A NEW DEFINITION FOR QUASISYMMETRIC FUNCTIONS

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1. INTRODUCTION

A continuous, strictly increasing function u mapping the line $(-\infty, \infty)$ onto itself is said to be ρ_0-*quasisymmetric* (or ρ_0 - QS for short) if $1 \leq \rho_0 < \infty$ and ρ_0 is the infimum of all ρ satisfying the condition

$$(1) \qquad 0 < \frac{1}{\rho} \leq \frac{u(x+t) - u(x)}{u(x) - u(x-t)} \leq \rho < \infty$$

for all x and all $t > 0$. The number $\rho_0 = \rho(u)$ is called the *quasisymmetric dilatation* of u on $(-\infty, \infty)$.

L. V. Ahlfors and A. Beurling proved in 1956 [1] that an autohomomorphism u of the real line can be extended to a quasiconformal autohomomorphism f of the upper half-plane if and only if u is quasisymmetric. Furthermore, if K(f) is the quasiconformal dilatation of f, then

$$K(f) \geq 1 + (0.2284) \log \rho(u)$$

for each quasiconformal extension f of u, and there exists an extension $\hat{f}$ for which

$$K(\hat{f}) \leq [\rho(u)]^2 .$$

A good bound on $\rho(u)$ is therefore of great importance in any investigation of the quasiconformal extensions of u to the upper half-plane.

We begin by showing that (1) is really a generalized convexity-concavity condition, and that we can weaken the assumptions in the definition of quasisymmetry significantly without altering the class of such functions.

We then use this to prove that the class of quasisymmetric functions is closed under the formation of sums, appropriate products, compositions, and inverses. These properties have previously been established by means of similar properties of quasiconformal mappings, but our proofs use only the new definition of quasisymmetry and some elementary real analysis.

Finally, we use the new definition to obtain sharp bounds for the quasisymmetric dilatation of sums, products, compositions and inverses of quasisymmetric functions on $(0, \infty)$.

2. A NEW DEFINITION

As we pointed out in the introduction, a homeomorphism u of $(-\infty, \infty)$ onto itself can be extended to a QC map of the upper half-plane onto itself if and only if u is

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QS according to (1). The following theorem gives an alternate formulation of quasi-symmetry, which is equivalent to (1) but better suited to the estimates we want to make in this paper.

THEOREM 1. *Let* u *be a nonconstant function defined on an interval* (a, b) *with* $-\infty \leq a < b \leq \infty$. *Then* u *is* QS *on* (a, b) *if and only if*

(i) u *is linear*

or

(ii) *there is some* λ $(1/2 < \lambda < 1)$ *such that*

$$\lambda u(x_1) + (1 - \lambda) u(x_2) \leq u\left(\frac{x_1 + x_2}{2}\right) \leq (1 - \lambda) u(x_1) + \lambda u(x_2) \tag{2}$$

for all x_1 *and* x_2 *with* $a < x_1 < x_2 < b$.

Definition 1. If u is a nonlinear QS function on (a, b), we define the *midpoint dilatation* $\lambda(u)$ of u to be the infimum of all numbers λ for which (2) holds. The relation (2) is called the *midpoint condition.* The relationship between $\lambda(u)$ and $\rho(u)$ is

$$\rho(u) = \frac{\lambda(u)}{1 - \lambda(u)}, \qquad \lambda(u) = \frac{\rho(u)}{1 + \rho(u)}. \tag{3}$$

Proof of Theorem 1. (i) Let u be QS on (a, b). If u is linear, then there is nothing left to prove; therefore assume that u is nonlinear. Since the condition $\rho = 1$ in (1) together with the continuity of u would imply linearity, we must have the inequality $\rho_0 > 1$. Hence we know that

$$\frac{1}{\rho_0} \leq \frac{u(x + t) - u(x)}{u(x) - u(x - t)} \leq \rho_0 \tag{4}$$

for all x and t satisfying the condition $a < x - t < x < x + t < b$, where $\rho_0 = \rho(u) > 1$. Multiplying (4) by the positive expression $u(x) - u(x - t)$ and solving for u(x), we see that

$$\frac{\rho_0}{1 + \rho_0} u(x - t) + \frac{1}{1 + \rho_0} u(x + t) \leq u(x) \leq \frac{1}{1 + \rho_0} u(x - t) + \frac{\rho_0}{1 + \rho_0} u(x + t).$$

This double inequality becomes (2) if we set $x_1 = x - t$, $x_2 = x + t$, and $\lambda(u) = \rho_0/(1 + \rho_0)$, so that $1/2 < \lambda(u) < 1$.

