A PROOF THAT SIMPLE-HOMOTOPY EQUIVALENT POLYHEDRA ARE STABLY HOMEOMORPHIC

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We give a new proof of the following result.

THEOREM. *If* $f\colon X \to Y$ *is a simple-homotopy equivalence between compact polyhedra, and if* Q *denotes the Hilbert cube, then* $f \times 1_Q\colon X \times Q \to Y \times Q$ *is homotopic to a homeomorphism of* $X \times Q$ *onto* $Y \times Q$.

Our proof is of interest because it assumes no facts from infinite-dimensional topology, because it uses the idea of *nearly-homeomorphic maps*, which may be of use in other contexts, and, above all, because it is based on a simple heuristic argument that explains why the result should be expected.

The theorem is a special case of a theorem of J. E. West [6], who achieved the same result under the assumption that X and Y are locally finite CW complexes. The importance of West's result was demonstrated by Chapman [3], who used it to prove its converse and thus to prove the topological invariance of Whitehead torsion. This has led to a spate of proofs of West's theorem. See [2], [5], and [7].

Here is the heuristic argument that explains why $X \times Q \approx Y \times Q$ if X and Y have the same simple-homotopy type (the symbol $\approx$ denotes homeomorphism).

Suppose $X \searrow_e Y$ (X collapses to Y by an elementary PL collapse). Then, by definition, $X = Y \cup Q^n$, where the pair $(Q^n, Y \cap Q^n)$ is homeomorphic to $(I^n, I^{n-1} \times 0)$. Thus, as was pointed out by P. Dierker [4], we may identify X with the subcomplex $(Y \times 0) \cup (Y \cap Q^n) \times I$ of $Y \times I$. Hence, if $X \searrow_e Y$, then $Y \times I \searrow X$. By induction, it follows trivially that if $X \searrow Y$, the collapse taking place in n_1 elementary steps, then $Y \times I^{n_1} \searrow X$. By the same reasoning, it follows that $X \times I^{m_1} \searrow Y \times I^{n_1}$ and then that $Y \times I^{n_1} \times I^{n_2} \searrow X \times I^{m_1}$, and so forth. In this way we get an inverse sequence

$$Y \swarrow X \swarrow Y \times I^{N_1} \swarrow X \times I^{M_1} \swarrow Y \times I^{N_2} \swarrow \cdots,$$

where $N_1 < N_2 < \cdots$ and $M_1 < M_2 < \cdots$. The inverse limit of this sequence is the same as that of the cofinal subsequence $\{Y \times I^{N_j}\}$, and (assuming that the bonding maps are the standard projections $Y \times I^{N_j} \to Y \times I^{N_{j-1}}$) the inverse limit of this subsequence is $Y \times Q$. Similarly, the inverse limit of the subsequence $\{X \times I^{M_j}\}$ is $X \times Q$. Hence $X \times Q \approx Y \times Q$.

There are two reasons why this argument is not rigorous.

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(1) The collapsing maps $X \times I^{M_{j-1}} \nearrow Y \times I^{N_j}$ and $Y \times I^{N_j} \nearrow X \times I^{M_j}$ have not been precisely defined.

(2) Even if the maps were defined, it would not be true that the composition $Y \times I^{N_j} \nearrow X \times I^{M_j} \nearrow Y \times I^{N_{j+1}}$ (or $X \times I^{M_{j-1}} \nearrow Y \times I^{N_j} \nearrow X \times I^{M_j}$) is the standard projection.

We resolve the first problem in Section 1 by defining the collapsing map $Y \times I^{N_j} \searrow X \times I^{M_{j-1}}$ to be the restriction of the natural projection $X \times I^{M_j} \to X \times I^{M_{j-1}}$. This requires the choice of an embedding of $Y \times I^{N_j}$ into $X \times I^{M_j}$, and some care must be taken to keep track of these embeddings. In Section 2, we resolve the second problem and give the proof that $X \times Q \approx Y \times Q$. Although the composition of collapsing maps does not give the standard projection, we show that the composition is nearly-homeomorphic to the standard projection, and this is good enough to yield the same inverse limit. In Section 3 we point out how our argument automatically yields the fact that simple-homotopy equivalences are stably homotopic to homeomorphisms.

1. THE SETTING

We denote by $I = [0, 1]$ the unit interval, by I^n the Cartesian product $I \times I \times \cdots \times I$ (n factors), and by Q the product of countably many intervals, that is, the Hilbert cube. A space is an n-*ball* if it is homeomorphic to I^n. The n-tuple $(0, 0, \cdots, 0)$ is denoted by 0_n. Unless it is otherwise stipulated, π_j or π_j' denotes the natural map $Z \times I^n \to Z \times I^j \times 0_{n-j}$, and π or π' denotes the natural map $Z \times I^n \to Z$.

By a *complex* we mean a finite CW complex such that the closure of each n-cell is an n-ball that is the underlying space of some subcomplex.

If X and Y are complexes, a *cellular map* $f: X \to Y$ is one in which the image of every subcomplex of X is a subcomplex of Y. (This is different from the usual definition of cellular map.)

