THE WHITEHEAD TORSION OF A FIBER-HOMOTOPY EQUIVALENCE

Douglas R. Anderson

1. INTRODUCTION AND STATEMENTS OF RESULTS

This paper is based on the observation that *if* $\xi = (E, p, B, F)$ *is a piecewise linear* (PL) *fiber bundle, then* p *induces a homomorphism* $p^*: \mathrm{Wh}\,\pi_1(B) \to \mathrm{Wh}\,\pi_1(E)$, *where* $\mathrm{Wh}\,\pi$ *denotes the Whitehead group of* π (see Proposition 2.3).

The definition of a PL fiber bundle is given in [1]. We can also completely determine the homomorphism p^* in many cases by using the results of [1].

We describe here the construction of the homomorphism p^*; for complete details we refer the reader to Section 2. Let $\tau_0 \in \mathrm{Wh}\,\pi_1(B)$ be arbitrary, and let $f: B' \to B$ be a PL homotopy equivalence such that $\pi(f) = \tau_0$, where $\tau(f)$ denotes the Whitehead torsion of f. Form the induced fiber space with total space

$$f^!(E) = \{(b', c) \in B' \times E \mid f(b') = p(e)\},$$

and notice that the map $g: f^!(E) \to E$ given by $g(b', e) = e$ is also a homotopy equivalence. Since f is PL, the space $f^!(E)$ inherits a PL structure in a natural way, and g has a Whitehead torsion $\tau(g)$. Define $p^*\tau_0 = \tau(g)$.

The following is our main result.

THEOREM A. *Let* $\xi_i = (E_i, p_i, B_i, F_i)$ $(i = 1, 2)$ *be* PL *fiber bundles with connected base and fiber, and let* $g: E_1 \to E_2$ *be a fiber-homotopy equivalence covering* $f: B_1 \to B_2$ *and inducing* $h: F_1 \to F_2$. *Then*

$$\tau(g) = p_2^*\,\tau(f) + \chi(B_2)\, j_{2*}\,\tau(h),$$

where $j_{2*}: \mathrm{Wh}\,\pi_1(F_2) \to \mathrm{Wh}\,\pi_1(E_2)$ *is induced by the inclusion* $j_2: F_2 \to E_2$.

We give the proof in Section 3. As a special case we obtain the following result, due to K. W. Kwun and R. H. Szczarba [7, Corollary 1.3].

COROLLARY B. *Let* $f: B_1 \to B_2$ *and* $h: E_1 \to E_2$ *be homotopy equivalences. Then*

$$\tau(f \times h) = \chi(F_2)\, k_{2*}\,\tau(f) + \chi(B_2)\, j_{2*}\,\tau(h),$$

where k_{2*} *is induced by the inclusion* $k_2: B_2 \to B_2 \times F_2$.

Proof. This follows from Theorem A if we set $g = f \times h$ and observe that the Product Theorem of [7] shows that $p_2^*\tau = \chi(F_2)\, k_{2*}\,\tau$ for each $\tau \in \mathrm{Wh}\,\pi_1(B_2)$, where $p_2: B_2 \times F_2 \to B_2$ is projection on the first factor.

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Suppose now that $\xi = (M_1^n, p_1, B_1, F_1)$ is a PL fiber bundle whose total space is a closed manifold of dimension $n \geq 5$ and whose base space and fiber are connected. Let W^{n+1} be an h-cobordism with $\partial W = M_1 \cup M_2$ and with torsion $\tau(W; M_1) \in Wh\pi_1(M_1)$. Let $i: M_2 \to W$ be the inclusion, and let $r: W \to M_1$ be a deformation retraction. One sometimes wants to know whether there exists a homotopy equivalence $f: B_1 \to B_2$ such that $p' = f p_1 r i$ is homotopic to a PL bundle map. (See [5], for example.)

COROLLARY C. *If* p' *is homotopic to a* PL *bundle map* $p_2: M_2 \to B_2$, *then there exist elements* $\tau_1 \in Wh\pi_1(F_1)$ *and* $\tau_2 \in Wh\pi_1(B_1)$ *such that*

$$\tau(W; M_1) + (-1)^{n+1}\tau(W; M_1)^* = \chi(B_1) j_* \tau_1 + p_1^* \tau_2,$$

where $*$ *denotes the duality involution of* $Wh\pi_1(M_1)$.

See [8; p. 373] for the definition of the duality involution.

