ON NORMAL AND AUTOMORPHIC FUNCTIONS

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1. INTRODUCTION

Let Γ be a Fuchsian group, that is, a discontinuous group of Moebius transformations of the unit disk $D = \{|z| < 1\}$ onto itself. The points $z, z' \in D$ are called *equivalent* if there exists a mapping $\phi \in \Gamma$ such that $z' = \phi(z)$. A domain $F \subset D$ is called a *fundamental domain* of Γ if it does not contain two equivalent points and if every point in D is equivalent to some point in $\overline{F}$.

The function $f(z)$ will be called *character-automorphic* (with respect to Γ) if it is meromorphic in D and if

$$(1.1)\qquad f(\phi(z)) = v(\phi) f(z), \quad \text{where } |v(\phi)| = 1 \quad (z \in D,\ \phi \in \Gamma).$$

It follows from (1.1) that $v(\phi \circ \psi) = v(\phi) v(\psi)$ for $\phi, \psi \in \Gamma$, so that v is a character of Γ, and (1.1) is equivalent to $|f \circ \phi| = |f|$ $(\phi \in \Gamma)$. If $v(\phi) = 1$ for all $\phi \in \Gamma$, then $f(z)$ is *automorphic.*

We use the notation

$$(1.2)\qquad f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

for the spherical derivative. It is invariant under spherical rotations. The meromorphic function $f(z)$ $(z \in D)$ is called *normal* [8] if

$$(1.3)\qquad \sup_{z \in D} (1 - |z|^2) f^{\#}(z) = M < \infty .$$

This quantity is invariant under Moebius transformations $\phi(z)$ of D onto D. For character-automorphic functions, the supremum can therefore be restricted to any fundamental domain F. In particular, every bounded analytic function is normal. If $f(z)$ is analytic and normal, then [5], [13]

$$\log^+|f(z)| \le 2(\log^+|f(0)| + M)(1 - |z|)^{-1} \quad (z \in D).$$

We denote the non-Euclidean distance by $d(z_1, z_2)$ $(z_1, z_2 \in D)$ and the spherical distance by

$$(1.4)\qquad d^*(w_1, w_2) = \arctan\left|\frac{w_1 - w_2}{1 + \bar{w}_1 w_2}\right| \quad (w_1, w_2 \in \hat{\mathbb{C}}).$$

If $f(z)$ is normal, then it follows from (1.3) by integration that

$$(1.5)\qquad d^*(f(z_1), f(z_2)) \le M\, d(z_1, z_2) \quad (z_1, z_2 \in D).$$

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We derive first a simple normality criterion for automorphic functions. Then we turn to character-automorphic functions and study the normality of a function connected with the Green's function; an approximation process allows us to deal with groups of divergence type. We shall see that for every Fuchsian group there ~~exists~~ a nonconstant normal character-automorphic function; it remains an open problem whether there is always a nonconstant normal automorphic function. In the last section, we show that every normal character-automorphic function has an angular limit at every parabolic fixed point.

2. A NORMALITY CRITERION

Let Γ be a Fuchsian group with fundamental domain F. Our first result is a slight extension of a normality criterion of P. Montel [10].

LEMMA 1. *If a meromorphic function assumes three values in* $\hat{\mathbf{C}}$ *in at most finitely many nonequivalent points, then it is normal.*

Proof. Let A be the set where the meromorphic function f(z) assumes the values a, b, c. Since A contains only finitely many equivalence classes, it easily follows from the discontinuity of Γ that there exists $\delta > 0$ such that

$$d(z, z') \geq 2\delta \qquad \text{for } z \in A,\ z' \in A,\ z \neq z'.$$

Hence every non-Euclidean disk

$$\{z \in D: d(z, \zeta) < \delta\} \qquad (\zeta \in D)$$

contains at most one point of A, and therefore, for every $\zeta \in D$, the function

$$g_\zeta(s) = f\left(\frac{\zeta + \delta s}{1 + \bar{\zeta}\delta s}\right) \qquad (|s| < 1)$$

assumes at most one of the values a, b, c. Hence it follows from a criterion of Montel [10] [4, p. 60] that $\{g_\zeta: \zeta \in D\}$ is a normal family, and Marty's criterion [1, p. 218] shows that

$$\delta(1 - |\zeta|^2) f^{\#}(\zeta) = g_\zeta^{\#}(0) \leq K \qquad (\zeta \in D)$$

for some constant K. Therefore the condition (1.3) for normality is satisfied.