If Y is a subcomplex of X, we say that there is an *elementary formal expansion* of Y to X - written $Y \nearrow^e X$ - if $X = Y \cup Q^n$, where Q^n is a closed n-cell of X, where $P^{n-1} \equiv Y \cap \partial Q^n$ is an $(n-1)$-ball in ∂Q^n (P^{n-1} is necessarily a subcomplex), and where $Q^{n-1} = \mathrm{Cl}(\partial Q^n - P^{n-1})$ is a closed $(n-1)$-cell of X. We say that there is a *formal expansion* $Y \nearrow X$ if there is a finite sequence of elementary formal expansions $Y = Y_0 \nearrow^e Y_1 \nearrow^e \cdots \nearrow^e Y_n = X$.

Each formal expansion $Y \nearrow X$ determines a decomposition

$$X = Y \bigcup_{P_1} Q_1 \bigcup_{P_2} Q_2 \bigcup \cdots \bigcup_{P_n} Q_n,$$

where

$$Y_0 = Y,$$

$$Y_j = Y \cup Q_1 \cup \cdots \cup Q_j \quad (j = 1, 2, \cdots),$$

$$P_{j+1} = Y_j \cap Q_{j+1}.$$

Give I^n and $Y \times I^n$ the product cell structures. We define an *expansion* corresponding to this formal expansion to be any cellular embedding β: $X \to Y \times I^n$ constructed inductively as follows:

a) Let $\beta_0 = 1_Y$.

b) Having defined a cellular embedding β_j: $Y_j \to Y \times I^j$, choose an embedding β_{j+1}: $Y_{j+1} \to Y \times I^{j+1}$ such that

$$\beta_{j+1} \mid Y_j = (\beta_j, 0) \quad \text{and} \quad \beta_{j+1}(Q_{j+1}) = \beta_j(P_{j+1}) \times I.$$

c) Let $\beta = \beta_n$.

Because β_j is cellular, $\beta_j(P_{j+1})$ is a subcomplex. Thus $\beta_{j+1}(Q_{j+1})$ and $\beta_{j+1}[\mathrm{Cl}(\partial Q_{j+1} - P_j)]$ are subcomplexes, and it follows that β_{j+1} is cellular.

The expansions β: $X \to Y \times I^n$ corresponding to a formal expansion $Y \nearrow X$ have the following basic properties:

E_1. $\beta(y) = (y, 0_n)$ for all $y \in Y \subset X$.

E_2. The image $\beta(B)$ is independent of which expansion β has been constructed (since, inductively, the image of each cell of Y_j is forced). Indeed, writing $\beta_j = \pi\beta$, where π: $Y \times I^n \to Y \times I^j$, we obtain the formula

$$\beta(X) = (Y \times 0_n) \cup (\beta_0(P_1) \times I \times 0_{n-1}) \cup (\beta_1(P_2) \times I \times 0_{n-2}) \cup \cdots \cup (\beta_{n-1}(P_n) \times I).$$

E_3. $\beta(X) \nearrow Y \times I^n$ by a naturally arising formal expansion. E_3 holds because, inductively,

$$\beta(X) \cup (Y \times I^j \times 0_{n-j}) \nearrow \beta(X) \cup (Y \times I^{j+1} \times 0_{n-j-1}).$$

Indeed, the entire expansion can be visualized if we represent $Y \times I^n$ as an ordered union

$$\begin{aligned} Y \times I^n = \beta(X) &\cup [(A_{11} \times I \times 0_{n-1}) \cup (A_{12} \times I \times 0_{n-1}) \cup \cdots \cup (A_{1p_1} \times I \times 0_{n-1})] \\ &\cup [(A_{21} \times I \times 0_{n-2}) \cup \cdots \cup (A_{2p_2} \times I \times 0_{n-2})] \\ &\cup [\quad] \cup \cdots \cup [(A_{n1} \times I) \cup \cdots \cup (A_{np_n} \times I)], \end{aligned}$$

where the A_{jk} are the cells of $Y \times I^{j-1} - \beta_{j-1}(P_j)$, and where

$$0 = \dim A_{j1} \leq \cdots \leq \dim A_{jp_j}.$$

(This explicit presentation of $Y \times I^n$ will be used in the proof of Lemma 2.)

Property E_1 of expansions leads to the following notation. Suppose we are given cellular embeddings

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} \alpha(A) \times I^n$$

such that $\beta\alpha = (\alpha, 0)$. Then we define $\pi_{\alpha,\beta}$: $B \to A$ by the equation

$$\pi_{\alpha,\beta} = \alpha^{-1}\beta^{-1}\pi_0\beta = (\alpha, 0)^{-1}\pi_0\beta.$$

(Intuitively, if $\alpha(A) \nearrow B$ then $\pi_{\alpha,\beta}$ is a choice of "collapsing map".)

Finally, we recall that if (Z_j, f_j) is a sequence

$$Z_0 \xleftarrow{f_1} Z_1 \xleftarrow{f_2} Z_2 \longleftarrow \cdots$$

of spaces and maps, then the *inverse limit* of the sequence is defined by

$$\operatorname{Lim}(Z_j, f_j) = \left\{ z \in \prod_{j=0}^{\infty} Z_j \;\middle|\; f_j(z_j) = z_{j-1} \text{ for all } j \right\},$$

where z_j is the jth coordinate of z. Corresponding to any two inverse sequences (Z_j, f_j) and (Z_j', f_j') and a sequence of maps $g_j: Z_j \to Z_j'$ such that $g_{j-1} f_j = f_j' g_j$, there is determined a map

$$g = \operatorname{Lim}(g_j): \operatorname{Lim}(Z_j, f_j) \to \operatorname{Lim}(Z_j', f_j')$$

defined by $g(z_0, z_1, z_2, \cdots) = (g_0(z_0), g_1(z_1), \cdots)$. Clearly, if each g_j is a homeomorphism, so also is g.