Proof. Let $p_2: M_2 \to B_2$ be a PL bundle map homotopic to p', and let $f': B_2 \to B_1$ be a homotopy inverse for f. Since $f'p_2 \simeq f'p' = f' f p_1 r i \simeq p_1(ri)$, the map ri is homotopic to a fiber homotopy equivalence $g': M_2 \to M_1$ covering f' and inducing $h': F_2 \to F_1$. Hence, by Theorem A,

$$\tau(ri) = \tau(g') = \chi(F_1) j_* \tau(h') + p_1^* \tau(f').$$

Since it is well known that $\tau(ri) = -[\tau(W; M_1) + (-1)^{n+1}\tau(W; M_1)^*]$, the corollary follows if we set $\tau_1 = -\tau(h')$ and $\tau_2 = -\tau(f')$.

The following example illustrates the use of Corollary C. Let

$$\xi = (M_1^{2n+1}, p_1, B_1^{2k}, S^{2\ell+1})$$

be a PL fiber bundle whose total space is a PL manifold, whose base space has $\pi_1(B_1) = Z/pZ$, where p is an odd prime, and whose fiber is a sphere of dimension $2\ell + 1$, where $\ell \geq 1$. Let W be an h-cobordism with $\partial W = M_1 \cup M_2$ and such that $\tau(W, M_1) \neq 0$. Then there is no homotopy equivalence $f: B_1 \to B_2$ for which the homotopy class of $f p_1 r i$ contains a PL bundle map.

To see this, we recall that every element of $Wh(Z/pZ)$ is self-conjugate, by [8, Lemma 6.7]. Hence, if there is a PL bundle map p_2 homotopic to $f p_1 r i$, then

$$2\tau(W, M_1) = \tau(W, M_1) + \tau(W, M_1) = \tau(W, M_1) + (-1)^{2n+2}\tau(W, M_1)^*$$

$$= \chi(B_1) j_{1*} \tau_1 + p_1^* \tau_2 = p_1^* \tau_2$$

for some $\tau_2 \in Wh(Z/pZ)$. Since $p_{1*}: Wh\pi_1(M_1) \to Wh\pi_1(B_1)$ is an isomorphism, Corollary B of [1] shows that $p_1^* \tau_2 = \chi(S^{2\ell+1})\tau_2 = 0$. Hence $2\tau(W, M_1) = 0$. Since $Wh(Z/pZ)$ is free abelian [8, p. 362], we conclude that $\tau(W, M_1) = 0$. This is a contradiction.

In attempting to construct new examples of homeomorphic but combinatorially distinct polyhedra by a refinement of Stallings' method of infinite repetition [10], the author was led to the following result.

COROLLARY D. *Let* $\xi_i = (M_i^n, p_i, B_i, F_i)$ $(i = 1, 2)$ *be* PL *fiber bundles whose total spaces* M_i^n *are manifolds of dimension* $n \geq 5$, *and whose base spaces and fibers are connected. Let* C_i *denote the open mapping cylinder of* p_i. *If there* *is a homeomorphism* $g: C_1 \to C_2$ *such that* $g(B_1) = B_2$, *then there exist a* PL *manifold* M_1' *homeomorphic to* M_1, *an* h*-cobordism* W *with* $\partial W = M_1' \cup M_2$, *and an element* $\tau_0 \in \mathrm{Wh}\,\pi_1(F_2)$ *such that*

$$\tau(W, M_2) + (-1)^{n+1}\,\tau(W, M_2)^* = \chi(B)\,j^*\,\tau_0,$$

where $j: F_2 \to M_2$ *is the inclusion.*

The open mapping cylinder C of a map $f: X \to Y$ is the space obtained from the disjoint union $X \times [0, \infty) \cup Y$ by the identification of $(x, 0)$ with $f(x)$. To points of C we assign coordinates (x, t) for $x \in X$ and $t \in [0, \infty)$ in the obvious way.

Proof. Observing that $M_2 \times (0, \infty) \subset C_2$ has a PL structure and that $g(M_1 \times 1) \subset M_2 \times (0, \infty)$ is topologically bicollared, and using the Product Structure Theorem of R. C. Kirby and L. C. Siebenmann [6, Corollary 7.2], we may assume that $M_1' = h(M_1 \times 1)$ is a PL submanifold of $M_2 \times (0, \infty)$. Let $0 < s < t$ be such that $M_1' \subset M_2 \times (s, t)$, set

$$W = C_2 - M_2 \times (t, \infty) - g(C_1 - M_1 \times (1, \infty)),$$

$$W' = g(C_1 - M_1 \times (1, \infty)) - [C_2 - M_2 \times (s, \infty)],$$

and notice that $W \cup W' = M_2 \times [s, t]$ while $W \cap W' = M_1'$.