P. Lappan [7] has shown that an automorphic function is normal if it assumes every value only finitely often in $\overline{F}$. Since an automorphic function assumes the same value at all equivalent points, we obtain the following generalization from Lemma 1:

THEOREM 1. *If an automorphic function assumes three values only finitely often in* $\overline{F}$, *then it is normal.*

A function automorphic with respect to Γ can be considered as a meromorphic function on the Riemann surface D/Γ. The hypothesis of Theorem 1 therefore means that this function assumes three values only finitely often on D/Γ. The next corollary follows at once from Theorem 1. Its hypothesis means that this function maps D/Γ onto a surface of finite spherical area.

COROLLARY 1. *If* f(z) *is automorphic and*

$$\iint_{\overline{F}} f^{\#}(z)^2 \, dx\, dy < \infty, \tag{2.1}$$

then f(z) *is normal.*

If Γ is finitely generated, then every simple automorphic function (in the sense of L. R. Ford [3, p. 87]) is normal because it assumes every value only finitely often in $\overline{F}$. More generally there exists a nonconstant, normal automorphic function if Γ is of finite genus (see, for instance, M. Mori [11]). On the other hand, there exist groups for which the integral (2.1) is infinite for all automorphic functions [15, p. 389]. (I want to thank E. Röding for these two references.) It remains an open problem whether, for every Fuchsian group, there is a nonconstant normal automorphic function.

3. NORMALITY AND THE GREEN'S FUNCTION

Let Γ be a Fuchsian group with identity ι for which 0 is not an elliptic fixed point. The group Γ is said to be of *convergence type* if

$$\sum_{\phi \in \Gamma} (1 - |\phi(z)|) < \infty \qquad (z \in D),$$

otherwise of *divergence type*. It is of convergence type if and only if the Riemann surface D/Γ has a Green's function [12][17, p. 522]. If Γ is of the second kind, that is, if the set of limit points of Γ is nowhere dense on ∂D, then Γ is of convergence type [3, p. 106].

THEOREM 2. *Let* $\{|z| < \rho\}$ *contain no* Γ*-equivalent points. If* $\Phi \subset \Gamma$ *is a subgroup of convergence type, then the function*

$$f(z) = z \prod_{\substack{\phi \in \Phi \\ \phi \neq \iota}} \frac{\phi(z)}{\phi(0)} = z + \cdots \qquad (z \in D) \tag{3.1}$$

is character-automorphic and satisfies the inequality

$$(1 - |z|^2) f^{\#}(z) < 40/\rho \qquad (z \in D). \tag{3.2}$$

If we consider the harmonic function

$$\log \frac{1}{|f(z)|} - \sum_{\substack{\phi \in \Phi \\ \phi \neq \iota}} \log |\phi(0)| = \sum_{\phi \in \Phi} \log \frac{1}{|\phi(z)|} \tag{3.3}$$

and identify Φ-equivalent points, we obtain the Green's function of D/Φ. We postpone the proof of Theorem 2 to the next section and derive first some consequences.

THEOREM 3. *Let* Γ *be infinitely generated. Then* Γ *is the union of an increasing sequence of finitely generated subgroups* Γ_n *such that the functions*

$$f_n(z) = z \prod_{\substack{\phi \in \Gamma_n \\ \phi \neq \iota}} \frac{\phi(z)}{\phi(0)} = z + \cdots \qquad (z \in D) \tag{3.4}$$

converge locally uniformly in D *to a normal function that is character-automorphic with respect to* Γ, *and*

$$\lim_{n \to \infty} \sum_{\phi \in \Gamma_n} (1 - |\phi(0)|^2)\phi(0)^k \tag{3.5}$$

exists for $k = 1, 2, \cdots$.