2. PROOF THAT $X \times Q \approx Y \times Q$

LEMMA 1. *If* A *is a topological ball of dimension* s *with metric* d *and if* A^c (read: A collared) *is the subset*

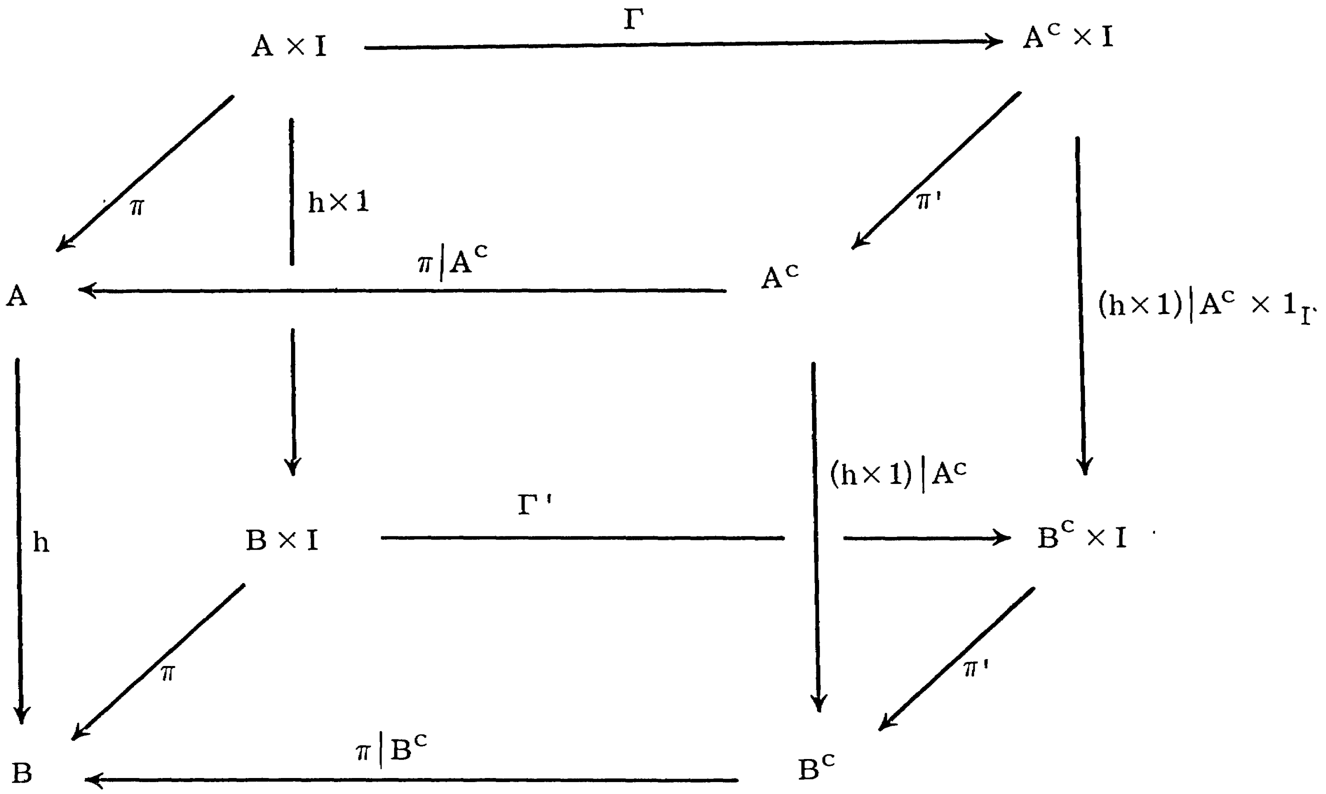
$$(A \times 0) \cup (\partial A \times I)$$

of $A \times I$, *then, for each* $\delta > 0$, *there exists a homeomorphism* $\Gamma: A \times I \to A^c \times I$ *such that* $\Gamma(x, t) = (x, t, 0)$ *if* $(x, t) \in A^c$ *and such that the diagram*

$$\begin{array}{ccc} A \times I & \xrightarrow{\Gamma} & A^c \times I \\ \downarrow \pi & & \downarrow \pi' \\ A & \xleftarrow{\pi | A^c} & A^c \end{array}$$

is δ-*commutative* (that is, $d(\pi, \pi\pi'\Gamma) < \delta$).