(ii) If the function u is nonconstant and linear, then it is obviously QS. Hence let us assume that u satisfies (2) with $\lambda_0 = \lambda(u)$ and $1/2 < \lambda_0 < 1$. It must be shown that

(a) u is continuous on (a, b),

(b) u is strictly increasing on (a, b),

(c) u satisfies (1) for some ρ $(1 \leq \rho < \infty)$.

Proof of (a). By (2) it is clear that if $a < x_1 < x_2 < b$, then the inequality

$$\lambda_0 u(x_1) + (1 - \lambda_0) u(x_2) \leq \lambda_0 u(x_2) + (1 - \lambda_0) u(x_1)$$

implies that $(2\lambda_0 - 1)u(x_1) \leq (2\lambda_0 - 1)u(x_2)$, or equivalently, since $1/2 < \lambda_0 < 1$, that $u(x_1) \leq u(x_2)$. Hence we know that u is nondecreasing. Now let x_0 be a point of (a, b). We shall show that u is continuous at x_0, and this will prove (a). Since u is nondecreasing, it must have a finite left-hand limit at x_0; therefore let $\lim_{x \to x_0^-} u(x) = M < \infty$. If $\varepsilon > 0$, then we can find some $\hat{x}_0$ $(a < \hat{x}_0 < x_0)$ for which $M - \varepsilon < u(x_0) \leq M$, and the monotonicity of u would imply that

$$M - \varepsilon < u\left(\frac{x_0 + \hat{x}_0}{2}\right) \leq M$$

as well. Now, using (2) with $x_1 = \hat{x}_0$ and $x_2 = x_0$, and solving for $u(x_0)$, we find that

$$M - \varepsilon = \frac{1}{\lambda_0}(M - \varepsilon) + \left(1 - \frac{1}{\lambda_0}\right)(M - \varepsilon) \leq \frac{1}{\lambda_0} u\left(\frac{x_0 + \hat{x}_0}{2}\right) + \left(1 - \frac{1}{\lambda_0}\right) u(\hat{x}_0) \leq u(x_0)$$

$$\leq \frac{1}{1 - \lambda_0} u\left(\frac{x_0 + \hat{x}_0}{2}\right) - \frac{\lambda_0}{1 - \lambda_0} u(\hat{x}_0) \leq \frac{1}{1 - \lambda_0} M - \frac{\lambda_0}{1 - \lambda_0}(M - \varepsilon) = M + \frac{\lambda_0}{1 - \lambda_0}\varepsilon .$$

Letting ε tend to 0, we see that $u(x_0) = M = \lim_{x \to x_0^-} u(x)$. Similarly, it can be shown that $u(x_0) = \lim_{x \to x_0^+} u(x)$. Therefore u is continuous at x_0.

Proof of (b). In (a) we showed that u is nondecreasing. It remains to prove that u is strictly increasing on (a, b); assume therefore that there are points x_1 and x_2 with $a < x_1 < x_2 < b$ and $u(x_1) = u(x_2) = M$. Since u is nondecreasing, this would imply that $u(x) = M$ identically on $[x_1, x_2]$. By assumption, u is not constant. Hence there exists an $x_3 \in (a, b)$ for which $u(x_3) \neq M$. Without loss of generality, we can assume that $x_2 < x_3$. Then, by the monotonicity of u, we see that $u(x_2) = M < u(x_3)$.

Define the set

$$S = \{x \mid x_2 \leq x \leq x_3 \text{ with } u(x) > u(x_2) = M\} . \tag{5}$$

Clearly, $x_3 \in S$, so that S is not empty. In addition, x_2 is a lower bound for S, by (5). Thus S must have a greatest lower bound (g.l.b.) $\bar{x}$ with $x_2 \leq \bar{x}$. If $u(\bar{x}) > M$, then by continuity there exists an $\varepsilon > 0$ such that $u(x) > M$ on $(\bar{x} - \varepsilon, \bar{x})$. But this contradicts the assumption that $\bar{x}$ is the g.l.b. of S. Hence $u(\bar{x}) = M$, and therefore $u(x) = M$ identically on $[x_1, \bar{x}]$. Pick $\varepsilon > 0$ so small that $u(\bar{x} + \varepsilon) > M$ and $u(\bar{x} - \varepsilon) = M$. This is possible, because $\bar{x}$ is the g.l.b. of S. Then, by (2), using the points $\bar{x} - \varepsilon$, $\bar{x}$, $\bar{x} + \varepsilon$, we obtain the inequalities

$$\lambda_0 u(\bar{x} - \varepsilon) + (1 - \lambda_0) u(\bar{x} + \varepsilon) \leq u(\bar{x}) \leq (1 - \lambda_0) u(\bar{x} - \varepsilon) + \lambda_0 u(\bar{x} + \varepsilon) .$$

The left inequality implies that

$$u(\bar{x}) \geq \lambda_0 u(\bar{x} - \varepsilon) + (1 - \lambda_0) u(\bar{x} + \varepsilon) > \lambda_0 M + (1 - \lambda_0) M = M .$$

But this obviously contradicts the condition $u(\bar{x}) = M$. That is, there cannot exist any $x_1 < x_2$ with $u(x_1) = u(x_2)$. Hence u must be strictly increasing on (a, b).