Let $g_1 = g \mid M_1 \times 1: M_1 \times 1 \to M_2 \times [s, t]$. From the observations above, it follows that g_1 is a homotopy equivalence whose torsion $\tau(g_1)$ equals

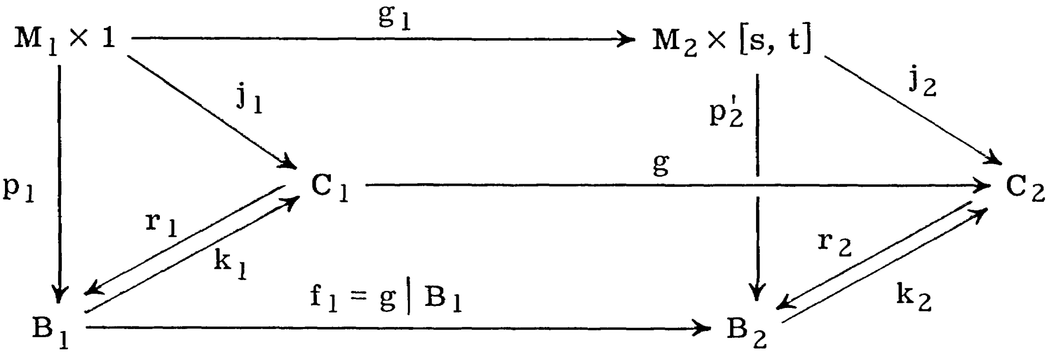
$$\tau(M_2 \times [s, t], M_1') = \tau(W, M_1') + \tau(W', M_1').$$

But now

$$\tau(W, M_1') = (-1)^n\,\tau(W, M_2)^* \quad \text{and} \quad \tau(W, M_2) + \tau(W', M_1') = \tau(W \cup W', M_2) = 0.$$

Hence $\tau(g_1) = -[\tau(W, M_2) + (-1)^{n+1}\,\tau(W, M_2)^*]$.

On the other hand, a consideration of the diagram



[where j_i and k_i $(i = 1, 2)$ are the inclusions, r_1 and r_2 are the obvious retractions, and p_2' is the composite of $p_2: M_2 \to B_2$ with the projection of $M_2 \times [s, t]$ on the first factor] shows that

$$p_2' g_1 = r_2 j_2 g_1 = r_2 g j_1 \simeq r_2 g (k_1 r_1) j_1 = (r_2 k_2) f_1 (r_1 j_1) \simeq f_1 p_1,$$

where $\simeq$ means "is homotopic to". Since p_2' is a bundle map, this implies that g_1 is homotopic to a fiber-homotopy equivalence g_1'. By Theorem A and the topological invariance of torsions (see [3] or [4]), $\tau(g_1) = \tau(g_1') = \chi(B_2)_{j*}\tau(h_1)$, where $h_1 = g_1' \mid F_1$. We complete the proof of the corollary by equating the two computations of $\tau(g_1)$ and setting $\tau_0 = -\tau(h_1)$.

2. THE HOMOMORPHISM p^*

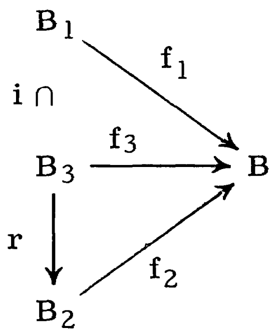
Let $p: E \to B$ be a PL fiber bundle with fiber F. It is the object of this section to construct a homomorphism $p^*: \mathrm{Wh}\,\pi_1(B) \to \mathrm{Wh}\,\pi_1(E)$ and to derive its main properties.

To define p^*, let $\tau_0 \in \mathrm{Wh}\,\pi_1(B)$ be an arbitrary element. Then there exist a polyhedron B_1 and a PL homotopy equivalence $f: B_1 \to B$ such that $\tau(f) = \tau_0$. Let $p_1: E_1 \to B_1$ be a PL bundle with fiber F such that there is a PL bundle map $g: E_1 \to E$ covering f. (For example, take $p_1: E_1 \to B_1$ to be the induced bundle.) Then g is also a homotopy equivalence, and we define $p^*\tau_0 = \tau(g)$.

