ALGEBRAICALLY SEPARABLE EXTENSIONS OF BANACH ALGEBRAS

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Let A be a commutative complex Banach algebra with identity, and let X(A) be its carrier space. In this paper we explore the relation between the finite-fibered covering spaces Y of X(A) and the faithful, commutative, separable algebras over the commutative ring A. (An algebra S over the commutative ring R is *separable* if S is a projective $S \otimes_R S$-module [8, p. 40].) We begin by showing that every such algebra B over A is also a Banach algebra, and that the induced mapping $X(B) \to X(A)$ on carrier spaces is a finite-fibered covering space projection. Thus separable algebras lead to covering spaces. Using Silov's idempotence theorem, we next show that the covering space mappings between any covering spaces of X(A) that are carrier spaces of separable algebras as above are induced from algebra homomorphisms. In other words, the functor $B \to X(B)$ above is full and faithful. For the functor to be a (contravariant) equivalance, we need to know that every covering space of X(A) comes from an algebra B. We show that this is the case if A is a regular Banach algebra without radical.

This equivalence, for the case of full rings of complex-valued continuous functions on compact spaces, was established by B. Wajnryb [13] and L. Childs [7]; since such function algebras are regular and without radical, this theorem is a consequence of our results here. Child's proof used the fact that rings of germs of continuous functions at a point are Henselian [7, Lemma, p. 32]. Since such rings are the local rings at maximal ideals of full function rings, the question arises whether the local rings at maximal ideals of an arbitrary Banach algebra are Henselian. We show that this is the case for regular Banach algebras without radical, but that there are examples to show that this is in general false.

We adopt the following conventions: $\mathbb{C}$ is the complex field, and all the rings we consider are commutative $\mathbb{C}$-algebras with identity. We use X(A) for the carrier space of the Banach algebra A, in the usual topology, although occasionally we also use the hull-kernel topology on X(A). If A is a Banach algebra and $a \in A$, we let $\hat{a}: X(A) \to \mathbb{C}$ denote the Gelfand transform of a.

For our purposes, the particular norm on a Banach algebra is not important, since we are concerned primarily with carrier spaces, and these do depend not on the norm chosen for the algebra, but only on the fact that the algebra is complete in some norm. Thus we refer throughout to *Banachable* algebras, by which we mean an algebra A such that there is some norm on A making A a Banach algebra.

If A is a Banachable algebra and $f \in A$, let $U_f = \{x \in X(A): \hat{f}(x) \neq 0\}$.

Our first goal is to show that a finitely generated projective faithful extension algebra of a Banachable algebra is also Banachable. We begin with some standard facts about Banach modules.

Received February 15, 1974.
This research was partially supported by NSF Grant GP-37051.

Michigan Math. J. 21 (1974).

Definition 1. Let A be a Banach algebra. A *Banach* A-*module* M is a unitary A-module that is a Banach space such that $\|am\| \leq \|a\| \, \|m\|$ for all a in A and all m in M. The free Banach A-module of rank n, $A^{(n)}$, is the free A-module $A^{(n)}$ with norm $\|(a_1, \cdots, a_n)\| = \|a_1\| + \cdots + \|a_n\|$. (It is an elementary exercise to verify that $A^{(n)}$ is in fact a Banach A-module in this norm.)

LEMMA 2. *Let* $T: A^{(n)} \to A^{(m)}$ *be an* A-*module homomorphism. Then* T *is bounded as a linear mapping of Banach spaces.*

Proof. Let (t_{ij}) be the matrix of T in the standard bases on $A^{(n)}$ and $A^{(m)}$, and let $M = \max(\|t_{ij}\|)$. It is easy to see that $\|Tx\| \leq mM\|x\|$ for each x in $A^{(n)}$.

LEMMA 3. *Let* P *be an* A-*module direct summand of* $A^{(n)}$. *Then* P *is a closed subspace of the Banach space* $A^{(n)}$.

Proof. There exists an A-module homomorphism $T: A^{(n)} \to A^{(n)}$ such that P is the kernel of T. (T denotes projection on a complementary summand to P.) By Lemma 2, T is continuous, and hence its kernel is closed.

The extension theorem will follow from Lemma 3 and the following result of R. Arens and K. Hoffman [2, 3.6, p. 205]: Let A be a Banach algebra, and let f in A[X] be a monic polynomial. Then there exists a norm on the A-algebra B = A[X]/(f) that extends the norm on A and under which B is a Banach algebra isometric, as a Banach A-module, to $A^{(n)}$.