This lemma is elementary, and its proof is an exercise. We include it for the sake of completeness. Let $h: A \to B = \{x \in R^s \mid \|x\| \leq 1\}$ be a homeomorphism. Let $B^c = (B \times 0) \cup (\partial B \times I)$. Choose $\delta' > 0$ so that if $z, z' \in B$ and $\|z - z'\| < \delta'$, then $d(h^{-1}(z), h^{-1}(z')) < \delta$. Suppose $\Gamma': B \times I \to B^c \times I$ is a homeomorphism with $\Gamma'(x, t) = (x, t, 0)$ for all $(x, t) \in B^c$, and suppose $\Gamma: A \times I \to A^c \times I$ is the induced homeomorphism. This gives the diagram



All the vertical squares are commutative, and one easily verifies that if the bottom square is δ'-commutative, then the top square is δ-commutative. Thus it suffices to show that $\Gamma'\colon B \times I \to B^c \times I$ can be constructed so that it is δ'-commutative.

Let $g\colon I^2 \to [(I \times 0) \cup (1 \times I)] \times I$ be a homeomorphism such that $g(\lambda, t) = (\lambda, t, 0)$ if $t = 0$ or $\lambda = 1$, and such that $g(\lambda, t) = (\lambda, 0, t)$ if $0 \le \lambda \le 1 - \delta'$. If $z \in \partial B$, define

$$T_z\colon \{(\lambda z, t) \mid (\lambda, t) \in I^2\} \to I^2$$

by $T_z(\lambda z, t) = (\lambda, t)$. Define

$$\Gamma'(\lambda z, t) = (T_z \times 1_I)^{-1} g T_z(\lambda z, t).$$

Then, if $0 \le \lambda \le 1 - \delta'$, we see that $\Gamma'(\lambda z, t) = (\lambda z, 0, t)$. Therefore $\pi\pi'\Gamma'(\lambda z, t) = \lambda z = \pi(\lambda z, t)$. If $1 - \delta' < \lambda \le 1$, then $\Gamma'(\lambda z, t) = (\lambda' z, t_1, t_2)$, where $1 - \delta' < \lambda' \le 1$ and $t_1 = 0$ or $\lambda' = 1$. Therefore $\pi\pi'\Gamma'(\lambda z, t) = \lambda' z$. Thus $d(\pi(\lambda z, t), \pi\pi'\Gamma'(\lambda z, t)) = d(\lambda' z, \lambda z) = |\lambda - \lambda'| < \delta'$. ∎

LEMMA 2. *Suppose* A *and* B *are complexes and* $\alpha\colon A \to B$ *is a cellular embedding such that* $\alpha(A) \nearrow B$ *by a formal expansion. Assume* $\beta\colon B \to \alpha(A) \times I^n$ *is a corresponding expansion. Let* $\varepsilon > 0$, *and let* d *be a metric on* A. *Then there exists an expansion* $\gamma\colon \alpha(A) \times I^n \to \beta(B) \times I^m$ *corresponding to the formal expansion* $\beta(B) \nearrow \alpha(A) \times I^n$ *of* E_3,

$$A \underset{\pi_{\alpha,\beta}}{\overset{\alpha}{\rightleftarrows}} B \underset{\pi_{\beta,\gamma}}{\overset{\beta}{\rightleftarrows}} \alpha(A) \times I^n \overset{\gamma}{\dashrightarrow} \beta(B) \times I^m,$$

such that $d(\pi_\alpha, \pi_{\alpha,\beta} \circ \pi_{\beta,\gamma}) < \varepsilon$, *where* $\pi_\alpha = (\alpha, 0)^{-1}\pi_0\colon \alpha(A) \times I^n \to A$.

Proof. As we pointed out following E_3 of Section 1, if

$$\alpha(A) \nearrow B = \alpha(A) \bigcup_{P_1} Q_1 \bigcup \cdots \bigcup_{P_n} Q_n,$$

then the formal expansion $\beta(B) \nearrow \alpha(A) \times I^n$ is carried by the ordered union

$$\alpha(A) \times I^n = \beta(B) \cup \bigcup_{1 \leq j \leq n} \Big[\bigcup_{1 \leq k \leq p_j} A_{jk} \times I \times 0_{n-j} \Big],$$

where $\{A_{jk} \times 0_{n-j+1} \mid 1 \leq k \leq p_j\}$ are the cells of $[\alpha(A) \times I^{j-1} \times 0_{n-j+1}] - \beta(P_j)$, and where $0 = \dim A_{j1} \leq \cdots \leq \dim A_{jp_j}$. Let us write $A_q = A_{jk}$ when $q = p_1 + \cdots + p_{j-1} + k$. Also, set $\mathscr{A}_0 = \beta(B)$ and

$$\mathscr{A}_q = \beta(B) \cup (A_1 \times I \times 0_{n-1}) \cup \cdots \cup (A_q \times I \times 0_{n-j})$$

when $A_q = A_{jk}$. Let $m = p_1 + p_2 + \cdots + p_n$. Thus

$$\beta(B) = \mathscr{A}_0 \nearrow^e \mathscr{A}_1 \nearrow^e \cdots \nearrow^e \mathscr{A}_m = \alpha(A) \times I^n.$$

Now we are ready to construct the desired expansion $\gamma: \alpha(A) \times I^n \to \beta(B) \times I^m$.