LEMMA 2.1. $p^*\tau_0$ *is well-defined.*

Proof. Let $f_i: B_i \to B$ be PL homotopy equivalences $(i = 1, 2)$ such that $\tau(f_1) = \tau(f_2)$. Let $p_i: E_i \to B_i$ $(i = 1, 2)$ be PL bundles with fiber F for which there exist PL bundle maps $g_i: E_i \to E$ covering f_i $(i = 1, 2)$. We shall show that $\tau(g_1) = \tau(g_2)$.

Let $s: B_1 \to B_2$ be a simple homotopy equivalence such that f_1 is homotopic to $f_2 s$. (For example, take s to be the composite of f_1 and a homotopy inverse for f_2.) By [12, Section 13] and [11, Theorem 5], there exists a polyhedron B_3 containing B_1 and B_2 such that B_1 expands to B_3 and B_3 collapses to B_2, and such that ri is homotopic to s, where $i: B_1 \subset B_3$ is the inclusion and $r: B_3 \to B_2$ is a deformation retraction associated with the collapse $B_3 \searrow B_1$ in the sense of [12, Section 13]. Letting $f_3 = f_2 r$, we obtain a diagram



in which the upper triangle homotopy commutes and the lower triangle commutes. Let $p_3: E_3 \to B_3$ be induced from $p_2: E_2 \to B_2$ by r, and let $g_3 = g_2 r': E_3 \to E$, where $r': E_3 \to E_2$ is the usual bundle map covering r. Since $r: B_3 \to B_2$ is associated with a collapse, $\tau(r) = 0$, and the arguments of [1, Section 2] show that $\tau(r') = 0$. Hence $\tau(g_3) = \tau(g_2 r') = g_{2*}\tau(r') + \tau(g_2) = \tau(g_2)$.

Similarly, since $f_3 i$ is homotopic to f_1, there exists a PL bundle map $i': E_1 \to E_3$ covering i such that $g_3 i'$ is homotopic to g_1. Hence $\tau(g_1) = g_{3*}\tau(i') + \tau(g_3)$. But $\tau(i') = 0$ by [1, Lemma 2.1]. Hence $\tau(g_1) = \tau(g_3) = \tau(g_2)$, and the lemma is established.

A PL homotopy equivalence $f: B_1 \to B$ *represents* the element $\tau_0 \in \mathrm{Wh}\,\pi_1(B)$ if $\tau(f) = \tau_0$. Before proving that p^* is a homomorphism, we show how to construct a representative for $\tau_1 + \tau_2$, given representatives for τ_1 and τ_2.

Let $f_i: B_i \to B$ represent $\tau_i \in \mathrm{Wh}\,\pi_1(B)$ for $i = 1, 2$; let $h_i: B \to B_i$ be a PL homotopy inverse for f_i; and let $F_i: B \times I \to B$ be a homotopy with $F_i \mid B \times 0 = 1_B$ and $F_i \mid B \times 1 = f_i h_i$ for $i = 1, 2$. Let B_3 be the double mapping cylinder $B \times [-1, 1] \cup B_1 \cup B_2$ with $(b, -1)$ identified with $h_1(b)$ and $(b, 1)$ identified with $h_2(b)$. Then B_3 may be given a PL structure via the simplicial mapping cylinder. Define a map $k': B_3 \to B \times [-1, 1]$ by

$$k'(x) = (f_1(x), -1) \qquad \text{if } x \in B_1,$$

$$k'(x, t) = \begin{cases} (F_1(x, -t), t) & \text{if } x \in B \text{ and } -1 \leq t \leq 0, \\ (F_2(x, t), t) & \text{if } x \in B \text{ and } 0 \leq t \leq 1, \end{cases}$$

$$k'(x) = (f_2(x), 1) \qquad \text{if } x \in B_2.$$

Let k be a PL approximation to k' such that $k \mid B \times 0 = 1_B$ and $k \mid B_i = f_i$ $(i = 1, 2)$; set $f_3 = qk: B_3 \to B$, where $q: B \times [-1, 1] \to B$ is projection on the first factor.

LEMMA 2.2. *The map* $f_3: B_3 \to B$ *represents* $\tau_1 + \tau_2$.