Proof of (c). It is sufficient to observe that all the steps in the proof of (i) are reversible.

Remark. In view of the statement of Theorem 1, it is reasonable to ask whether we can omit condition (i) by simply changing condition (ii) to allow $1/2 \leq \lambda_0 < 1$. Obviously, if u is QS and linear, then (2) holds with $\lambda_0 = 1/2$. The converse, however, is not true. It is possible for a nonconstant function to satisfy (2) with $\lambda_0 = 1/2$ without being QS on (a, b). To show this, we shall exhibit an example.

In [2, p. 150], a function f is constructed, by means of a Hamel basis $x_1, x_2, \cdots, x_\alpha, \cdots$ $(\alpha \in \Omega)$, with the property $f(x+y) = f(x) + f(y)$ for all x, y. Taking x = y, we obtain the equation $f(2x) = 2f(x)$, which leads to $f(x) = f(2x)/2$ or $f(x/2) = f(x)/2$. Thus, for all x and y,

$$f\left(\frac{x+y}{2}\right) = f\left(\frac{x}{2}+\frac{y}{2}\right) = f\left(\frac{x}{2}\right) + f\left(\frac{y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y).$$

But this is simply (2) with $\lambda_0 = 1/2$. As was indicated in [2, p. 150], f can be defined arbitrarily on the Hamel basis. Since a Hamel basis has an infinite number of elements, we can choose three elements x_1, x_2, x_3 from this particular basis. Assume without loss of generality that in the usual ordering of the reals, $x_1 < x_2 < x_3$, and define $f(x_1) = 0$, $f(x_2) = 1$, $f(x_3) = 0$, and $f = 0$ for all the other elements in this basis. Let u = f. Then u satisfies (2) with $\lambda_0 = 1/2$, because f does. The function u is not constant, since $u(x_1) \neq u(x_2)$. Yet u cannot be QS, because it is not even monotonic, since $x_1 < x_2 < x_3$ with $u(x_1) < u(x_2)$ and $u(x_2) > u(x_3)$. Hence condition (i) cannot be omitted in the statement of Theorem 1.

3. UPPER AND LOWER BOUNDS

In order to obtain upper and lower bounds for a QS function u on the interior of an interval in terms of the values of the function at the endpoints, we define the following two functions $P(\lambda)$ and $p(\lambda)$ on [0, 1]. If

$$\lambda = \frac{\theta_1}{2^1} + \frac{\theta_2}{2^2} + \cdots + \frac{\theta_n}{2^n} + \cdots \qquad (\theta_i = 0 \text{ or } 1)$$

is the binary expansion of $\lambda \in [0, 1]$, then

$$(6)\qquad \begin{cases} P(\lambda) = P_{\lambda_0}(\lambda) = \lambda_0[\theta_1 + \lambda_{\theta_1}\theta_2 + \lambda_{\theta_1}\lambda_{\theta_2}\theta_3 + \cdots + \lambda_{\theta_1}\lambda_{\theta_2}\cdots\lambda_{\theta_{n-1}}\theta_n + \cdots], \\ p(\lambda) = p_{\lambda_0}(\lambda) = \lambda_1[\theta_1 + \lambda_{1-\theta_1}\theta_2 + \lambda_{1-\theta_1}\lambda_{1-\theta_2}\theta_3 + \cdots], \end{cases}$$

where λ_0 $(1/2 \leq \lambda_0 < 1)$ is the midpoint dilatation of u (see Definition 1) and $\lambda_1 = 1 - \lambda_0$. When there is no chance of confusion, we shall use P and p in place of P_{λ_0} and p_{λ_0}.

To see that the function $P(\lambda)$ is not itself QS, take $x = 1/2$ and $t = 1/2^n$ (n an integer greater than 1) in (1) with u = P. Then (6) implies that

$$(7)\qquad \frac{P(x+t) - P(x)}{P(x) - P(x-t)} = \left(\frac{\lambda_0}{\lambda_1}\right)^{n-2}.$$

Since $\lambda_0/\lambda_1 > 1$, it is clear that the right-hand side of (7) goes to ∞ as $n \to \infty$. Hence $P(\lambda)$ is not QS. A similar proof shows that $p(\lambda)$ is not QS.