The only interesting case is the case where Γ is of divergence type. Here (3.5) means that the series

$$\sum_{\phi \in \Gamma} (1 - |\phi(0)|^2)\phi(0)^k \qquad (k = 1, 2, \cdots) \tag{3.6}$$

converges in a very weak sense, namely if we restrict ourselves to suitable partial sums; of course, (3.6) does not converge absolutely.

Proof. Since Γ is countable, we can write Γ as the union of an increasing sequence of finitely generated subgroups Γ_n. Now the normal fundamental domain (with respect to 0) has finite non-Euclidean area if and only if the group is finitely generated and of the first kind [16, p. 39]. Since the normal fundamental domain F_n of Γ_n contains that of Γ, it follows that F_n likewise has infinite non-Euclidean area. Hence Γ_n is of the second kind and therefore of convergence type.

Thus we can apply Theorem 2 with $\Phi = \Gamma_n$. Since 0 is not an elliptic fixed point, some disk $\{|z| < \rho\}$ contains no Γ-equivalent points, and we deduce from (3.2) that

$$(1 - |z|^2) f_n^{\#}(z) < 40/\rho \qquad (z \in D,\ n = 1, 2, \cdots). \tag{3.7}$$

By Marty's criterion [1, p. 218], we can therefore find $\{n_\nu\}$ such that

$$f_{n_\nu}(z) \to f(z) \quad \text{as } \nu \to \infty, \text{ locally uniformly in } D. \tag{3.8}$$

The limit function f(z) is character-automorphic with respect to each Γ_n and therefore with respect to the union Γ. It is normal, because it follows from the condition $f'(0) = 1$ and from (3.7) that

$$1 \leq \sup_{z \in D} (1 - |z|^2) f^{\#}(z) \leq 40/\rho\ . \tag{3.9}$$

We change the notation and write Γ_n and f_n instead of Γ_{n_ν} and f_{n_ν}. By logarithmic differentiation, we deduce from (3.4) and (3.8) that

$$z \frac{f_n'(z)}{f_n(z)} = \sum_{\phi \in \Gamma_n} z \frac{\phi'(z)}{\phi(z)} \to z \frac{f'(z)}{f(z)} \qquad (n \to \infty) \tag{3.10}$$

uniformly near $z = 0$. Writing

$$\phi(z) = c\,\frac{a - z}{1 - \bar{a}z} \qquad (|c| = 1,\ |a| < 1),$$

we see that, near $z = 0$,

$$z\,\frac{f_n'(z)}{f_n(z)} = 1 + \sum_{k=1}^{\infty}\left[\sum_{\substack{\phi\in\Gamma_n\\ \phi\neq\iota}}\left(\bar{a}^k - \frac{1}{a^k}\right)\right]z^k.$$

By (3.10), each coefficient of the power series on the right tends to a limit as $n \to \infty$. Since

$$\sum_{\phi\in\Gamma} (1 - |\phi(0)|)^2 < \infty \qquad \text{and} \qquad a = \phi^{-1}(0)$$

(see [3, p. 104]), we easily deduce that

$$\lim_{n\to\infty}\ \sum_{\phi\in\Gamma_n} (1 - |a|^2)\bar{a}^k \qquad (k = 1, 2, \cdots)$$

exists. Since $a = \phi^{-1}(0)$ and since Γ_n is a group, it follows that the limit (3.5) exists.