THEOREM 4. *Let* A *be a Banachable algebra and* B *a faithful* A-*algebra, finitely generated and projective as an* A-*module. Then* B *is Banachable.*

Proof. Since B is integral over A and finitely generated, there exist elements $b_1, \cdots, b_n$ in B that generate B as an A-algebra, and each b_i satisfies a monic polynomial f_i in A[X] of degree d_i. Let $B_0 = A$ and $B_i = B_{i-1}[X]/(f_i)$ for $i = 1, \cdots, n$. If B_{i-1} is normed so that it is a Banach algebra, then by the result of Arens and Hoffman we can extend this norm to B_i so that it is a Banach algebra isometric (as a Banach B_{i-1}-module) to $B_{i-1}^{(d_i)}$. By induction, we see that every Banach norm on A extends to a Banach norm on B_n, where B_n is isometric, as a Banach A-module, to $A^{(d)}$ $(d = d_1 d_2 \cdots d_n)$. By construction, we have an A-algebra homomorphism $f: B_n \to B$. Since B is projective, the kernel of f is a direct summand of B_n. By Lemma 3, this kernel is closed. Thus if we norm B by using the quotient seminorm, this seminorm is actually a norm making B a Banach algebra.

Next, we want to consider the case where the A-algebra B is A-separable, that is, projective as a $B \otimes_A B$-module. We refer the reader to [8] for information on separable algebras.

THEOREM 5. *Let* A *be a Banachable algebra, and let* B *be a faithful, separable* A-*algebra, finitely generated and projective as an* A-*module. Then the continuous mapping* $X(B) \to X(A)$ *is a covering space projection.*

Proof. By [12, p. 166], there exists a faithful B-algebra C that is a Galois extension of A with (finite) group G (in the sense of [8, p. 84]), and there exists a subgroup H of G such that B is the set of elements of C invariant under H. By Theorem 4, C and B are Banachable. By [5, Theorem 2, p. 331] and [8, p. 81], G acts without fixed points on X(C), and X(C)/G = X(A) and X(C)/H = X(B). Since G acts continuously without fixed points on a compact Hausdorff space, $X(C) \to X(A)$ is a covering space, and hence so is $X(B) = X(C)/H \to X(A)$.

D. Brown [4, Remark 2.1, p. 588] and J. Lindberg [10, Proposition 1.3, p. 358] have proved special cases of Theorem 5. They require that the algebra B of the theorem be of the form A [X]/(f), where f is a monic polynomial in A [X].

Next, we consider to what extent the covering projection of Theorem 5 determines the algebra B.

LEMMA 6. *Let* A *be a Banachable algebra, and let* B *be a faithful* A*-algebra finitely generated and projective as an* A*-module. Then* $B \otimes_A B$ *is Banachable, and there exists a homeomorphism* $X(B \otimes_A B) \to X(B) \times_{X(A)} X(B)$.

Proof. The first assertion follows immediately from Theorem 4. Concerning the second, we observe that the maps $b \to b \otimes 1$ and $b \to 1 \otimes b$ induce a continuous bijection $X(B \otimes_A B) \to X(B) \times_{X(A)} X(B)$. Since both spaces are compact Hausdorff spaces, the mapping is a homeomorphism.

We also need Silov's idempotence theorem [9, 4.1, p. 51]: If A is a Banachable algebra and X(A) is the disjoint union of open-closed sets U and V, there exists an idempotent e of A with $0 = \hat{e}(U)$ and $1 = \hat{e}(V)$.

THEOREM 7. *Let* A *be a Banachable algebra, and let* B *and* C *be faithful separable* A*-algebras, finitely generated and projective as* A*-modules. Suppose there exists a continuous function* f: X(B) → X(C) *commuting with the projections on* X(A). *Then* f *is induced by a unique* A*-algebra homomorphism* g: C → B.