Let $\gamma_0 = 1_{\beta(B)}$. Let $\gamma_1: \mathscr{A}_1 \to \beta(B) \times I$ be defined by setting $\gamma_1(x) = (x, 0)$ if $x \in \beta(B)$ and $\gamma_1(A_1, t, 0_{n-1}) = (A_1, t) = (\gamma_0(A_1), t)$. (This makes sense because A_1 is a point.) Assume inductively that an expansion $\gamma_{q-1}: \mathscr{A}_{q-1} \to \beta(B) \times I^{q-1}$ has been constructed corresponding to the formal expansion $\beta(B) \nearrow \mathscr{A}_{q-1}$ and satisfying the condition that $d(\pi_\alpha \mid \mathscr{A}_{q-1}, \pi_{\alpha,\beta} \circ \pi_{\beta,\gamma_{q-1}}) \leq (q-1)\varepsilon/m$.

To construct $\gamma_q: \mathscr{A}_q \to \beta(B) \times I^q$, we of course set $\gamma_q \mid \mathscr{A}_{q-1} = (\gamma_{q-1}, 0)$. We must define γ_q on $A_q \times I \times 0_{n-j}$ (where $A_q = A_{jk}$). If $\dim A_q = 0$, then γ_q is defined like γ_1 above, and we leave this case to the reader. Otherwise, set

$$A_q^c = (A_q \times 0) \cup (\partial A_q \times I) \subset A_q \times I.$$

By our ordering of the cells,

$$A_q^c \times 0_{n-j} = \mathscr{A}_{q-1} \cap (A_q \times I \times 0_{n-j});$$

therefore $\gamma_{q-1} \mid A_q^c \times 0_{n-j}$ is already defined. Give B a metric, $\alpha(A) \subset B$ the induced metric, and $\alpha(A) \times I^n$ a standard product metric. (All metrics will be denoted by the letter d). Using the uniform continuity of π_α, fix a $\delta > 0$ such that if $y, y' \in \alpha(A) \times I^n$ and $d(y, y') < \delta$, then $d(\pi_\alpha(y), \pi_\alpha(y')) < \varepsilon/m$. By Lemma 1, we can find a homeomorphism Γ such that in the following diagram the left-hand square is δ-commutative. (Each map should be understood as being restricted to the appropriate domain.)

$$\begin{array}{ccccc}
A_q \times I \times 0_{n-j} & \xrightarrow{\Gamma} & A_q^c \times 0_{n-j} \times I & \xrightarrow{\gamma_{q-1} \times 1_I} & \beta B \times I^q \\
\downarrow \pi_{j-1} & & \downarrow \pi' & & \downarrow \pi_{q-1} \\
A_q \times 0_{n-j+1} & \xleftarrow[\pi_{j-1}]{} & A_q^c \times 0_{n-j} & \xleftarrow[\gamma_{q-1}^{-1}]{} & \beta B \times I^{q-1} \times 0
\end{array}$$

Clearly, the right-hand square is commutative. We define

$$\gamma_q \,|\, (A_q \times I \times 0_{n-j}) = (\gamma_{q-1} \times 1_I)\Gamma .$$

It follows that $d(\pi_{j-1} \,|\, \mathscr{A}_q, \pi_{j-1}\gamma_{q-1}^{-1}\pi_{q-1}\gamma_q) < \delta$. Now

$$\begin{aligned}\pi_{\beta,\gamma_q} &\equiv (\beta, 0)^{-1}\pi_0\gamma_q = (\beta, 0)^{-1}\pi_0\pi_{q-1}\gamma_q \\ &= (\beta, 0)^{-1}\pi_0\gamma_{q-1}\gamma_{q-1}^{-1}\pi_{q-1}\gamma_q = \pi_{\beta,\gamma_{q-1}}\gamma_{q-1}^{-1}\pi_{q-1}\gamma_q .\end{aligned}$$

Thus, if $x \in \mathscr{A}_q$ and $x' \equiv \gamma_{q-1}^{-1}\pi_{q-1}\gamma_q(x)$, we have the relations

$$\begin{aligned}d(\pi_\alpha(x), \pi_{\alpha,\beta}\pi_{\beta,\gamma_q}(x)) &= d(\pi_\alpha(x), \pi_{\alpha,\beta}\pi_{\beta,\gamma_{q-1}}(x')) \\ &\leq d(\pi_\alpha(x), \pi_\alpha(x')) + d(\pi_\alpha(x'), \pi_{\alpha,\beta}\pi_{\beta,\gamma_{q-1}}(x')) \\ &< d(\pi_\alpha(x), \pi_\alpha(x')) + \frac{\varepsilon(q-1)}{m} \quad \text{(by the induction hypothesis)} \\ &= d(\pi_\alpha\pi_{j-1}(x), \pi_\alpha\pi_{j-1}(x')) + \frac{\varepsilon(q-1)}{m}, \quad \text{where } d(\pi_{j-1}(x), \pi_{j-1}(x')) < \delta \\ &< \frac{\varepsilon}{m} + \frac{\varepsilon(q-1)}{m}, \text{ by choice of } \delta .\end{aligned}$$

This completes the induction step. In the end, we set $\gamma = \gamma_m$ and have the inequality $d(\pi_\alpha, \pi_{\alpha,\beta}\pi_{\beta,\gamma}) < \varepsilon$. The proof of Lemma 2 is complete. ∎

Definition. Two maps f, g: $X \to Y$ (where X and Y are metric spaces) are *nearly homeomorphic* if there exists a sequence of homeomorphisms h_j: $X \to X$ such that fh_j converges uniformly to g.