COROLLARY 2. *For every infinitely generated Fuchsian group, there exists a character-automorphic function* $g(z)$ *with*

$$1 \leq \sup_{z\in D}\ (1 - |z|^2)\,g^{\#}(z) \leq K_0 < \infty, \tag{3.11}$$

where K_0 *is an absolute constant.*

Proof. A. Marden [9] has shown that there exists an absolute constant $\rho_0 > 0$ such that, for every Fuchsian group Γ and a suitable Moebius transformation ψ of D onto D, the disk $\{|z| < \rho_0\}$ contains no points equivalent under the conjugate group $\Gamma^* = \psi\circ\Gamma\circ\psi^{-1}$. We consider again the function $f(z)$ constructed in the proof of Theorem 3, with Γ replaced by Γ^*. It is character-automorphic with respect to Γ^* and satisfies (3.9) with $\rho = \rho_0$. The function

$$g(z) = f(\psi(z)) \qquad (z \in D)$$

is character-automorphic with respect to $\Gamma = \psi^{-1}\circ\Gamma^*\circ\psi$ and satisfies (3.11) with $K_0 = 40/\rho_0$, because this supremum is invariant under ψ.

Our method of proof forces us to exclude the case where Γ is finitely generated and of the first kind, because there Γ cannot be written as the union of finitely generated subgroups. On the other hand, we have seen in Section 2 that for every finitely generated group there exist normal automorphic functions. We do not know, however, whether $\sup(1 - |z|^2)\,f^{\#}(z)$ is bounded by an absolute constant for these functions.

In the case where Γ is of divergence type and the normal fundamental domain F is not relatively compact, we obtain another example of a normal character-automorphic function if we complete the Evans-Selberg function of D/Γ [15, p. 352] to an analytic function and then exponentiate. The resulting function satisfies the condition

$$|f(z)| \to \infty \quad \text{as } |z| \to 1 \ (z \in F).$$

4. PROOF OF THEOREM 2

We need some lemmas that are already known. A function $h(\zeta)$ analytic in D is called *(circumferentially) mean univalent* if

$$\int_{|w|=R} \nu(w)\,|dw| \leq 2\pi R \qquad (0 < R < +\infty), \tag{4.1}$$

where $\nu(w)$ $(w \in \mathbf{C})$ denotes the number of zeros of $h(\zeta) - w$ in D. W. K. Hayman [6, p. 99] has proved the following.

LEMMA 2. *If* $h(\zeta) = \zeta + \cdots$ *is mean univalent in* D, *then*

$$\frac{|\zeta|}{(1+|\zeta|)^2} \leq |h(\zeta)| \leq \frac{|\zeta|}{(1-|\zeta|)^2} \qquad (\zeta \in D), \tag{4.2}$$

$$|h'(\zeta)| \leq \frac{1+|\zeta|}{(1-|\zeta|)^3} \qquad (\zeta \in D). \tag{4.3}$$

The next lemma follows easily by elementary transformations from the lemma in [13, p. 8].

LEMMA 3. *Let* $f(z)$ *be analytic in* D. *Let there exist a set* $G \subset D$ *such that*

$$|f(z)| \geq \alpha \quad (z \in G), \qquad (1-|z|^2)\,|f'(z)| \leq \beta \quad (z \in D \setminus G) \tag{4.4}$$

for some constants α *and* β $(0 < \alpha \leq 1/2,\ \beta > 0)$. *Then*

$$(1-|z|^2) f^{\#}(z) \leq \frac{\beta+2}{2\alpha} \qquad (z \in D). \tag{4.5}$$

Proof of Theorem 2. The product (3.1) converges, because Φ is of convergence type and because each factor has the value 1 when $z = 0$. It is character-automorphic because, for $\psi \in \Phi$,

$$|f(\psi(z))| = |\psi(z)| \prod_{\substack{\phi\in\Phi \\ \phi\neq\iota}} \frac{|\phi\circ\psi(z)|}{|\phi(0)|} = \frac{\prod_{\chi\in\Phi} |\chi(z)|}{\prod_{\substack{\phi\in\Phi \\ \phi\neq\iota}} |\phi(0)|} = |f(z)|.$$

It remains to prove the estimate (3.2). We may restrict ourselves to the case where Φ is finitely generated; the general case follows if we write Φ as the union of such groups, as in the proof of Theorem 3.