Proof. A-algebra homomorphisms of C to B are the same as B-algebra homomorphisms of $B \otimes_A C$ to B, and continuous maps of X(B) to X(C) commuting with the projections on X(A) are the same as continuous sections of $X(B) \times_{X(A)} X(C) \to X(B)$. By using Lemma 6, then, we replace C by $B \otimes_A C$, and we can assume B = A. Since X(C) → X(A) is a covering projection, the image of f is open and closed in X(C). By Silov's idempotence theorem, there is an idempotent element e of C with $\hat{e} = 0$ on the image of f and $\hat{e} = 1$ elsewhere. If we replace C by C/Ce, then f is a homeomorphism. Now, if M is a maximal ideal of A, then C/MC is a separable A/M-algebra with only one maximal ideal, so that C/MC = A/M. This is true for all such M, and since C is finitely generated and projective over A, we see that A = C. Backing up, we see further that C/Ce = A, and the projection C → C/Ce induces the section f: X(A) → X(C). By [6, 1.2, p. 3], every other A-algebra map C → A inducing f is of the form C → C/Ce', for some idempotent e' of C. It is easy to see that $\hat{e} = \hat{e}'$, so that $e \equiv e' \pmod{\mathrm{rad}(C)}$. But idempotents congruent modulo the radical are equal, by a well-known result. It follows that the map inducing f is unique.

We can rephrase Theorem 7 categorically:

COROLLARY 8. *Let* A *be a Banachable algebra. Then the functor* B → X(B) *from the category of faithful separable* A*-algebras that are finitely generated and projective as* A*-modules to the category of finite covering spaces of* X(A) *is full and faithful.*

We would like to know that the functor is actually an equivalence of categories, in other words, that every covering space of X(A) comes from a separable algebra. We shall show that this is the case if A is regular and has no radical (recall that a Banach algebra is regular if the hull-kernel topology on its carrier space is the same as the Gelfand topology [9, Section 3.1, p. 54]). We shall need to deal with the sheaf of functions locally in a Banach algebra, and we begin with a definition.

Definition 9. Let A be a Banachable algebra, Y a subspace of X(A), and s: Y → $\mathbb{C}$ a continuous function. We say s is *locally in* A if for all y ∈ Y there exist a neighborhood U of y in X and an element a of A such that $\hat{a} \mid Y \cap U = s \mid Y \cap U$. Let $\mathscr{S}_A(Y)$ denote the set (actually algebra) of all functions on Y locally in A.

PROPOSITION 10. *Let* A *be a regular Banach algebra, and let* F *be a closed subset of* X(A). *Then the restriction mapping* $\mathscr{S}_A(X(A)) \to \mathscr{S}_A(F)$ *is surjective.*

Proof. For each Y, the set $\mathscr{S}_A(Y)$ depends only on the Gelfand transforms of the elements of A; therefore, we can assume A has radical zero. Let h(F) be the hull of F in A. Clearly, s: $F \to \mathbf{C}$ is locally in A if and only if it is locally in $B = A/h(F)$. Note that B is also regular and without radical, and that $X(B) = F$. By [9, Theorem 5.1, p. 56], $\mathscr{S}_B(X(B)) = B$ and $\mathscr{S}_A(X(A)) = A$. Since $A \to B$ is surjective, so is $\mathscr{S}_A(X(A)) \to \mathscr{S}_B(X(B)) = \mathscr{S}_A(F)$.

We want to restate Proposition 10 in a sheaf-theoretical language. First we recall that with each Banach algebra A there is associated a sheaf on X(A) whose stalk at x is the algebra of germs of transforms of elements of A (see [9, Section 5.4, p. 61]). Clearly, the set of sections of this sheaf over a subset Y of X(A) is the set $\mathscr{S}_A(Y)$ above. Thus Proposition 10 asserts that if A is regular, the sheaf $\mathscr{S}_A$ is soft [3, 9.1, p. 47]. Since X(A) is compact, the set of all closed subsets of X(A) is paracompactifying [3, 6.1, p. 15], and hence by [3, 9.12, p. 50], if A is regular, each (sheaf) module over the sheaf $\mathscr{S}_A$ of rings is also soft. We shall use this remark in our next theorem.

THEOREM 11. *Let* A *be a regular Banach algebra without radical, and let* p: $Y \to X(A)$ *be a covering space with finite-fibers. Then there exist a faithful separable* A*-algebra* B *(finitely generated and projective as an* A*-module) and a homeomorphism* c: $Y \to X(B)$ *commuting with the projections on* X(A).