THEOREM 2. *Let* u *be a* QS *function on* (a, b), *and let* $x_1, x_2 \in (a, b)$. *Then, for each* $\lambda \in [0, 1]$,

$$(8) \qquad \begin{cases} u[(1-\lambda)x_1 + \lambda x_2] \leq [1 - P(\lambda)]u(x_1) + P(\lambda)u(x_2), \\ u[(1-\lambda)x_1 + \lambda x_2] \geq [1 - p(\lambda)]u(x_1) + p(\lambda)u(x_2). \end{cases}$$

Proof. In a paper by R. Salem [4], P and p are shown to be continuous, strictly increasing functions mapping the segment [0, 1] onto itself. If v is QS with $v(0) = 0$ and $v(1) = 1$, then by (2) and the geometric construction of P and p in [4], we see that $p(\lambda) \leq v(\lambda) \leq P(\lambda)$ for all $\lambda \in [0, 1]$ with a finite binary expansion. But the set of these numbers is dense in [0, 1]. Hence, by the continuity of v, P, and p, $p(\lambda) \leq v(\lambda) \leq P(\lambda)$ for all $\lambda \in [0, 1]$.

If we now take any QS function u and set

$$v(\lambda) = \frac{u[(1-\lambda)x_1 + \lambda x_2] - u(x_1)}{u(x_2) - u(x_1)},$$

then v is QS with $v(0) = 0$, $v(1) = 1$. Hence

$$p(\lambda) \leq v(\lambda) = \frac{u[(1-\lambda)x_1 + \lambda x_2] - u(x_1)}{u(x_2) - u(x_1)} \leq P(\lambda).$$

Solving for $u[(1-\lambda)x_1 + \lambda x_2]$, we obtain the desired bounds.

THEOREM 3. $P(t) + p(1 - t) = 1$ *identically on* [0, 1].

Proof. Define $f(t) = P(t) + p(1 - t)$ on [0, 1]. Since P and p are both continuous [4], so is f. We shall show by induction on n that $f(t) = 1$ when t is of the form $t = m/2^n$ ($n = 0, 1, 2, \cdots$ and $m = 0, 1, \cdots, 2^n$). Since the set of all such t is dense in [0, 1], the continuity of f will then imply $f(t) = 1$ identically on [0, 1].

(i) Let $n = 0$. Then m can be either 0 or 1, and

$$f(0) = P(0) + p(1 - 0) = 0 + 1 = 1, \qquad f(1) = P(1) + p(1 - 1) = 1.$$

(ii) Let $n = 1$. Then m can be either 0, 1, or 2. But the cases $m = 0$ and $m = 2$ are covered by (i), while $m = 1$ gives

$$f(1/2) = P(1/2) + p(1 - (1/2)) = \lambda_0 + \lambda_1 = 1.$$

(iii) Assume $f(t) = 1$ for all t of the form $t = m/2^{N_0}$, and let $t_0 = m/2^{N_0+1}$, where $0 \leq m \leq 2^{N_0+1}$. If $m = 0$ or $m = 2^{N_0+1}$, then $t = 0$ or 1, respectively. These cases have been treated in (i).

Let m be an even number. Then $m = 2k$, where $1 \leq k < 2^{N_0} - 1$, and therefore

$$t = m/2^{N_0+1} = 2k/2^{N_0+1} = k/2^{N_0}.$$

By the induction hypothesis, this would imply $f(t) = f(k/2^{N_0}) = 1$.

Let m be an odd number. Then $m - 1$ and $m + 1$ are both even. Hence $m - 1 = 2k_1$ and $m + 1 = 2k_2$ for some k_1 and k_2 with $0 \leq k_1 < k_2 \leq 2^{N_0}$. Therefore

$$f(t) = f(m/2^{N_0+1}) = f\left(\left(\frac{m-1}{2^{N_0+1}} + \frac{m+1}{2^{N_0+1}}\right)/2\right)$$

$$= P\left(\left(\frac{m-1}{2^{N_0+1}} + \frac{m+1}{2^{N_0+1}}\right)/2\right) + p\left(1 - \left(\frac{m-1}{2^{N_0+1}} + \frac{m+1}{2^{N_0+1}}\right)/2\right)$$

$$(9) \qquad = P([(k_1/2^{N_0}) + (k_2/2^{N_0})]/2) + p([(1 - k_1/2^{N_0}) + (1 - k_2/2^{N_0})]/2)$$

$$= [\lambda_1 P(k_1/2^{N_0}) + \lambda_0 P(k_2/2^{N_0})] + [\lambda_1 p(1 - k_1/2^{N_0}) + \lambda_0 p(1 - k_2/2^{N_0})]$$

$$= \lambda_1 f(k_1/2^{N_0}) + \lambda_0 f(k_2/2^{N_0}) .$$

Hence, by the induction hypothesis, (9) reduces to $\lambda_1 + \lambda_0 = 1$. The induction proof is completed.

