THE DERIVATIVE OF A HOLOMORPHIC FUNCTION IN THE DISK

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1. INTRODUCTION

J. E. McMillan pointed out that the derived function of a univalent holomorphic function in the disk D: $|z| < 1$ is normal in D, in the sense of O. Lehto and K. I. Virtanen [4, p. 47]; for the proof, see [5] or [6]. In this note, we first improve McMillan's result and then investigate relations between local univalency of a holomorphic function f in D and boundary properties of f'. For $z, w \in D$, write

$$\delta(z, w) = \left|\frac{z - w}{1 - \bar{w}z}\right|, \qquad \gamma(z, w) = \frac{1}{2}\log\frac{1 + \delta(z, w)}{1 - \delta(z, w)}.$$

If $z \in D$ and $f'(z) \neq 0$, let $\tau(z) \equiv \tau(z, f)$ be the greatest value γ such that f is univalent in the hyperbolic disk $\{\zeta \in D: \gamma(\zeta, z) < \gamma\}$; if $f'(z) = 0$, we set $\tau(z, f) = 0$. Our first result is the following; it has McMillan's theorem as a corollary.

THEOREM 1. *Let* f *be holomorphic in* D, *and suppose that*

$$\inf_{z \in R(r)} \tau(z, f) > 0,$$

where R(r) *is the annulus* $r < |z| < 1$ $(0 < r < 1)$. *Then* f' *is normal in* D.

2. PROOF OF THEOREM 1

Let $\rho(z) \equiv \rho(z, f)$ $(z \in D)$ be the greatest value δ $(0 < \delta < 1)$ such that f is univalent in $\{\zeta \in D: \delta(\zeta, z) < \delta\}$; if $f'(z) = 0$, we set $\rho(z, f) = 0$.

LEMMA 1. *At each point* $z \in D$ *where* $\rho(z, f) > 0$, *we have the inequality*

$$\left|\frac{f''(z)}{f'(z)} - \frac{2\bar{z}}{1 - |z|^2}\right| \leq \frac{4}{\rho(z, f)(1 - |z|^2)}. \tag{1}$$

Proof. For a fixed δ $(0 < \delta < \rho(z))$, we set $f_z(\xi) \equiv f(\zeta)$, where $(\zeta - z)/(1 - \bar{z}\zeta) = \delta\xi$ for all $\xi \in D$. Then the function

$$f_z(\xi) = f\left(\frac{z + \delta\xi}{1 + \delta\bar{z}\xi}\right)$$

is univalent in $|\xi| < 1$. Applying the Bieberbach inequality $|b_2| \leq 2$ to the coefficient of ξ^2 in the expansion in powers of ξ of the function

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$$\frac{f_z(\xi) - f_z(0)}{\left[\dfrac{df_z(\xi)}{d\xi}\right]_{\xi=0}} = \frac{f_z(\xi) - f(z)}{\delta f'(z)(1 - |z|^2)},$$

we obtain the bound

$$\left|\frac{f''(z)}{f'(z)} - \frac{2\bar{z}}{1 - |z|^2}\right| \leq \frac{4}{\delta(1 - |z|^2)}.$$

Since δ is arbitrary, this implies (1).

Proof of Theorem 1. Since $\rho(z) > 0$, it follows from (1) and the inequality $t + t^{-1} \geq 2$ for $t > 0$ that

$$\frac{|f''(z)|}{1 + |f'(z)|^2} \leq \left\{|z| + \frac{2}{\rho(z, f)}\right\} \frac{1}{1 - |z|^2}. \tag{2}$$

Set

$$\phi(z) = (1 - |z|^2)\frac{|f''(z)|}{1 + |f'(z)|^2},$$

for $z \in D$. By the assumption of the theorem, $\inf_{z \in R(r)} \rho(z, f) > 0$; combined with (2), this means that ϕ is bounded in $R(r)$. On the other hand, the continuous function ϕ is bounded in the disk $|z| \leq (1 + r)/2$, whence ϕ is bounded in D. It follows from a well-known theorem [4, Theorem 3] that f' is normal in D.

Remark. It is an open question whether there exists a holomorphic function f in D such that $\tau(z, f) > 0$ in D, $\inf_{z \in D} \tau(z, f) = 0$, and f' is normal in D.