Proof. Consider $k: B_3 \to B \times I$, and note that $B_3 = Z_{h_1} \cup Z_{h_2}$, where $Z_{h_1} \cap Z_{h_2} = B \times 0$ and Z_{h_i} is the mapping cylinder of h_i. Furthermore, the restrictions

$$k_1 = k \mid Z_{h_1}: Z_{h_1} \to B \times [-1, 0], \qquad k_2 = k \mid Z_{h_2}: Z_{h_2} \to B \times [0, 1],$$

$$k_0 = k \mid B \times 0: B \times 0 \to B \times 0$$

are all homotopy equivalences. By the Sum Theorem ([7] or [9, Theorem 6.9]), k is a homotopy equivalence and

$$\tau(k) = j_{1*}\tau(k_1) + j_{2*}\tau(k_2) - j_{0*}\tau(k_0),$$

where j_0, j_1, j_2 are the inclusions of $B \times 0$, $B \times [-1, 0]$, and $B \times [0, 1]$ into $B \times [-1, 1]$, respectively. Since $k_0 = k \mid B \times 0 = 1_B$, we see that $\tau(k_0) = 0$; also,

$$\tau(f_3) = \tau(qk) = q_*\tau(k) + \tau(q) = q_* j_{1*}\tau(k_1) + q_* j_{2*}\tau(k_2),$$

since q is simple. Finally, since the diagram

$$\begin{array}{ccccc} Z_{h_1} & \xrightarrow{k_1} & B \times [-1, 0] & \overset{j_1}{\subset} & B \times [-1, 1] \\ \cup & & & & \big\downarrow q \\ B_1 & & \xrightarrow{\quad f_1 \quad} & & B \end{array}$$

commutes, and the inclusions and q are simple, $q_* j_{1*}\tau(k_1) = \tau(f_1) = \tau_1$. Similarly, $q_* j_{2*}\tau(k_2) = \tau(f_2) = \tau_2$. Hence $\tau(f_3) = \tau_1 + \tau_2$. This completes the proof.

PROPOSITION 2.3. *The function* $p^*: \mathrm{Wh}\,\pi_1(B) \to \mathrm{Wh}\,\pi_1(E)$ *is a homomorphism.*

Proof. For $i = 1, 2$ let $f_i: B_i \to B$ represent $\tau_i \in Wh\pi_1(B)$, and let $f_3: B_3 \to B$ be the map representing $\tau_1 + \tau_2$ constructed above. Let $p_3: E_3 \to B_3$ be the bundle induced from $p: E \to B$ by f_3, and let $g_3: E_3 \to E$ be a PL bundle map covering f_3. Let

$$E_3' = p_3^{-1}(Z_{h_1}), \quad E_3'' = p_3^{-1}(Z_{h_2}), \quad g_3' = g_3 \mid E_3', \quad g_3'' = g_3 \mid E_3''.$$

An application of the Sum Theorem similar to the one above shows that $\tau(g_3) = \tau(g_3') + \tau(g_3'')$. Since the argument above shows that

$$f_3 \mid Z_{h_1} = q j_1 k_1 : Z_{h_1} \to B$$

represents τ_1, we see by Lemma 2.1 that $\tau(g_3') = p^* \tau_1$. Similarly, $\tau(g_3'') = p^* \tau_2$. Hence $p^*(\tau_1 + \tau_2) = p^* \tau(f_3) = \tau(g_3) = p^* \tau_1 + p^* \tau_2$ and p^* is a homomorphism.

LEMMA 2.4. *Let* $p_i: E_i \to B_i$ $(i = 1, 2)$ *be* PL *bundles with fiber* F, *and let* $k: E_1 \to E_2$ *be a* PL *bundle map covering the* PL *homotopy equivalence* $h: B_1 \to B_2$. *Then* $k_* p_1^* = p_2^* h_*$.

Proof. Let $f: B_0 \to B_1$ represent $\tau \in Wh\pi_1(B_1)$, and let $g: f^! E_1 \to E_1$ be a PL bundle map covering f. Then g represents $p_1^* \tau$. Now the equations

$$\begin{aligned} p_2^* \tau(h) + p_2^* h_* \tau &= p_2^*(\tau(h) + h_* \tau(f)) = p_2^* \tau(hf) = \tau(kg) \\ &= k_* \tau(g) + \tau(k) = k_* p_1^* \tau + p_2^* \tau(h) \end{aligned}$$

show that $p_2^* h_* = k_* p_1^*$; this completes the proof.

Let $p: E \to B$ be a PL bundle, and consider the PL bundle $q: E \times I^n \to B$, where $I^n = [-1, 1]^n$ is the n-cube, $q = p\pi$, and $\pi: E \times I^n \to E$ is projection on the first factor. Let $k: E \to E \times I^n$ be the inclusion $k(e) = (e, 0)$.

LEMMA 2.5. $q^* = k_* p^*$.