Proof. Let $\mathscr{A} = \mathscr{S}_A$, and let $\mathscr{B} = p_* p^* \mathscr{A}$. The sheaf $\mathscr{B}$ is a sheaf of $\mathscr{A}$-algebras, and we shall show that $\mathscr{B}(X(A))$ is the desired algebra. By the remarks above, $\mathscr{B}$ is soft; therefore, if F is a closed subset of X(A), then the mapping $\mathscr{B}(X) \to \mathscr{B}(F)$ is surjective. If U is a neighborhood in X(A) evenly covered by p, then as rings $\mathscr{B} \mid_U = (\mathscr{A} \mid_U)^{(n)}$ (for suitable n). Thus, if F is a closed subset of an evenly covered neighborhood in X(A), it follows from [9, Theorem 5.1, p. 56] that $\mathscr{B}(F) = (\mathscr{A}(F))^{(n)}$.

Now let $B = \mathscr{B}(X(A))$, and let $x \in X(A)$. There exist an evenly covered neighborhood U of x and an element f of A such that $x \in U_f$ and $F = \overline{U}_f$ is contained in U. Let I be the hull of F in A. Then, by the argument in the preceding paragraph, we have a surjection $B_f \to \mathscr{B}(F)_f$, and $\mathscr{B}(F) = (A/I)^{(n)}$ (since A is regular and without radical, $\mathscr{A}(F) = A/I$ as in the proof of Proposition 10), and hence a surjection $B_f \to (A/I)_f^{(n)}$. Suppose b/1 is in the kernel. Then, for some m, the global section $f^m b$ vanishes on F. Since $U_f \subseteq F$, the section $f^m b$ vanishes outside F also, and hence $f^m b = 0$. Thus $b/1 = 0$, and the surjection $B_f \to (A/I)_f^{(n)}$ is an isomorphism. Finally, $(A/I)_f = A_f$, and therefore B_f is isomorphic to $(A_f)^{(n)}$. We can carry out this construction for any x in X(A). By [5] and [6], then, B is a faithful separable A-algebra, finitely generated and projective as an A-module.

Now $p^* \mathscr{A}$ is a sheaf on Y with $p^* \mathscr{A}(p^{-1}(U)) = \mathscr{B}(U)$ for each open set U of X(A). In particular, $B = p^* \mathscr{A}(Y)$. Let c: $Y \to X(B)$ be the mapping defined by $c(y)(b) = b(y)$ for a point y of Y and a section b in B. We want to show that c is a homeomorphism. If $f \in A$ is such that U_f is evenly covered, then by the first paragraph of the proof the restriction of c to $p^{-1}(U_f)$ is a homeomorphism. To complete the proof, it will be sufficient to show that each pair y, z of points of Y identified by c must lie in such a set $p^{-1}(U_f)$. If $p(y) \neq p(z)$, there is an element a of A such that $\hat{a}(p(y)) \neq \hat{a}(p(z))$. Then the function $b = \hat{a}p$ belongs to B, and $c(y)(b) \neq c(z)(b)$. Thus c is a homeomorphism, and the theorem follows.

We summarize Theorems 8 and 11 in categorical language as follows:

COROLLARY 12. *Let* A *be a regular Banach algebra without radical. Then the category of faithful, separable* A*-algebras that are finitely generated projective* A*-modules and the category of finite-fibered covering spaces of* X(A) *are contravariantly equivalent.*

Corollary 12 applies, in particular, to rings of all continuous functions on a compact space, and hence we recover [7, Theorem 2, p. 30]. There are, of course, many regular algebras without radical that are not full function rings (see [9, Section 5.1, p. 54], for example).

Next, we examine a technical question arising from the proof of Corollary 12 for full function algebras given in [7]. The technique used there to prove the analogue for full function algebras of our Theorem 5 is essentially to use the fact that the ring of germs of continuous functions is strictly Henselian (that is, is Henselian with algebraically closed residue field; see [11, Proposition 4, p. 78] for the assertion; note that the proof of [7, p. 32] is similar to the proof of [11, p. 78]). This raises the question: are the local rings of every Banachable algebra strictly Henselian? We shall show that the answer is positive for regular algebras, but negative in general.

We shall need the following theorem of Arens and Calderon [1, 4.1, p. 208]: Let A be a Banach algebra. If $g = \sum b_i X^i$ is a polynomial in A[X], we use $\hat{g}$ to denote the polynomial $\sum \hat{b}_i X^i$. Now suppose that f is a polynomial in A[X] such that there is a continuous function $h: X(A) \to \mathbb{C}$ such that $\hat{f}(h) = 0$, while $\hat{f}'(h)$ is a unit. Then there exists an a in A such that $\hat{a} = h$ and $f(a) = 0$.