3. THE BOUNDARY BEHAVIOR OF τ

By an *angular domain* at a point ζ of the circle Γ: $|z| = 1$ we mean a triangular domain whose vertices are ζ and two points of D. By an *admissible arc at* ζ we mean a continuous curve Λ: $z = z(t)$ in D $(0 \leq t < 1)$, with $\lim_{t \to 1} z(t) = \zeta$, and tangent at ζ to a chord of Γ at ζ. Let f be holomorphic in D. We say that a point ζ on Γ is of the *first, second,* or *third* kind if

$$\liminf_{z \to \zeta} \tau(z, f) = 0 \text{ on each admissible arc at } \zeta,$$

$$\liminf_{z \to \zeta} \tau(z, f) > 0 \text{ in each angular domain at } \zeta,$$

$$\liminf_{z \to \zeta,\, z \in D} \tau(z, f) > 0,$$

respectively. Let $\Gamma_1(f)$, $\Gamma_2(f)$, and $\Gamma_3(f)$ be the sets of all points of the first, second, and third kind, respectively. Plainly, $\Gamma_3(f) \subset \Gamma_2(f)$ and $\Gamma_1(f) \cap \Gamma_2(f) = \emptyset$.

THEOREM 2. *Let* f *be holomorphic in* D. *Then* $\Gamma_1(f) \cup \Gamma_2(f)$ *has linear Lebesgue measure* 2π *and is a residual set on* Γ, *and* $\Gamma_1(f) \cup \Gamma_3(f)$ *is residual on* Γ.

LEMMA 2. *The inequality* $|\rho(z, f) - \rho(w, f)| \leq \delta(z, w)$ *holds for all* $z, w \in D$.

Proof. If $\rho(z) = \rho(w)$, we have nothing to prove. By the symmetry of z and w in the inequality, we need only consider the case $\rho(z) < \rho(w)$. It is enough to prove that $\delta - \rho(z) \leq \delta(z, w)$ for each δ in the interval $\rho(z) < \delta < \rho(w)$. If z is not contained in the disk

$$P(w, \delta) = \{ \xi \in D: \delta(\xi, w) < \delta \},$$

then $\delta \leq \delta(z, w)$, hence $\delta - \rho(z) \leq \delta(z, w)$. If $z \in P(w, \delta)$, then the disk $\{\xi \in D: \delta(\xi, z) < \delta - \delta(z, w)\}$ is contained in $P(w, \delta)$, by the triangle inequality of $\delta(\cdot, \cdot)$ [7, pp. 510-511]. Hence $\delta - \delta(z, w) \leq \rho(z)$, that is, $\delta - \rho(z) \leq \delta(z, w)$. This completes the proof.

We continue the study of $\rho(z, f)$ in terms of cluster sets. Let E be a subset of D whose closure $\mathrm{cl}\, E$ in the plane contains a point $\zeta \in \Gamma$. Then we set

$$C_E(\rho, \zeta) = \bigcap_U \mathrm{cl}\, \rho(E \cap U),$$

where U ranges over all open disks containing ζ, and where the closure of the image $\rho(E \cap U)$ of $E \cap U$ by ρ is taken in the plane. We set

$$C_{\mathscr{A}}(\rho, \zeta) = \bigcup_\Delta C_\Delta(\rho, \zeta) \quad \text{and} \quad \Pi_T(\rho, \zeta) = \bigcap_\Lambda C_\Lambda(\rho, \zeta),$$

where Δ ranges over all angular domains at ζ and Λ ranges over all admissible arcs at ζ. Let $K(\rho)$ be the set of all points $\zeta \in \Gamma$ where $C_{\mathscr{A}}(\rho, \zeta) = C_\Delta(\rho, \zeta)$ for each angular domain Δ at ζ, and let $J(\rho)$ be the set of all points $\zeta \in K(\rho)$ where $C_{\mathscr{A}}(\rho, \zeta) = C_D(\rho, \zeta)$. Finally, let $L(\rho)$ be the set of all points $\zeta \in \Gamma$ where $C_{\mathscr{A}}(\rho, \zeta) = \Pi_T(\rho, \zeta)$.

LEMMA 3. $J(\rho) \subset K(\rho) = L(\rho)$.