Proof. Let $f: B_1 \to B$ represent $\tau \in Wh\pi_1(B)$, and let $g: f'E \to E$ be a PL bundle map covering f. Since $g \times 1: f^! E \times I^n \to E \times I^n$ is also a PL bundle map covering f, we see that $q^* \tau = \tau(g \times 1)$. But since the diagram

$$\begin{array}{ccc} f^! E & \xrightarrow{g} & E \\ j \cap & & \cap k \\ f^! E \times I^n & \xrightarrow{g \times 1} & E \times I^n \end{array}$$

commutes and j and k are simple,

$$\tau(g \times 1) = \tau(g \times 1) + (g \times 1)_* \tau(j) = \tau((g \times 1)j) = \tau(kg) = k_* \tau(g) + \tau(k) = k_* \tau(g).$$

Hence $q^* \tau = \tau(g \times 1) = k_* \tau(g) = k_* p^* \tau$; this completes the proof.

3. THE PROOF OF THEOREM A

The proof is based on an analysis of the commutative diagram

$$\begin{array}{ccccccc}
F_1 & \xrightarrow{h} & F_2 & = & F_2 & \overset{k_2}{\subset} & F_2 \times I^n \\
{\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle j_3} & & \downarrow{\scriptstyle j_4} \\
E_1 & \xrightarrow{g} & E_2 & \overset{i_1}{\subset} & E_2 \times I^m & \overset{k_1}{\subset} & E_2 \times I^m \times I^n \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle (p_2 \times 1)} & & \downarrow{\scriptstyle p_3} \\
B_1 & \xrightarrow{f} & B_2 & \overset{i_2}{\subset} & B_2 \times I^m & = & B_2 \times I^m
\end{array}$$

where the two left-hand squares come from the hypothesis of Theorem A; i_1, i_2, k_1, k_2 are all the zero-section inclusions; and $p_3 = (p_2 \times 1)\pi$, where $\pi: E_2 \times I^m \times I^n \to E_2 \times I^m$ is the projection on the first factor.

LEMMA 3.1. *If* $\tau(k_1 i_1 g) = p_3^* \tau(i_2 f) + \chi(B_2) j_{4*} \tau(k_2 h)$, *then Theorem* A *holds.*

Proof. Since i_1, i_2, k_1, and k_2 are all simple equivalences, we have the relations

$$\begin{aligned}
k_{1*} i_{1*} \tau(g) &= \tau(k_1 i_1 g) = p_3^* \tau(i_2 f) + \chi(B_2) j_{4*} \tau(k_2 h) \\
&= k_{1*}(p_2 \times 1)^* \tau(i_2 f) + \chi(B_2) j_{4*} k_{2*} \tau(h) \\
&= k_{1*} i_{1*} p_2^* \tau(f) + \chi(B_2) k_{1*} i_{1*} j_{2*} \tau(h) = k_{1*} i_{1*}(p_2^* \tau(f) + \chi(B_2) j_{2*} \tau(h)),
\end{aligned}$$

by Lemmas 2.4 and 2.5 and the commutativity of the diagram. Since $k_{1*} i_{1*}$ is an isomorphism, the lemma follows.

THEOREM 3.2. *If* $m \geq 2 \dim B_1 + 1$ *and* $n \geq 2 \dim E_1 + 1$, *then*

$$\tau(k_1 i_1 g) = p_3^* \tau(i_2 f) + \chi(B_2) j_{4*} \tau(k_2 h).$$

The proof of Theorem 3.2 depends on several lemmas.

LEMMA 3.3. *Let* $m \geq 2 \dim B_1 + 1$, *and let* $b_0 \in B_1$ *be a base point. Then* $i_2 f$ *is homotopic, relative to* b_0, *to a* PL *embedding* $f': B_1 \to B_2 \times I^m$.

Proof. Let $F: B_1 \times I \to B_2$ be a homotopy between f and a PL map f_1 approximating f, and note that F may be taken to be relative to b_0. Similarly, let $G: B_1 \times I \to I^m$ be a homotopy relative to b_0 between the constant map to the origin and a PL embedding $f_2: B_1 \to I^m$. Then $H(x, t) = (F(x, t), G(x, t))$ gives the needed homotopy.

By the Covering Homotopy Theorem, the homotopy between $i_2 f$ and f' may be covered by a homotopy $H': E_1 \times I \to E_2 \times I^m$ that starts at $i_1 g$ and is stationary with H. In particular, if $b_0 \in B_1$ is the base point and $F_1 = p_1^{-1}(b_0)$, then $H' \mid F_1 \times t = h$ for all t. Setting $g_1 = H' \mid E_1 \times 1$, we obtain the commutative diagram

$$\begin{array}{ccc} F_1 & \xrightarrow{h} & F_2 \\ \downarrow & & \downarrow \\ E_1 & \xrightarrow{g_1} & E_2 \times I^m \\ \downarrow & & \downarrow \\ B_1 & \xrightarrow{f'} & B_2 \times I^m \end{array}$$

in which f' and g_1 are homotopic to $i_2 f$ and $i_1 g$, respectively, and where f' is a PL embedding.