COROLLARY 3. $P^{-1}(1/2) + p^{-1}(1/2) = 1$.

Proof. Since $P(0) = 0$, $P(1) = 1$, and P is continuous, there must be some $t_1 \in (0, 1)$ with $P(t_1) = 1/2$ or, equivalently, $t_1 = P^{-1}(1/2)$. Similarly, the conditions $p(0) = 0$, and $p(1) = 1$ together with the continuity of p imply the existence of a $t_2 \in (0, 1)$ with $t_2 = p^{-1}(1/2)$. By Theorem 3, $1/2 = P(t_1) = 1 - p(1 - t_1)$. Hence $p(1 - t_1) = 1/2 = p(t_2)$. Since p is strictly increasing, the relation $p(1 - t_1) = p(t_2)$ implies that $1 - t_1 = t_2$, in other words, $t_1 + t_2 = 1$. Substituting the values $t_1 = P^{-1}(1/2)$ and $t_2 = p^{-1}(1/2)$, we see that $P^{-1}(1/2) + p^{-1}(1/2) = 1$.

CLOSURE PROPERTIES

4. THE SUM OF QS FUNCTIONS

THEOREM 4. *Let* $u_1, u_2, \cdots, u_n$ *be* QS *functions on* (a, b) *with* QS *dilatations* $\rho_1, \rho_2, \cdots, \rho_n$, *respectively. Then the function* v *defined as* $v(x) = \sum_{i=1}^{n} u_i(x)$ *is also* QS *on* (a, b), *and*

$$\rho(v) \leq \max\{\rho_i \mid 1 \leq i \leq n\} .$$

This result is sharp.

Proof. The proof is by induction on n. Let the midpoint dilatation of u_i be represented by the symbol $\lambda^{(i)}$ $(i = 1, \cdots, n)$.

(i) Assume $n = 2$, and let $c = \max\{\lambda^{(1)}, \lambda^{(2)}\}$. Then, for $x_1, x_2 \in (a, b)$ it follows from (2) that

$$v\left(\frac{x_1 + x_2}{2}\right) = u_1\left(\frac{x_1 + x_2}{2}\right) + u_2\left(\frac{x_1 + x_2}{2}\right)$$

$$\leq [(1 - \lambda^{(1)}) u_1(x_1) + \lambda^{(1)} u_1(x_2)] + [(1 - \lambda^{(2)}) u_2(x_1) + \lambda^{(2)} u_2(x_2)]$$

$$\leq [(1 - c) u_1(x_1) + c u_1(x_2)] + [(1 - c) u_2(x_1) + c u_2(x_2)]$$

$$= (1 - c) v(x_1) + c v(x_2) .$$

Similarly,

$$v\left(\frac{x_1+x_2}{2}\right) = u_1\left(\frac{x_1+x_2}{2}\right) + u_2\left(\frac{x_1+x_2}{2}\right)$$
$$\geq [\lambda^{(1)} u_1(x_1) + (1 - \lambda^{(1)}) u_1(x_2)] + [\lambda^{(2)} u_2(x_1) + (1 - \lambda^{(2)}) u_2(x_2)]$$
$$\geq c\, v(x_1) + (1 - c)\, v(x_2).$$

Hence

$$c\, v(x_1) + (1 - c)\, v(x_2) \leq v\left(\frac{x_1+x_2}{2}\right) \leq (1 - c)\, v(x_1) + c\, v(x_2).$$

By Theorem 1, v is QS on (a, b), while Definition 1 shows that

$$\lambda(v) \leq c = \max\{\lambda^{(1)}, \lambda^{(2)}\}.$$

Since $\rho(u) = \lambda(u)/(1 - \lambda(u))$ is an increasing function of $\lambda(u)$, this is equivalent to the inequality $\rho(v) \leq \max\{\rho_1, \rho_2\}$.

(ii) Assume the theorem is true for $n = N_0$, and let $\hat{v} = \sum_{i=1}^{N_0} u_i$. Then $v = \hat{v} + u_{N_0+1}$. But u_{N_0+1} is assumed to be QS, and $\hat{v}$ is QS by the induction hypothesis, with $\rho(\hat{v}) \leq \max\{\rho_i \mid 1 \leq i \leq N_0\}$. Hence, by part (i), $v = \hat{v} + u_{N_0+1}$ is also QS on (a, b), with

$$\rho(v) \leq \max\{\rho(\hat{v}), \rho_{N_0+1}\} \leq \max\{\rho_i \mid 1 \leq i \leq N_0 + 1\}.$$