Proof. The relation $J(\rho) \subset K(\rho)$ is trivial. Since each angular domain at $\zeta \in L(\rho)$ contains a terminal part

$$\Lambda_0: z = z(t) \quad (r_0 \leq t < 1),$$

where $0 \leq r_0 < 1$, of an admissible arc

$$\Lambda: z = z(t) \quad (0 \leq t < 1)$$

at ζ, we see that $L(\rho) \subset K(\rho)$. Since ρ is uniformly continuous as a map from D endowed with the metric $\gamma(\cdot, \cdot)$ into the Euclidean plane, as a consequence of Lemma 2, the proof that $L(\rho) \supset K(\rho)$ is in spirit the same as that of [1, Lemma].

Proof of Theorem 2. First we remark that $\liminf_{z \to \zeta,\, z \in E} \tau(z, f)$ is zero or positive according as $\liminf_{z \to \zeta,\, z \in E} \rho(z, f)$ is zero or positive, where E is one of the sets described in the paragraph before Lemma 3. It follows from results of E. P. Dolženko [2, Theorem 1] and P. Lappan [3] that $K(\rho)$ is a residual set of measure 2π. By Lemma 3, $K(\rho) = L(\rho)$. If $\zeta \in K(\rho)$ and $0 \in C_{\mathscr{A}}(\rho, \zeta)$, then $0 \in \Pi_T(\rho, \zeta)$, hence $\zeta \in \Gamma_1(f)$. If $\zeta \in K(\rho)$ and $0 \notin C_{\mathscr{A}}(\rho, \zeta)$, then $0 \notin C_\Delta(\rho, \zeta)$ for each Δ at ζ, hence $\zeta \in \Gamma_2(f)$. We thus have the relation $K(\rho) \subset \Gamma_1(f) \cup \Gamma_2(f)$, which proves the first assertion. If $\zeta \in J(\rho)$ and $0 \in C_D(\rho, \zeta)$, then $0 \in \Pi_T(\rho, \zeta)$ for $J(\rho) \subset L(\rho)$. Therefore $\zeta \in \Gamma_1(f)$. If $\zeta \in J(\rho)$ and $0 \notin C_D(\rho, \zeta)$, then $\zeta \in \Gamma_3(f)$. We thus obtain the relation $J(\rho) \subset \Gamma_1(f) \cup \Gamma_3(f)$, and this completes the proof of Theorem 2.

Finally, we investigate local properties of f' near points of $\Gamma_2(f)$ and $\Gamma_3(f)$. Let $\chi(\cdot, \cdot)$ denote chordal distance. Let $\mathscr{D}$ be a simply connected subdomain of D, and let $\gamma_{\mathscr{D}}(\cdot, \cdot)$ denote the hyperbolic distance in $\mathscr{D}$. Thus

$$(1 - |z|^2)^{-1} |dz| = d\gamma(z) \equiv d\gamma_D(z) .$$

THEOREM 3. *Let* f *be holomorphic in* D. *If* Δ *is an angular domain at* $\zeta \in \Gamma_2(f)$, *then there exists a constant* $k_\Delta > 0$ *such that* $\chi(f'(z), f'(w)) \leq k_\Delta \gamma_\Delta(z, w)$ *for all* $z, w \in \Delta$. *If* $\zeta \in \Gamma_3(f)$, *then there exist an open disk* U *containing* ζ *and a constant* $k_U > 0$ *such that* $\chi(f'(z), f'(w)) \leq k_U \gamma_{D \cap U}(z, w)$ *for all* $z, w \in D \cap U$.

Proof. Let U_1 be an open disk $|z - \zeta| < \varepsilon$ $(\varepsilon < 1)$ such that

$$\inf_{z \in \Delta \cap U_1} \rho(z, f) > 0 .$$

Since the function ϕ considered in Section 2 is continuous in $\mathrm{cl}(\Delta - U_2)$, where $U_2 = \{z: |z - \zeta| < \varepsilon/2\}$ and since the inequality (2) holds in the present case, ϕ is bounded in Δ by a constant $k_\Delta > 0$. On the other hand, by the principle of hyperbolic metrics, $d\gamma(z) \leq d\gamma_\Delta(z)$ for each point $z \in \Delta \subset D$. The inequality

$$\frac{|f''(z)|\,|dz|}{1 + |f'(z)|^2} \leq k_\Delta\, d\gamma(z) \leq k_\Delta\, d\gamma_\Delta(z) \qquad (z \in \Delta)$$

proves the first assertion. The proof of the second assertion is similar.

As a final remark, we note that the horocyclic versions of Lemma 3 and Theorems 2 and 3 are valid.

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