

for all orthonormal vectors X and Y that span an antiholomorphic 2-plane. For such vectors X and Y, it is easily seen from (2.5) that $\frac{X+Y}{\sqrt{2}}$ and $\frac{JX - JY}{\sqrt{2}}$ also span an antiholomorphic 2-plane. Therefore, using (2.1), (2.2), (2.6), and (4.1), we obtain the relation

$$(4.2)\qquad \begin{aligned} K(X+Y, JX-JY) &= g\left(R\left(\frac{X+Y}{\sqrt{2}}, \frac{JX-JY}{\sqrt{2}}\right)\frac{JX-JY}{\sqrt{2}}, \frac{X+Y}{\sqrt{2}}\right) \\ &= \frac{1}{4}\{H(X)+H(Y)\}\,. \end{aligned}$$

On the other hand, (4.1) implies that

$$K(X, Y) + K(X, JY) = 0;$$

from this we obtain the relation

$$K(X+Y, JX-JY) = -K(X+Y, X-Y) = -K(X, Y)\,.$$

Together with (4.2), this implies that

$$\frac{1}{4}\{H(X)+H(Y)\} = -K(X, Y) = K(X, JY) = -\frac{1}{4}\{H(X)+H(JY)\}\,,$$

so that $H(X) + H(Y) = 0$. Since $m \geq 3$, it follows that $H(X) = 0$; this completes the proof of Theorem 2.

REFERENCES

1. É. Cartan, *Leçons sur la géométrie des espaces de Riemann.* Second ed. Gauthier-Villars, Paris, 1946.

2. B.-Y. Chen, *Geometry of submanifolds.* M. Dekker, New York, 1973.

3. B.-Y. Chen and K. Ogiue, *Some characterizations of complex space forms.* Duke Math. J. 40 (1973), 797-799.

4. ———, *On totally real submanifolds.* Trans. Amer. Math. Soc. 193 (1974), 257-266.

5. K. Nomizu, *Conditions for constancy of the holomorphic sectional curvature.* J. Differential Geometry 8 (1973), 335-339.

6. K. Ogiue, *On invariant immersions.* Ann. Mat. Pura Appl. (4) 80 (1968), 387-397.

7. K. Yano and I. Mogi, *On real representations of Kaehlerian manifolds.* Ann. of Math. (2) 61 (1955), 170-189.

Michigan State University
East Lansing, Michigan 48823