SYLVESTER'S PARTITION THEOREM, AND A RELATED RESULT

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For $k > 0$, let $\Pi_d(k)$ denote the set of partitions of k into distinct parts. For $\Pi \in \Pi_d(k)$, let $s(\Pi)$ be the number of sequences of consecutive integers in Π, and let $g(\Pi)$ be the number of gaps in Π. That is, let

$$g(\Pi) = s(\Pi) - 1$$

if the smallest part in Π is 1, while

$$g(\Pi) = s(\Pi)$$

if the smallest part in Π is greater than 1.

For $k > 0$ and $r \geq 0$, let $A(k, r)$ denote the number of partitions of k into odd parts (repetitions allowed) exactly r of which are distinct, $B(k, r)$ the number of $\Pi \in \Pi_d(k)$ with $s(\Pi) = r$, $C(k, r)$ the number of partitions of k into even parts (repetitions allowed) exactly r of which are distinct, and $D(k, r)$ the number of $\Pi \in \Pi_d(k)$ with $g(\Pi) = r$, and let

$$A(0, 0) = B(0, 0) = C(0, 0) = D(0, 0) = 1,$$

$$A(k, r) = B(k, r) = C(k, r) = D(k, r) = 0 \qquad \text{otherwise.}$$

We shall prove the following two results.

THEOREM 1. $B(k, r) = A(k, r)$ *for all* k *and* r.

THEOREM 2. $D(k, r) = C(k, r) + C(k - 1, r) + C(k - 3, r) + C(k - 6, r) + \cdots$ *for all* k *and* r.

Theorem 1 was proved arithmetically by J. J. Sylvester [4, Section 46]. Recently, G. E. Andrews [1, Section 2] gave a proof of Theorem 1 by means of generating functions. Our proofs also make use of generating functions; but they are more direct.

V. Ramamani and K. Venkatachaliengar [3, Section 2] have given a combinatorial proof of Theorem 1. A similar proof is available for Theorem 2.

For $k > 0$ and $r, n \geq 0$, let $B(k, r, n)$ denote the number of $\Pi \in \Pi_d(k)$ with $s(\Pi) = r$, and with no part greater than n, let

$$B(0, 0, n) = 1 \qquad \text{for } n \geq 0,$$

$$B(k, r, n) = 0 \qquad \text{otherwise},$$

and let

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$$B_n(a, q) = \sum_{k,r} B(k, r, n)\, a^r q^k,$$

$$B(a, q) = \lim_{n\to\infty} B_n(a, q) = \sum_{k,r} B(k, r)\, a^r q^k.$$

Andrews has shown [1, equation (2.3)] that

$$(1) \qquad B_n(a, q) = (1+q^n) B_{n-1}(a, q) + (a-1) q^n B_{n-2}(a, q) \quad \text{for } n \geq 2,$$

with

$$B_0(a, q) = 1 \quad \text{and} \quad B_1(a, q) = 1 + aq.$$

Multiplying both sides of (1) by z^n and summing over n, we see that if

$$B(a, q, z) = \sum_n B_n(a, q) z^n,$$

then

$$(1-z) B(a, q, z) = (1 + (a-1)qz) + qz(1 + (a-1)qz) B(a, q, qz).$$

That is,

$$B(a, q, z) = \frac{(1 + (a-1)qz)}{(1-z)} + \frac{qz(1 + (a-1)qz)}{(1-z)} B(a, q, qz).$$

It follows by iteration that

$$B(a, q, z) = \sum_{r\geq 0} \frac{q^{r(r+1)/2} z^r (-(a-1)qz)_{r+1}}{(z)_{r+1}},$$

where

$$(a)_r = (1-a)(1-aq)\cdots(1-aq^{r-1}).$$

Employing Abel's lemma [2, p. 101] and the well-known identity

$$\sum_{r\geq 0} \frac{q^{r(r+1)/2} (b)_r}{(q)_r} = \prod_{r\geq 0} \frac{1 - bq^{2r+1}}{1 - q^{2r+1}},$$