To prove that this result is sharp, choose a finite sequence of numbers $\rho_1, \rho_2, \cdots, \rho_n$ $(1 \leq \rho_1 \leq \cdots \leq \rho_n)$. If we let $\alpha_i = \log_2(1 + \rho_i)$ $(i = 1, \cdots, n)$, then by [1, p. 133], the function $u_i(x) = x^{\alpha_i}$ for $x \in [0, \infty)$ is QS on $[0, \infty)$ with dilatation

$$\rho(u_i) = 2^{\alpha_i} - 1 = \rho_i \qquad (i = 1, \cdots, n).$$

By the first part of this proof, the function

$$v(x) = \sum_{i=1}^{n} u_i(x) = x^{\alpha_1} + \cdots + x^{\alpha_n}$$

is QS on $[0, \infty)$, and $\rho(v) \leq \max\{\rho_i \mid 1 \leq i \leq n\} = \rho_n$. If we now choose $t = x$ in (1) for $u = v$, however, then

$$\frac{v(x+t) - v(x)}{v(x) - v(x-t)} = \frac{v(2x) - v(x)}{v(x) - v(0)} = \frac{v(2x)}{v(x)} - 1$$

and

$$\lim_{x \to \infty} \frac{v(2x) - v(x)}{v(x) - v(0)} = \lim_{x \to \infty} \frac{v(2x)}{v(x)} - 1 = 2^{\alpha_n} - 1 = \rho_n.$$

Hence, by (1) and the definition of $\rho(v)$, we must have the inequality $\rho(v) \geq \rho_n$ as well. That is, $\rho(v) = \rho_n = \max\{\rho_i \mid 1 \leq i \leq n\}$.

An obvious analogue of Theorem 4 is the assertion that

$$\rho(v) \geq \min\{\rho_i \mid 1 \leq i \leq n\}.$$

This statement, however, is false even in the simple case $n = 2$, as the following corollary shows.

COROLLARY 4. *There exist functions* u_1 *and* u_2, QS *on* $(-\infty, \infty)$, *such that* $\rho(u_1 + u_2) < \min\{\rho_1, \rho_2\}$.

Proof. For each λ_0 $(1/2 < \lambda_0 < 1)$ and $\lambda_1 = 1 - \lambda_0$, define $\hat{u}_1$ and $\hat{u}_2$ on $[0, 1]$ as

$$\hat{u}_1(x) = \begin{cases} 2\lambda_0 x & \text{if } 0 \leq x \leq 1/2, \\ (2 - 2\lambda_0)x + (2\lambda_0 - 1) & \text{if } 1/2 < x \leq 1, \end{cases}$$

$$\hat{u}_2(x) = \begin{cases} 2\lambda_1 x & \text{if } 0 \leq x \leq 1/2, \\ (2 - 2\lambda_1)x + (2\lambda_1 - 1) & \text{if } 1/2 < x \leq 1. \end{cases}$$

It is easy to see that (1) is satisfied for both $\hat{u}_1$ and $\hat{u}_2$ on $[0, 1]$ with $\rho = \lambda_0/\lambda_1$, and that $\hat{v}(x) = \hat{u}_1(x) + \hat{u}_2(x) = 2x$ for all $x \in [0, 1]$. Next, let us define functions u_1 and u_2 on $(-\infty, \infty)$ by

$$u_i(x) = \hat{u}_i(x) \quad \text{for } 0 \leq x \leq 1, \qquad u_i(x + 2) = u_i(x) + 2 \qquad (i = 1, 2).$$

Then, by Theorem 5 of [3, p. 239], u_1 and u_2 are both QS on $(-\infty, \infty)$. Also,

$$\frac{u_i(3/4) - u_i(1/2)}{u_i(1/2) - u_i(1/4)} = \begin{cases} \dfrac{\lambda_0}{\lambda_1} & \text{if } i = 1, \\ \dfrac{\lambda_1}{\lambda_0} & \text{if } i = 2. \end{cases}$$

Therefore $\rho_1 = \rho(u_1) \geq \lambda_0/\lambda_1$ and $\rho_2 = \rho(u_2) \geq \lambda_0/\lambda_1$. But it follows from the construction of v that $v(x) = u_1(x) + u_2(x) = 2x$ identically on $(-\infty, \infty)$, so that $\rho(v) = 1$. Hence $\rho(v) = 1 < \lambda_0/\lambda_1 < \min\{\rho_1, \rho_2\}$.

5. THE PRODUCT OF QS FUNCTIONS

THEOREM 5. *Let* $u_1, u_2, \cdots, u_n$ *be* QS *functions on* (a, b) *with* $u_i(a) = 0$ *for each* i. *Then the function* v *defined as* $v(x) = \prod_{i=1}^n u_i(x)$ *is also* QS *on* (a, b), *and*

$$\rho(v) \leq \prod_{i=1}^{n} (1 + \rho_i) - 1.$$

This result is sharp.