LEMMA 3.4. *The map* g_1 *is fiberwise homotopic to a* PL *map* $g_2: E_1 \to E_2 \times I^m$.

We defer the proof to the end of this section.

LEMMA 3.5. *If* $n \geq 2 \dim E_1 + 1$, *then* $k_1 g_2$ *is fiberwise homotopic to a* PL *embedding* $g': E_1 \to E_2 \times I^m \times I^n$.

Proof. The proof is an obvious modification of the proof of Lemma 3.3.

By combining Lemmas 3.3, 3.4, and 3.5, we obtain a commutative diagram

$$\begin{array}{ccc} F_1 & \xrightarrow{h'} & F_2 \times I^n \\ {\scriptstyle j_1}\downarrow & & {\scriptstyle j_4}\downarrow \\ E_1 & \xrightarrow{g'} & E_2 \times I^m \times I^n \\ {\scriptstyle p_1}\downarrow & & {\scriptstyle p_3}\downarrow \\ B_1 & \xrightarrow{f'} & B_2 \times I^m \end{array}$$

in which f', g', h' are all PL embeddings and are homotopic to $i_2 f$, $k_1 i_1 g$, and $k_2 h$, respectively.

We are now ready for the proof of Lemma 3.2. By the remarks above, it suffices to prove that $\tau(g') = p^* \tau(f') + \chi(B_2) j_{4*} \tau(h')$. To prove this, consider the commutative diagram

$$\begin{array}{ccccc} F_1 & \xrightarrow{h'} & F_2 \times I^n & \xrightarrow{1} & F_2 \times I^n \\ {\scriptstyle j_1}\downarrow & & {\scriptstyle j}\downarrow & & {\scriptstyle j_4}\downarrow \\ E_1 & \xrightarrow{i} & E_3 & \xrightarrow{\bar{g}} & E_2 \times I^m \times I^n \\ {\scriptstyle p_1}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle p_3}\downarrow \\ B_1 & \xrightarrow{1} & B_1 & \xrightarrow{f'} & B_2 \times I^m \end{array}$$

where $E_3 = f'^{!}(E_2 \times I^m \times I^n)$, $\bar{g}$ and q are the usual maps, and $i(x) = (p_1(x), g'(x))$. Since $g = \bar{g}i$, we see that $\tau(g) = \bar{g}_* \tau(i) + \tau(\bar{g})$. But $\tau(\bar{g}) = p_3^* \tau(f')$, by the definition of p_3^*; also, $\tau(i) = \chi(B_1) j_* \tau(h')$, by [2], and $\chi(B_1) = \chi(B_2)$, since B_1 and B_2 have the same homotopy type. Hence

$$\tau(g) = \bar{g}_* \tau(i) + \tau(\bar{g}) = p_3^* \tau(f') + \bar{g}_* \chi(B_1) j_* \tau(h') = p_3^* \tau(f') + \chi(B_2) j_{4*} \tau(h'),$$

and the proof of Lemma 3.2 is complete.

We return now to the proof of Lemma 3.4, which requires two lemmas. To set notation, let Δ^n denote the standard n-simplex.

LEMMA 3.6. *Let* $g'\colon \Delta^n \times F \to F'$ *be any map, and let* $G'\colon \dot{\Delta}^n \times F \times I \to F'$ *be a homotopy of* $g' \mid \dot{\Delta}^n \times F$ *such that* $G' \mid \dot{\Delta}^n \times 1$ *is* PL. *Then* G' *extends to a homotopy of* g', $H'\colon \Delta^n \times F \times I \to F'$, *such that* $H' \mid \Delta^n \times F \times 1$ *is* PL.