we obtain the relations

$$\sum_{k,r} B(k, r)\, a^r q^k = B(a, q) = \lim_{n\to\infty} B_n(a, q) = \lim_{z\to 1-} (1-z) B(a, q, z)$$

$$= \sum_{r\geq 0} \frac{q^{r(r+1)/2} (-(a-1)q)_{r+1}}{(q)_r}$$

$$= (1 + (a-1)q) \sum_{r\geq 0} \frac{q^{r(r+1)/2} (-(a-1)q^2)_r}{(q)_r}$$

$$= (1 + (a - 1)q) \prod_{r \geq 0} \frac{1 + (a - 1)q^{2r+3}}{1 - q^{2r+1}}$$

$$= \prod_{r \geq 0} \frac{1 + (a - 1)q^{2r+1}}{1 - q^{2r+1}} = \prod_{r \geq 0} \left(1 + \frac{aq^{2r+1}}{1 - q^{2r+1}}\right) = \sum_{k,r} A(k, r) a^r q^k .$$

Therefore

$$B(k, r) = A(k, r),$$

and this is Theorem 1.

To prove Theorem 2, we proceed similarly.

For $k > 0$ and $r, n \geq 0$, let $D(k, r, n)$ denote the number of $\Pi \in \Pi_d(k)$ with $g(\Pi) = r$ and with no part greater than n, let

$$D(0, 0, n) = 1 \quad \text{for } n \geq 0,$$

$$D(k, r, n) = 0 \quad \text{otherwise},$$

and let

$$D_n(a, q) = \sum_{k,r} D(k, r, n) a^r q^k,$$

$$D(a, q) = \lim_{n \to \infty} D_n(a, q) = \sum_{k,r} D(k, r) a^r q^k .$$

Exactly as Andrews proved (1), we may establish that

$$D_n(a, q) = (1 + q^n) D_{n-1}(a, q) + (a - 1) q^n D_{n-2}(a, q) \quad \text{for } n \geq 2,$$

with

$$D_0(a, q) = 1 \quad \text{and} \quad D_1(a, q) = 1 + q .$$

If

$$D(a, q, z) = \sum_n D_n(a, q) z^n,$$

then

$$(1 - z) D(a, q, z) = 1 + qz(1 + (a - 1)qz) D(a, q, qz) .$$

That is,

$$D(a, q, z) = \frac{1}{(1 - z)} + \frac{qz(1 + (a - 1)qz)}{(1 - z)} D(a, q, qz) .$$

It follows by iteration that

$$D(a, q, z) = \sum_{r \geq 0} \frac{q^{r(r+1)/2} z^r (-(a - 1)qz)_r}{(z)_{r+1}} .$$

Hence,

$$\sum_{k,r} D(k, r)\, a^r q^k = D(a, q) = \lim_{n\to\infty} D_n(a, q) = \lim_{z\to 1-} (1 - z) D(a, q, z)$$

$$= \sum_{r\geq 0} \frac{q^{r(r+1)/2}(-(a - 1)q)_r}{(q)_r} = \prod_{r\geq 0} \frac{1 + (a - 1)q^{2r+2}}{1 - q^{2r+1}}$$

$$= \prod_{r\geq 0} \frac{1 - q^{2r+2}}{1 - q^{2r+1}} \prod_{r\geq 0} \frac{1 + (a - 1)q^{2r+2}}{1 - q^{2r+2}}$$

$$= \prod_{r\geq 0} \frac{1 - q^{2r+2}}{1 - q^{2r+1}} \prod_{r\geq 0} \left(1 + \frac{aq^{2r+2}}{1 - q^{2r+2}}\right)$$

$$= \prod_{r\geq 0} \frac{1 - q^{2r+2}}{1 - q^{2r+1}} \sum_{k,r} C(k, r)\, a^r q^k = \sum_{r\geq 0} q^{r(r+1)/2} \sum_{k,r} C(k, r)\, a^r q^k .$$

It follows that

$$D(k, r) = C(k, r) + C(k - 1, r) + C(k - 3, r) + C(k - 6, r) + \cdots,$$

and this is Theorem 2.

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