Proof. The proof is by induction on n.

(i) Assume $n = 2$, and let $c = 1 - (1 - \lambda^{(1)})(1 - \lambda^{(2)})$. Then, for any $x_1, x_2 \in (a, b)$, we see from (2) that

$$\begin{aligned} v\left(\frac{x_1 + x_2}{2}\right) &= u_1\left(\frac{x_1 + x_2}{2}\right) u_2\left(\frac{x_1 + x_2}{2}\right) \\ &\leq [(1 - \lambda^{(1)}) u_1(x_1) + \lambda^{(1)} u_1(x_2)][(1 - \lambda^{(2)}) u_2(x_1) + \lambda^{(2)} u_2(x_2)] \\ &= (1 - \lambda^{(1)})(1 - \lambda^{(2)}) u_1(x_1) u_2(x_1) + (1 - \lambda^{(1)})\lambda^{(2)} u_1(x_1) u_2(x_2) \\ &\qquad + \lambda^{(1)}(1 - \lambda^{(2)}) u_1(x_2) u_2(x_1) + \lambda^{(1)}\lambda^{(2)} u_1(x_2) u_2(x_2) \\ &\leq (1 - \lambda^{(1)})(1 - \lambda^{(2)}) v(x_1) + (1 - \lambda^{(1)})\lambda^{(2)} v(x_2) \\ &\qquad + \lambda^{(1)}(1 - \lambda^{(2)}) v(x_2) + \lambda^{(1)}\lambda^{(2)} v(x_2) \\ &= (1 - \lambda^{(1)})(1 - \lambda^{(2)}) v(x_1) + (1 - (1 - \lambda^{(1)})(1 - \lambda^{(2)})) v(x_2) \\ &= (1 - c) v(x_1) + c\, v(x_2). \end{aligned}$$

Similarly,

$$v\left(\frac{x_1 + x_2}{2}\right) = u_1\left(\frac{x_1 + x_2}{2}\right) u_2\left(\frac{x_1 + x_2}{2}\right) \geq c\, v(x_1) + (1 - c) v(x_2).$$

Hence

$$c\, v(x_1) + (1 - c) v(x_2) \leq v\left(\frac{x_1 + x_2}{2}\right) \leq (1 - c) v(x_1) + c\, v(x_2).$$

By Theorem 1, v is QS on (a, b), while $\lambda(v) \leq c$. Changing to the QS dilatation, we see that $\rho(v) \leq (\rho_1 + 1)(\rho_2 + 1) - 1$.

(ii) Assume the theorem true for $n = N_0$, and let $\hat{v} = \prod_{i=1}^{N_0} u_i$. Then $v = \hat{v} \cdot u_{N_0+1}$. But u_{N_0+1} is assumed to be QS, and $\hat{v}$ is QS by the induction hypothesis, with $\rho(\hat{v}) \leq \prod_{i=1}^{N_0} (\rho_i + 1) - 1$. Hence, by part (i), $v = \hat{v} \cdot u_{N_0+1}$ is also QS on (a, b), with

$$\rho(v) \leq (\rho(\hat{v}) + 1)(\rho_{N_0+1} + 1) - 1 = \rho(\hat{v})(\rho_{N_0+1} + 1) + \rho_{N_0+1} \leq \prod_{i=1}^{N_0+1} (\rho_i + 1) - 1.$$

Equality holds when $u_i(x) = x^{\alpha_i}$ $(i = 1, \cdots, n)$ on $[0, \infty)$, for each choice of $\alpha_1, \cdots, \alpha_n$ all greater than or equal to 1. By [1, p. 133], this choice of the u_i gives the equation

$$\prod_{i=1}^{n} (\rho_i + 1) - 1 = \prod_{i=1}^{n} 2^{\alpha_i} - 1 = 2^{\sum \alpha_i} - 1 = \rho(x^{\sum \alpha_i}) = \rho(v).$$

6. THE INVERSE OF A QS FUNCTION

THEOREM 6. *Let* u *be a* QS *function from* (a, b) *onto* (c, d) *with midpoint dilatation* $\lambda_0 = \lambda(u)$. *Then* u^{-1} *is a* QS *function of* (c, d) *onto* (a, b) *whose* QS *dilatation satisfies the inequality*

$$\rho(u^{-1}) \leq \frac{p_{\lambda_0}^{-1}(1/2)}{P_{\lambda_0}^{-1}(1/2)},$$

where p *and* P *denote the Salem functions in* (6). *There is equality when* $u(x) = x^{1/n}$ *on* $[0, \infty)$, *for any integer* $n \geq 1$.