The function (3.3) becomes the Green's function of D/Φ, if we identify Φ-equivalent points. Hence we obtain from results of M. Brelot and G. Choquet [2] [15,

p. 197-202] the existence of a fundamental domain G of Φ (the Green's star domain) that is mapped by f(z) one-to-one onto a disk of center 0 from which a finite number of radial slits has been removed.

By hypothesis, the disk $\{|z| < \rho\}$ contains no Γ-equivalent points and therefore no Φ-equivalent points. For $0 < R < \infty$, every point in $\{|z| < \rho, \ |f(z)| = R\}$ (with a finite number of exceptions) is equivalent to exactly one point $z^* \in \overline{G}$, and $|f(z^*)| = |f(z)| = R$. If $\nu(w)$ and $\nu^*(w)$ denote the number of zeros of $f(z) - w$ in $\{|z| < \rho\}$ and G, respectively, we conclude that

$$(4.6) \qquad \int_{|w|=R} \nu(w)\,|dw| = \int_{|w|=R} \nu^*(w)\,|dw| \le 2\pi R;$$

because f(z) is univalent in G, we have the inequality $\nu^*(w) \le 1$. Hence, by (4.1), the function

$$h(\zeta) = \rho^{-1} f(\rho\zeta) = \zeta + \cdots \qquad (|\zeta| < 1)$$

is mean univalent, and we deduce from Lemma 2 that

$$(4.7) \qquad |f(z)| \ge \frac{|z|}{4}, \qquad |f'(z)| \le \frac{1 + \rho^{-1}|z|}{(1 - \rho^{-1}|z|)^3} \qquad (|z| < \rho).$$

Let $B = \{|z| \le \rho/3\}$. We denote the elements of Φ by ϕ_k $(k = 0, 1, \cdots)$, with $\phi_0 = \iota$. For $m = 1, 2, \cdots$, let

$$G_m = D \setminus \bigcup_{k=1}^{m} \phi_k(B), \qquad G = D \setminus \bigcup_{k=1}^{\infty} \phi_k(B).$$

We see from (3.1) that, for $z \in D \cap \partial G_m$,

$$\prod_{k=0}^{m} |\phi_k(z)| \ge |f(z)| \prod_{k=1}^{\infty} |\phi_k(0)| \ge \frac{\rho}{12} \prod_{k=1}^{\infty} |\phi_k(0)|$$

because, by (4.7), $|f(\phi(z))| = |f(z)| \ge \rho/12$ for $z \in \partial B$, $\phi \in \Phi$. Since the product $\prod_{k=0}^{m} |\phi_k(z)|$ has the value 1 for $z \in \partial D$ and is different from 0 for $z \in G_m$, we conclude from the minimum principle that

$$\left| \prod_{k=0}^{m} \phi_k(z) \right| \ge \frac{\rho}{12} \prod_{k=1}^{\infty} |\phi_k(0)| \qquad \text{for } z \in G_m,$$

hence for $z \in G$. Letting $m \to \infty$, we deduce that

$$(4.8) \qquad |f(z)| \ge \rho/12 \qquad (z \in G).$$

Since $|\phi' f'(\phi)| = |f'(z)|$ for $\phi \in \Phi$, we see by (4.7) that

$$(1 - |\phi(z)|^2)\,|f'(\phi(z))| = (1 - |z|^2)\,|f'(z)| \le 9/2 \qquad (z \in B),$$

and hence that

$$(4.9) \qquad (1 - |z|^2)\,|f'(z)| \leq 9/2 \qquad \text{for } z \in \bigcup_{\phi \in \Gamma} \phi(B) = D \setminus G.$$

We obtain inequality (3.2) from Lemma 3 by (4.8) and (4.9).

5. THE BEHAVIOR AT PARABOLIC FIXED POINTS

A function $f(z)$ $(z \in D)$ is said to have the *angular limit* $a \in \hat{\mathbf{C}}$ at $\zeta \in \partial D$ if

$$f(z) \to a \qquad \text{as } z \to \zeta, \ |\arg(1 - \bar{\zeta}z)| < \pi/2 - \delta$$

for every $\delta > 0$. A normal function $f(z)$ has the following important property: If $f(z) \to a$ as z tends to the point $\zeta \in \partial D$ along some Jordan arc in D, then $f(z)$ has the angular limit a at ζ (see [8]); in particular, existence of the radial limit implies existence of the angular limit.