It is easily verified that the relation of being nearly homeomorphic is an equivalence relation on the set of maps from X to Y. Notice that the fact that f and g are nearly homeomorphic does not imply that the homeomorphisms h_j themselves converge. Thus there need not exist a near-homeomorphism h (that is, a uniform limit of homeomorphisms as defined in [1]) such that $fh = g$.

LEMMA 3. *Consider the diagram*

$$A \underset{\pi_{\alpha,\beta}}{\overset{\alpha}{\rightleftarrows}} B \underset{\pi_{\beta,\gamma}}{\overset{\beta}{\rightleftarrows}} \alpha A \times I^n \xrightarrow{\ \gamma\ } \beta B \times I^m,$$

where α *is a cellular embedding such that* $\alpha A \nearrow B$, *where* β *is a corresponding expansion, and where* γ *is an expansion corresponding to the formal expansion* $\beta B \nearrow \alpha A \times I^n$. *Let* $\pi_\alpha = (\alpha, 0)^{-1}\pi_0$: $\alpha A \times I^n \to A$. *Then* π_α *and* $\pi_{\alpha,\beta}\pi_{\beta,\gamma}$ *are nearly homeomorphic.*

Proof. For each integer $j > 0$, choose γ_j: $\alpha A \times I^n \to \beta B \times I^m$ so that $d(\pi_\alpha, \pi_{\alpha,\beta}\pi_{\beta,\gamma_j}) < 1/j$. This is possible, by Lemma 2. Let $h_j = \gamma^{-1}\gamma_j$. Then

$$\pi_{\beta,\gamma}h_j = (\beta, 0)^{-1}\pi_0\gamma\gamma^{-1}\gamma_j = \pi_{\beta,\gamma_j}.$$

Therefore $d(\pi_\alpha, \pi_{\alpha,\beta}\pi_{\beta,\gamma}h_j) = d(\pi_\alpha, \pi_{\alpha,\beta}\pi_{\beta,\gamma_j}) < 1/j$. Thus $\{\pi_{\alpha,\beta}\pi_{\beta,\gamma}h_j\}$ converges to π_α. ■

From now on we assume that a formal expansion $Y \nearrow X$ is preassigned. Set $Y_0 = Y$ and $X_0 = X$, and let $\alpha_0: Y_0 \to X_0$ be the inclusion. This leads to a sequence

$$Y_0 \xrightarrow{\alpha_0} X_0 \xrightarrow{\alpha_1} Y_0 \times I^{n_1} \equiv Y_1 \xrightarrow{\alpha_2} \alpha_1 X_0 \times I^{m_1} \equiv X_1 \longrightarrow$$

$$\cdots \xrightarrow{\alpha_{2i-1}} Y_i \xrightarrow{\alpha_{2i}} X_i \xrightarrow{\alpha_{2i+1}} Y_{i+1} \longrightarrow \cdots,$$

where $Y_{i+1} = \alpha_{2i}(Y_i) \times I^{n_{i+1}}$, $X_i = \alpha_{2i-1}(X_{i-1}) \times I^{m_i}$, and each α_j is an expansion corresponding to the fact that (image α_{j-1}) $\nearrow$ (domain α_j). In particular, $\alpha_{j+1}\alpha_j = (\alpha_j, 0)$. The direct sequence gives rise to an inverse sequence

$$Y_0 \xleftarrow[\pi_{0,1}]{} X_0 \xleftarrow[\pi_{1,2}]{} Y_1 \xleftarrow[\pi_{2,3}]{} X_1 \longleftarrow \cdots,$$

where $\pi_{j,j+1} \equiv \pi_{\alpha_j,\alpha_{j+1}}$. We write $\pi_{j,j+2} = \pi_{j,j+1}\pi_{j+1,j+2}$.

LEMMA 4. *There exist homeomorphisms* $h_i: Y_i \to Y_i$ $(i > 0)$ *such that* $\operatorname{Lim}(Y_i, \pi_{2i-2,2i}) \approx \operatorname{Lim}(Y_i, \pi_i h_i)$, *where* (changing from earlier notation)

$$\pi_i \equiv (\alpha_{2i-2}, 0)^{-1}\pi_0: Y_i = \alpha_{2i-2}(Y_{i-1}) \times I^{n_i} \to Y_{i-1}.$$

Proof. By Lemma 3, there exists for each fixed i a sequence $h_{ik}: Y_i \xrightarrow{\approx} Y_i$ such that $\operatorname{Lim}_k(\pi_i h_{ik}) = \pi_{2i-2,2i}$. By the approximation theorem for inverse limits (Theorem 3 of [1]), we may choose, for each i, one of these homeomorphisms h_{ik} - call it h_i - in such a way that $\operatorname{Lim}(Y_i, \pi_i h_i) \approx \operatorname{Lim}(Y_i, \pi_{2i-2,2i})$. ■

LEMMA 5. *Let* $N_i = n_1 + n_2 + \cdots + n_i$. *Let* $p_i: Y_0 \times I^{N_i} \to Y_0 \times I^{N_{i-1}}$ *be the natural projection. Then* $\operatorname{Lim}(Y_i, \pi_i h_i) \approx \operatorname{Lim}(Y_0 \times I^{N_i}, p_i) \approx Y_0 \times Q$.