We also require some notation: if X is a topological space, $x \in X$, and $f: X \to \mathbb{C}$ is a continuous function, we let $[f]_x$ denote the germ of f at x. If A is a Banach algebra and $x \in X(A)$, we let $[A]_x = \{[\hat{a}]_x : a \in A\}$. If $M = \{a \in A: \hat{a}(x) = 0\}$, we have a homomorphism $A_M \to [A]_x$ [9, Section 5.4, p. 59] that is an isomorphism if A is regular and without radical [9, Section 5.4, p. 60]. We can now prove our basic result.

THEOREM 13. *Let* A *be a regular Banach algebra without radical, and let* M *be a maximal ideal of* A. *Then the local ring* A_M *is strictly Henselian.*

Proof. We use the criteria of [11, Proposition 3, p. 76] to show that A_M is a Henselian ring: let $P \in A_M[T]$ be a monic polynomial whose image $\overline{P}$ in $\mathbb{C}[T]$ has a simple root $\bar{a}$. Then P has a root in A_M that lifts $\bar{a}$. Our hypotheses on A imply that A_M can be identified with $[A]_x$, where $x \in X(A)$ is such that $M = \{a \in A: \hat{a}(x) = 0\}$; with this identification, P becomes

$$T^n + [\hat{a}_1]_x T^{n-1} + \cdots + [\hat{a}_n]_x,$$

for suitable $a_i \in A$. Let B be the ring of all germs at x of continuous functions $X(A) \to \mathbb{C}$, and regard $[A]_x$ as a subring. Since B is Henselian, the assumptions on P imply that there exists a continuous function $h: X(A) \to \mathbb{C}$ such that $P([h]_x) = 0$ and $h(x) = \bar{a}$. Since $\overline{P}'(h(x)) \neq 0$, we also see that $P'([h]_x)$ is a unit in B. Let $Q = T^n + \hat{a}_1 T^{n-1} + \cdots + \hat{a}_n$. The argument above implies that there exists an $s \in A$

with the following properties: let $U_s = \{y \in X(A): \hat{s}(y) \neq 0\}$. Then x belongs to U_s, and for all $y \in U_s$, we see that $Q(h)(y) = 0$ while $Q'(h)(y) \neq 0$. Choose $t \in A$ so that $x \in U_t$, and so that the closure F of U_t is contained in U_s. Let

$$I = h(F) = \{a \in A: a(y) = 0 \text{ for all } y \in F\}.$$

By restriction, regard Q as an element of $A/I[T]$. By the theorem of Arens and Hoffman quoted above, there exists an element b of A/I such that $\hat{b} = h$. Let a in A map to b. Then $[\hat{a}]_x = [h]_x$; we have thus produced the desired root of P, and A_M is Henselian. It is strictly Henselian because its residue class field is $\mathbb{C}$.

Finally, we give an example to show that Theorem 13 may fail for nonregular algebras.

Example 14. Let $D = \{z \in \mathbb{C}: 1 \leq |z| \leq 5\}$, and let A be the Banach algebra of all continuous functions on D analytic on the interior of D. We identify X(A) with D in the usual manner. Let z denote the identity function of D, and let B denote the A-algebra $A[X]/(X^2 - z)$. The algebra B is clearly faithful, and also B is finitely generated and projective as an A-module. Since the element z of A is a unit, B is also a separable A-algebra (see [8, p. 113]). By Theorem 5, the mapping $p: X(B) \to X(A)$ is a covering space projection, which must be equivalent to the standard double covering of D. Let $M = \{a \in A: \hat{a}(3) = 0\}$, and suppose that A_M is strictly Henselian. Then A_M is separably closed, and hence $B_M = A_M^{(2)}$. Therefore there exists an $s \in A$ such that $B_s = A_s^{(2)}$, and hence such that $p^{-1}(U_s)$ is a disjoint union of two copies of U_s. Let $D' = \{z \in \mathbb{C}: 2 \leq |z| \leq 4\}$. Then, since s is analytic on a neighborhood of D', the set $\hat{s}^{-1}(0) \cap D'$ is finite. It follows that we can find a mapping ϕ of the circle into D', of degree 1, whose range is contained in U_s. But then ϕ lifts to a mapping of the circle to $X(B)$. But a mapping of the circle of degree 1 onto D can not be lifted to the standard double covering of D. This contradiction shows that our assumption is false; that is, A_M is not a Henselian ring.

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