THE HAUSDORFF METRIC AND CONVERGENCE IN MEASURE

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1. INTRODUCTION

Let m denote n-dimensional Lebesgue measure in R^n. If $\{C_k\}$ is a sequence of compact sets in R^n, convergent in the Hausdorff metric to a compact set C, the sequence $\{m(C \triangle C_k)\}$ may fail to converge to zero. For example, the unit disc in the plane is the Hausdorff limit of a sequence of finite sets. Equivalently, the sequence of characteristic functions $\{\chi_{C_k}\}$ may fail to converge in measure to the characteristic function of C. We characterize the sequences $\{C_k\}$ for which $\lim_{k\to\infty} m(C \triangle C_k) = 0$.

2. PRELIMINARIES

Let $B_\varepsilon(x)$ denote the closed ε-ball about a point x in R^n.

Definition. Let C be a compact set in R^n. The ε-*parallel body* $B_\varepsilon(C)$ is the compact set $\bigcup_{x\in C} B_\varepsilon(x)$. The ε-*annulus* $A_\varepsilon(C)$ is the compact set $B_\varepsilon(C) \setminus \text{int } C$.

If C and K are compact subsets of R^n, the Hausdorff distance of C from K is

$$d(C, K) = \inf\{\varepsilon: B_\varepsilon(C) \supset K \text{ and } B_\varepsilon(K) \supset C\}.$$

If $\mathscr{A}$ denotes the collection of compact subsets of R^n, then $\langle \mathscr{A}, d \rangle$ is a complete metric space. Each closed and bounded subspace of $\langle \mathscr{A}, d \rangle$ is compact [1]. If $\{C_k\}$ is a sequence of compact sets such that $\lim_{k\to\infty} d(C_k, C) = 0$, then for each $\varepsilon > 0$, C_k is contained in $B_\varepsilon(C)$ for all sufficiently large integers k. Since $\lim_{\varepsilon\to 0+} m(B_\varepsilon(C)) = m(C)$, the assignment $C \to m(C)$ is an upper-semicontinuous function. In addition,

$$\lim_{k\to\infty} m(C \triangle C_k) = 0 \quad \text{if and only if} \quad \lim_{k\to\infty} m(C \setminus C_k) = 0.$$

3. RESULTS

To establish our characterization theorem, we shall use the following theorem of Dini. Let $\{f_k\}$ be a sequence of upper-semicontinuous nonnegative functions defined on a compact metric space Y. Suppose for each x in Y, the sequence $\{f_k(x)\}$ converges monotonically to zero. Then $\{f_k\}$ converges uniformly to the zero function on Y.

For $\ell = 1, 2, \cdots$ and for each compact set C in R^n, let $m_\ell(C)$ denote $m(B_{1/\ell}(C))$. Of course, the assignment $C \to m_\ell(C)$ determines an upper-semicontinuous function on $\langle \mathscr{A}, d \rangle$ for each ℓ.

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THEOREM 1. *Let* $\{C_k\}$ *be a sequence of compact sets in* R^n, *convergent in the Hausdorff metric to a compact set* C. *Then* $\lim_{k\to\infty} m(C \triangle C_k) = 0$ *if and only if* $\{m_\ell\}$ *converges to* m *uniformly on* $\{C_k: k \in Z^+\}$ *as* ℓ *approaches infinity.*

Proof. First, suppose that $\lim_{k\to\infty} m(C \triangle C_k) = 0$. Clearly,

$$\mathscr{B} = \{C\} \cup \{C_k: k \in Z^+\}$$

is a compact subset of $\langle \mathscr{A}, d \rangle$. Since C is the only limit point of $\mathscr{B}$, the function m is continuous on $\mathscr{B}$. Thus, the sequence $\{(m_\ell - m) \mid \mathscr{B}\}$ is a sequence of upper-semicontinuous nonnegative functions converging monotonically to zero. By Dini's Theorem, the convergence must be uniform on $\mathscr{B}$. Certainly, the convergence is then uniform on $\{C_k: k \in Z^+\}$.

Conversely, suppose that $\lim_{k\to\infty} m(C_k \triangle C) \neq 0$, so that

$$\lim\inf_{k\to\infty} m(C_k) < m(C).$$

By passing to a subsequence, we may assume that for some $\varepsilon > 0$, $d(C_k, C) < 1/k$ and $m(C_k) < m(C) - \varepsilon$ for all k. Since $B_{1/k}(C_k)$ includes C, we have the inequalities

$$m_k(C_k) \geq m(C) > m(C_k) + \varepsilon.$$

Hence, $\{m_\ell\}$ cannot converge uniformly to m on $\{C_k: k \in Z^+\}$.

Let $\mathscr{C}$ denote the subfamily of $\mathscr{A}$ consisting of the compact convex sets in R^n. The continuity of Lebesgue measure on $\langle \mathscr{C}, d \rangle$ is often established by means of polyhedral approximations [2, Part XII]. The previous result offers a different approach.

THEOREM. *Let* C_k *be a sequence of compact convex sets in* R^n *convergent to a compact convex set* C. *Then* $\lim_{k\to\infty} m(C_k \triangle C) = 0$.

Proof. For each positive integer ℓ, let

$$\mathscr{B}_\ell = \{A_{1/\ell}(C)\} \cup \{A_{1/\ell}(C_k): k \in Z^+\}.$$

Since the terms of $\{C_k\}$ are convex, it follows that

$$\lim_{k\to\infty} d(A_{1/\ell}(C), A_{1/\ell}(C_k)) = 0.$$

Hence, $\mathscr{B}_\ell$ is compact relative to the Hausdorff metric. Since Lebesgue measure is an upper-semicontinuous function on $\mathscr{B}_\ell$, there exists an element F_ℓ of $\mathscr{B}_\ell$ of maximal Lebesgue measure. Clearly, $\{F_\ell\}$ has a convergent subsequence whose limit is either the boundary of C or the boundary of C_k for some k. This implies that $\lim_{\ell\to\infty} m(F_\ell) = 0$. Since $m(F_\ell) \geq \sup_k m(B_{1/\ell}(C_k)) - m(C_k)$ for $\ell = 1, 2, \cdots$, the sequence $\{m_\ell\}$ converges uniformly to m on $\{C_k: k \in Z^+\}$. Theorem 1 now applies.

REFERENCES

1. E. Michael, *Topologies on spaces of subsets.* Trans. Amer. Math. Soc. 71 (1951), 152-182.

2. F. A. Valentine, *Convex sets.* McGraw-Hill, New York, 1964.

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