Proof. We construct a vertical sequence of homeomorphisms g_i such that the diagram

$$\begin{array}{ccccccc}
Y_0 \longleftarrow & Y_1 \longleftarrow & \cdots & Y_i & \xleftarrow{\pi_{i+1}h_{i+1}} & Y_{i+1} = \alpha_{2i}(Y_i) \times I^{n_{i+1}} & \longleftarrow \cdots \\
\downarrow g_0 & \downarrow g_1 & & \downarrow g_i & & \downarrow g_{i+1} & \\
Y_0 \longleftarrow & Y_0 \times I^{N_1} \longleftarrow & \cdots & Y_0 \times I^{N_i} & \xleftarrow[p_{i+1}]{} & Y_0 \times I^{N_{i+1}} = (Y_0 \times I^{N_i}) \times I^{n_{i+1}} &
\end{array}$$

commutes. Let $g_0 = 1$. Having constructed g_i, let $g_{i+1} = [(g_i\alpha_{2i}^{-1}) \times 1]h_{i+1}$. Then

$$p_{i+1}g_{i+1} = p_{i+1}[g_i\alpha_{2i}^{-1} \times 1]h_{i+1} = g_i(\alpha_{2i}, 0)^{-1}\pi_0 h_{i+1} = g_i(\pi_{i+1}h_{i+1}).$$

The map $g = \operatorname{Lim}(g_i)$ is a homeomorphism of $\operatorname{Lim}(Y_i, \pi_i h_i)$ onto $\operatorname{Lim}(Y_0 \times I^{N_i}, p_i)$.

Finally, the map $f: \operatorname{Lim}(Y_0 \times I^{N_i}, p_i) \to Y_0 \times Q$ given by the equation

$$f(y_0, (y_0, t_1, \cdots, t_{N_1}), (y_0, t_1, \cdots, t_{N_1}, \cdots, t_{N_2}), \cdots) = (y_0, t_1, t_2, \cdots)$$

is obviously a homeomorphism. ■

Proof that $X_0 \times Q \approx Y_0 \times Q$. Let Z be the inverse limit of the sequence

$$Y_0 \xleftarrow{\pi_{0,1}} X_0 \xleftarrow{\pi_{1,2}} Y_1 \xleftarrow{\pi_{2,3}} X_1 \longleftarrow \cdots.$$

Then Z is homeomorphic to the inverse limit of its cofinal subsequence $(Y_i, \pi_{2i-2,2i})$. By Lemmas 4 and 5, $\operatorname{Lim}(Y_i, \pi_{2i-2,2i}) \approx Y_0 \times Q$. Similarly, using Lemmas 4 and 5 stated in terms of the X_i (call these Lemmas 4' and 5'), we see that $Z \approx X_0 \times Q$. Thus $X_0 \times Q \approx Y_0 \times Q$. ■

3. PROOF OF THE THEOREM

If X and Y are simplicial complexes, then a simple-homotopy equivalence $f: X \to Y$ is by definition a map homotopic to a composition

$$X = X_0 \xrightarrow{f_1} X_1 \longrightarrow \cdots \xrightarrow{f_n} X_n = Y,$$

where either there exists an elementary simplicial collapse $X_i \searrow X_{i-1}$ and f_i is the inclusion map, or else there exists an elementary simplicial collapse $X_{i-1} \searrow X_i$ and f_i is a homotopy inverse to the inclusion map. Thus, to prove that $f \times 1_Q$ is homotopic to a homeomorphism of $X \times Q$ onto $Y \times Q$, it suffices to prove that if $X_0 \searrow Y_0$ by an elementary simplicial collapse and if $\alpha_0: Y_0 \to X_0$ is the inclusion, then $\alpha_0 \times 1_Q$ is homotopic to a homeomorphism of $Y_0 \times Q$ onto $X_0 \times Q$. We shall show that the final homeomorphism constructed in Section 2 has this property.

In Section 2 we constructed homeomorphisms (unlabelled there) as follows:

$F: \operatorname{Lim}(Y_i, \pi_{2i-2,2i}) \to \operatorname{Lim}(Y_i, \pi_i h_i)$ (Lemma 4),

$G: \operatorname{Lim}(Y_i, \pi_i h_i) \to Y_0 \times Q$ (Lemma 5),

$H: \operatorname{Lim}(X_i, \pi_{2i-1,2i+1}) \to \operatorname{Lim}(Y_i, \pi_{2i-2,2i})$,

$\hat{F}: \operatorname{Lim}(X_i, \pi_{2i-1,2i+1}) \to \operatorname{Lim}(X_i, \hat{\pi}_i \hat{h}_i)$ (Lemma 4'),

$\hat{G}: \operatorname{Lim}(X_i, \hat{\pi}_i \hat{h}_i) \to X_0 \times Q$ (Lemma 5').

Let $T = GFH\hat{F}^{-1}\hat{G}^{-1}: X_0 \times Q \to Y_0 \times Q$. Let each of the maps

$\pi_Y: Y_0 \times Q \to Y_0$, $\pi_X: X_0 \times Q \to X_0$,

$p_Y: \operatorname{Lim}(Y_i, \pi_{2i-2,2i}) \to Y_0$, $p_X: \operatorname{Lim}(X_i, \pi_{2i-1,2i+1}) \to X_0$,

$p'_Y: \operatorname{Lim}(Y_i, \pi_i h_i) \to Y_0$, $p'_X: \operatorname{Lim}(X_i, \hat{\pi}_i \hat{h}_i) \to X_0$,

be a standard projection onto the first coordinate.