Let Γ be a Fuchsian group. Then

$$S = (D \cup \{\text{parabolic fixed points}\})/\Gamma$$

is a Riemann surface, and we obtain the Riemann surface D/Γ from S by removing certain isolated points. Thus the parabolic elements of Γ give rise to punctures. If an automorphic function has an angular limit at all parabolic fixed points, then it becomes a meromorphic function on S.

We prove now that a normal character-automorphic function has an angular limit at all parabolic fixed points. More generally, we assert the following.

THEOREM 4. *Let* $f(z)$ *be normal in* D. *Let* ϕ *be a parabolic Moebius transformation with* $\phi(D) = D$ *and fixed point* ζ, *and let*

$$(5.1) \qquad f(\phi(z)) = \psi(f(z)) \qquad (z \in D)$$

for some Moebius transformation ψ. *Then* $f(z)$ *has an angular limit at* ζ. *Furthermore, if* $\psi(w) \not\equiv w$, *then this angular limit is a fixed point of* ψ.

Proof. We may assume that $\zeta = 1$. Let A be the radial cluster set at 1, so that $f(z)$ has the radial (and therefore angular) limit a at 1 if and only if $A = \{a\}$. Let now $a \in A$. Then there exists a sequence $\{x_k\}$ such that $x_k \to 1 - 0$ and $f(x_k) \to a$ as $k \to \infty$.

(a) We consider first the case where $\psi(w) \not\equiv w$. Since ϕ is parabolic, we easily see that the non-Euclidean distance satisfies the condition $d(\phi(x_k), x_k) \to 0$ as $k \to \infty$. Hence, by (5.1) and (1.5), the spherical distance satisfies the condition

$$d^*(\psi(f(x_k)), f(x_k)) = d^*(f(\phi(x_k)), f(x_k)) \leq M\, d(\phi(x_k), x_k) \to 0.$$

It follows that $d^*(\psi(a), a) = 0$ and therefore that $\psi(a) = a$. Thus a is a fixed point of ψ. Hence A contains at most two points, and since A is connected, we conclude that $A = \{a\}$.

(b) We consider now the case where $\psi(w) \equiv w$. We can make $a = 0$ by a spherical rotation. The circle

$$C_k = \{|z - (1 + x_k)/2| = (1 - x_k)/2\} \qquad (k = 1, 2, \cdots)$$

through x_k is tangent to ∂D at 1, and $\phi(C_k) = C_k$. For every $z \in C_k$, there exists $\nu \in \mathbb{Z}$ such that $d(z, \phi^\nu(x_k)) < d(x_k, \phi(x_k))$. Since $f(\phi^\nu(x_k)) = f(x_k) \to 0$ $(k \to \infty)$, we deduce from (1.5) that

$$d^*(f(z), 0) \leq d^*(f(z), f(\phi^\nu(x_k))) + d^*(f(\phi^\nu(x_k)), 0) \to 0$$

uniformly for $z \in C_k$ $(k \to \infty)$ and therefore that

$$\sup_{z \in C_k} |f(z)| \to 0 \qquad (k \to \infty). \tag{5.2}$$

Since the segment $(x_k, 1)$ is a diameter of C_k, it follows from (5.2) and the maximum principle of O. Lehto and K. I. Virtanen [8] [14, Theorem 9.1] for normal functions that

$$\sup_{x_k < x < 1} |f(x)| \to 0 \qquad (k \to \infty).$$

Thus $f(z)$ has the radial limit $a = 0$ at 1.