Proof. Let $h\colon \Delta^n \times I \to \Delta^n \times I$ be a PL homeomorphism such that

$$h(\Delta^n \times 0 \cup \dot{\Delta}^n \times I) = \Delta^n \times 0 \quad \text{and} \quad h(\dot{\Delta}^n \times 1) = \dot{\Delta}^n \times 0.$$

Let $\tau\colon F \times I \to I \times F$ be the switching map. Then

$$f = (g' \cup G')(1 \times \tau)(h^{-1} \times 1)(1 \times \tau)\colon \Delta^n \times F \times 0 \to F'$$

is a continuous map that is PL on $\dot{\Delta}^n \times F \times 0$. By [13], there exists a homotopy $H\colon \Delta^n \times F \times I \to F'$ relative to $\dot{\Delta}^n \times F$ such that $H \mid \Delta^n \times F \times 0 = f$ and $H \mid \Delta^n \times F \times 1$ is PL. Let

$$H' = H(1 \times \tau)(h \times 1)(1 \times \tau).$$

Then, for each point $(x, z, t) \in \Delta^n \times F \times 0 \cup \dot{\Delta}^n \times F \times I$, we have the relation $H'(x, z, t) = (g' \cup G')(x, z, t)$, and therefore $H' \mid \Delta^n \times F \times 1$ is PL.

LEMMA 3.7. *Let* $p_i\colon E_i \to B_i$ *be a* PL *fiber bundle with fiber* F_i $(i = 1, 2)$, *and let* $g\colon E_1 \to E_2$ *be a fiberwise map covering a* PL *embedding* $f\colon B_1 \to B_2$. *Then* g *is fiberwise homotopic to a* PL *map.*

Lemma 3.4 is an obvious consequence of Lemma 3.7.

Proof. The proof is by induction on the dimension of B_1. Suppose the lemma holds when $\dim B_1 \leq n - 1$, and let $\dim B_1 = n$. Let K_1 and K_2 be triangulations of B_1 and B_2, respectively, such that $f\colon K_1 \to K_2$ is simplicial. Let K_0 be the $(n - 1)$-skeleton of K_1, let $B_0 = |K_0|$, and let $E_0 = p_1^{-1}(B_0)$. By the induction hypothesis, $g \mid E_0\colon E_0 \to E_2$ is fiberwise homotopic to a PL map. Let $G\colon E_0 \times I \to E_2$ be a fiberwise homotopy. We shall establish the lemma by extending G.

Let $\Delta_1^n \in K_1$ be an n-simplex, and set $\Delta_2^n = f(\Delta_1^n) \in K_2$. Let

$$h_i\colon \Delta_i^n \times F_i \to p_i^{-1}(\Delta_i^n) \qquad (i = 1, 2)$$

be PL homeomorphisms such that $p_i h_i(x, y) = x$ for all $(x, y) \in \Delta_i^n \times F_i$, and set $g' = \pi h_2^{-1} g h_1\colon \Delta_1^n \times F_1 \to F_2$, where $\pi\colon \Delta_2^n \times F_2 \to F_2$ is projection on the second factor. Consider the commutative diagram

$$\begin{array}{ccccccccc}
\dot{\Delta}_1^n \times F_1 \times I & \xrightarrow{h_1 \times 1} & E_0 \times I & \xrightarrow{G} & E_2 & \xleftarrow{h_2} & \dot{\Delta}_2^n \times F_2 & \xrightarrow{\pi'} & F_2 \\
\downarrow & & \downarrow{\scriptstyle (p_1 | E_0) \times 1} & & \downarrow{\scriptstyle p_2} & & \downarrow & & \\
\dot{\Delta}_1^n \times I & \xrightarrow{i \times 1} & B_0 \times I & \xrightarrow{f'} & B_2 & \xleftarrow{j} & \Delta_2^n & &
\end{array}$$

where i and j are the inclusions and $f'(x, t) = f(x)$ for all t. The map $G' = \pi' h_2^{-1} G((h_1 | \dot{\Delta}_1^n \times F) \times 1)$ is a homotopy from $g' | \dot{\Delta}_1^n \times F_1$ to a PL map. Let $H': \Delta_1^n \times F_1 \times I \to F_2$ be the homotopy of g' extending the map G' given by Lemma 3.6, and define $G_1: (E_0 \cup p_1^{-1}(\Delta_1^n)) \times I \to E_2$ by

$$G_1(x, t) = \begin{cases} G(x, t) & \text{if } (x, t) \in E_0 \times I, \\ h_2(j^{-1} f p_1(x), H'(h_1^{-1}(x), t)) & \text{if } (x, t) \in p_1^{-1}(\dot{\Delta}_1^n) \times I. \end{cases}$$

Then G_1 is well-defined and gives a fiberwise homotopy of $g | E_0 \cup p_1^{-1}(\Delta_1^n)$ to a PL map covering f.

An obvious inductive argument over the n-simplices of K_1 shows that G_1 extends to a fiberwise homotopy of g; this completes the proof.

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Syracuse University
Syracuse, New York 13210