ASSERTION: $\pi_Y T \simeq \pi_{0,1} \pi_X: X_0 \times Q \to Y_0$.

This assertion will suffice, because $\pi_{0,1}\pi_X = \pi_Y(\pi_{0,1} \times 1_Q)$. Therefore our assertion implies that $\pi_Y T \simeq \pi_Y(\pi_{0,1} \times 1_Q)$, whence $T \simeq \pi_{0,1} \times 1_Q$. Since $\pi_{0,1}$ is a homotopy inverse to $\alpha_0\colon Y_0 \overset{\subseteq}{\to} X_0$, it follows that $\alpha_0 \times 1_Q \simeq T^{-1}$.

To prove the assertion, we first show that $p'_Y F \simeq p_Y\colon \mathrm{Lim}(Y_i, \pi_{2i-2,2i}) \to Y_0$. The homeomorphism F is obtained from Theorem 3 of [1], and it is defined there so that if

$$s = (s_0, s_1, \cdots) \in \mathrm{Lim}(Y_i, \pi_{2i-2,2i}),$$

then $p'_Y F(s) = F_0(s) = \mathrm{Lim}_n[(\pi_1 h_1) \cdots (\pi_n h_n)(s_n)]$. But by the proof of Lemmas 3 and 4, h_i is of the form $h_i = \gamma_{j_i}^{-1}\gamma$, for some expansions γ, γ_{j_i}. Thus, by E_2 of Section 1, h_i takes each cell of Y_i onto itself. Therefore $\pi_1 h_1 \cdots \pi_n h_n(s_n)$ lies in the same open simplex $\mathring{\sigma} = \mathring{\sigma}(s_n)$ of Y_0 as $\pi_1\pi_2 \cdots \pi_n(s_n)$. On the other hand, if $y \in \alpha_{2i-2}(Q_0) \times I^{n_i}$, where Q_0 is a closed cell of Y_{i-1}, then

$$\pi_{2i-2,2i}(y) \in Q_0 = \pi_i[\alpha_{2i-2}(Q_0) \times I^{n_i}].$$

Hence, proceeding inductively, we see that the point $s_0 = \pi_{0,2} \cdots \pi_{2n-2,2n}(s_n)$ and the point $\pi_1 \cdots \pi_n(s_n)$ lie in a common simplex. Therefore, for all n,

$$(\pi_1 h_1) \cdots (\pi_n h_n)(s_n)$$

lies in the closed simplicial star of s_0 (that is, in the union of all closed simplexes containing s_0), and the limit of this sequence $p'_Y F(s)$ must lie in this star. Since $s_0 = p_Y(s)$, this shows that p_Y and $p'_Y F$ are contiguous and, consequently, homotopic maps.

Similarly, $p'_X \hat{F} \simeq p_X\colon \mathrm{Lim}(X_i, \pi_{2i-1,2i+1}) \to X_0$.

Next notice that $\pi_Y G = p'_Y\colon \mathrm{Lim}(Y_i, \pi_i h_i) \to Y_0$. For G was defined in the proof of Lemma 5 by the formula $G = f \circ \mathrm{Lim}(g_i)$, where $g_0 = 1_{Y_0}$ and f preserves the first coordinate.

Similarly, $\pi_X \hat{G} = p'_X$.

The homeomorphism H was not explicitly defined. Its existence was merely inferred from cofinality. We define H explicitly as the map induced from the vertical map

$$\begin{array}{ccccccc}
X_0 & \xleftarrow{\pi_{1,2}\pi_{2,3}} & X_1 & \xleftarrow{\pi_{3,4}\pi_{4,5}} & X_2 & \longleftarrow & \\
\downarrow{\scriptstyle \pi_{0,1}} & & \downarrow{\scriptstyle \pi_{2,3}} & & \downarrow{\scriptstyle \pi_{4,5}} & & \\
Y_0 & \xleftarrow[\pi_{0,1}\pi_{1,2}]{} & Y_1 & \xleftarrow[\pi_{2,3}\pi_{3,4}]{} & Y_2 & \longleftarrow &
\end{array}$$

of inverse sequences. It is clear that H is a homeomorphism with

$$H^{-1}(y_0, y_1, y_2, \cdots) = (\pi_{1,2}(y_1), \pi_{3,4}(y_2), \cdots).$$

Obviously, $p_Y H = \pi_{0,1} p_X$.

Finally, all of the information above gives the relations

$$\pi_Y T = \pi_Y GFH\hat{F}^{-1}\hat{G}^{-1} = p'_Y FH\hat{F}^{-1}\hat{G}^{-1} \simeq p_Y H\hat{F}^{-1}\hat{G}^{-1} = \pi_{0,1} p_X \hat{F}^{-1}\hat{G}^{-1}$$

$$\simeq \pi_{0,1} p'_X \hat{G}^{-1} = \pi_{0,1}\pi_X .$$

This completes the proof of our assertion.

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