COROLLARY 3. *Let Γ be finitely generated and of the first kind. A character-automorphic function is normal if and only if it has an angular limit at each parabolic fixed point.*

Proof. In one direction this is a special case of Theorem 4. Conversely, let the character-automorphic function $f(z)$ have an angular limit $a_\nu \in \hat{\mathbf{C}}$ at each of the finitely many parabolic vertices ζ_ν $(\nu = 1, \cdots, n)$ of the normal fundamental domain F. Then

$$(1 - |z|^2) f^\#(z) \to 0 \quad \text{as } z \to \zeta_\nu \ (z \in F). \tag{5.3}$$

Hence $f(z)$ is normal, because $\overline{F} \cap \partial D = \{\zeta_1, \cdots, \zeta_n\}$ and

$$\sup_{z \in D} (1 - |z|^2) f^\#(z) = \sup_{z \in F} (1 - |z|^2) f^\#(z) < \infty.$$

We give the standard proof of (5.3). We may assume that $a_\nu \neq \infty$. If $z_k \to \zeta_\nu$ and $z_k \in F$, then, as $k \to \infty$,

$$f_k(z) \equiv f(z_k + (1/2)(\zeta_\nu - z_k)s) \to a_\nu \qquad \text{uniformly in } |s| < 1.$$

It follows that

$$(1 - |z_k|)\,|f'(z_k)| \leq |(\zeta_\nu - z_k) f'(z_k)| = 2\,|f_k'(0)| \to 0.$$

REFERENCES

1. L. V. Ahlfors, *Complex analysis: An introduction to the theory of analytic functions of one complex variable.* Second Edition. McGraw-Hill, New York, 1966.
2. M. Brelot and G. Choquet, *Espaces et lignes de Green.* Ann. Inst. Fourier, Grenoble 3 (1951), 199-263.
3. L. R. Ford, *Automorphic functions.* Second Edition. Chelsea, New York, 1951.

4. G. M. Golusin, *Geometrische Funktionentheorie.* Deutscher Verlag Wiss., Berlin, 1957.

5. W. K. Hayman, *Uniformly normal families.* Lectures on functions of a complex variable, University of Michigan Press, Ann Arbor, 1955, 199-212.

6. ———, *Multivalent functions.* Cambridge Tracts in Mathematics and Mathematical Physics, No. 48. Cambridge Univ. Press, Cambridge, 1958.

7. P. Lappan, *Some sequential properties of normal and non-normal functions with applications to automorphic functions.* Comment. Math. Univ. St. Paul. 12 (1964), 41-57.

8. O. Lehto and K. I. Virtanen, *Boundary behaviour and normal meromorphic functions.* Acta Math. 97 (1957), 47-65.

9. A. Marden, *Universal properties of Fuchsian groups in the Poincaré metric.* Discontinuous Groups and Riemann Surfaces. Proc. of the 1973 Conference at the University of Maryland; Annals of Mathematics Studies 79, Princeton, 1974; 315-339.

10. P. Montel, *Leçons sur les familles normales de fonctions analytiques et leurs applications.* Gauthier-Villars, Paris, 1927.

11. M. Mori, *Canonical conformal mappings of open Riemann surfaces.* J. Math. Kyoto Univ. 31 (1963/64), 169-192.

12. P. J. Myrberg, *Über die Existenz der Greenschen Funktionen auf einer gegebenen Riemannschen Fläche.* Acta Math. 61 (1933), 39-79.

13. Ch. Pommerenke, *Estimates for normal meromorphic functions.* Ann. Acad. Sci. Fenn. Ser. AI No. 376 (1970), 1-10.

14. ———, *Univalent functions.* Vandenhoeck u. Ruprecht, Göttingen, 1974.

15. L. Sario and M. Nakai, *Classification theory of Riemann surfaces.* Grundlehren, Vol. 164, Springer-Verlag, Berlin, 1970.

16. C. L. Siegel, *Topics in complex function theory.* Vol. 2, *Automorphic functions and abelian integrals.* Wiley-Interscience, New York, 1971.

17. M. Tsuji, *Potential theory in modern function theory.* Maruzen Comp., Tokyo, 1959.

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