Proof. Let $y_1, y_3 \in (c, d)$, with $y_1 < y_3$, and let $y_2 = (y_1 + y_3)/2$. Let $x_j = u^{-1}(y_j)$ (j = 1, 2, 3), and let $\bar{\lambda} = P^{-1}(1/2) = 1 - p^{-1}(1/2)$. Then

$$u[(1 - \bar{\lambda})x_1 + \bar{\lambda}x_3] \leq (1 - P(\bar{\lambda}))u(x_1) + P(\bar{\lambda})u(x_3) = (u(x_1) + u(x_3))/2 = y_2,$$

by Theorem 2. Since u is increasing, this gives the inequality

$$x_2 \geq (1 - \bar{\lambda})x_1 + \bar{\lambda}x_3.$$

Similarly,

$$\begin{aligned} u[\bar{\lambda}x_1 + (1 - \bar{\lambda})x_3] &= u((1 - (1 - \bar{\lambda}))x_1 + (1 - \bar{\lambda})x_3) \\ &\geq (1 - p(1 - \bar{\lambda}))u(x_1) + p(1 - \bar{\lambda})u(x_3) = (u(x_1) + u(x_3))/2 = y_2. \end{aligned}$$

By the monotonicity of u, this implies that $x_2 \leq \bar{\lambda}x_1 + (1 - \bar{\lambda})x_3$. Hence

$$(1 - \bar{\lambda})u^{-1}(y_1) + (1 - (1 - \bar{\lambda}))u^{-1}(y_3) \leq u^{-1}(y_2) \leq (1 - (1 - \bar{\lambda}))u^{-1}(y_1) + (1 - \bar{\lambda})u^{-1}(y_3),$$

and by Theorem 1, u^{-1} must be QS on (c, d), with $\lambda(u^{-1}) < 1 - \bar{\lambda} = p^{-1}(1/2)$. By Definition 1, this becomes, in terms of the QS dilatation,

$$\rho(u^{-1}) \leq \frac{p^{-1}(1/2)}{P^{-1}(1/2)}.$$

Equality holds when $u(x) = x^{1/n}$ on $[0, \infty)$, for any integer $n \geq 1$. By [1, p. 133], $\rho(u) = 1/(2^{1/n} - 1)$. Hence, by Definition 1, $\lambda(u) = 1/2^{1/n}$. From (6) it is clear that

$$p(1 - (1/2^n)) = 1 - (\lambda(u))^n = 1 - (1/2^{1/n})^n = 1 - 1/2 = 1/2,$$

or $p^{-1}(1/2) = 1 - 1/2^n$. For this choice of u, the inverse u^{-1} is given by the formula $u^{-1}(y) = y^n$, and it is again easy to see that $\rho(u^{-1}) = 2^n - 1$. Hence

$$\rho(u^{-1}) = 2^n - 1 = \frac{1 - 1/2^n}{1/2^n} = \frac{p^{-1}(1/2)}{1 - p^{-1}(1/2)} = \frac{p^{-1}(1/2)}{P^{-1}(1/2)}.$$

For computations, the following bound for $\rho(u^{-1})$ is probably easier to use than the bound in Theorem 6.

COROLLARY 6. *Under the conditions of Theorem* 6,

$$\rho(u^{-1}) \leq 2 \cdot 2^{\frac{\log 2}{\log(1+1/\rho(u))}} - 1.$$

Proof. Let $\lambda_0 = \lambda(u)$ and $P = P_{\lambda_0}$. Since $P(1/2^n) = \lambda_0^n$ for each $n \geq 0$, and since $P(1/2^0) = P(1) = 1 > 1/2$, there exists an integer k such that

$$P(1/2^{k+1}) = \lambda_0^{k+1} \leq 1/2 < \lambda_0^k = P(1/2^k). \tag{10}$$

Since P is strictly increasing, (10) is equivalent to the relation

$$1/2^{k+1} \leq P^{-1}(1/2) < 1/2^k,$$

which implies that

$$1 - 1/2^k < 1 - P^{-1}(1/2) = p^{-1}(1/2) \leq 1 - 1/2^{k+1}. \tag{11}$$

Solving (10) for k, we find that

$$k \leq \frac{\log 2}{\log(1/\lambda_0)} = \frac{\log 2}{\log(1 + 1/\rho(u))} < k + 1,$$

and thus

$$2^k \leq 2^{\frac{\log 2}{\log(1+1/\rho(u))}}.$$

Hence, by Theorem 6 and (11), we find the desired bound

$$\rho(u^{-1}) \leq \frac{p^{-1}(1/2)}{P^{-1}(1/2)} \leq \frac{1 - 1/2^{k+1}}{1/2^{k+1}} = 2^{k+1} - 1 = 2 \cdot 2^k - 1 \leq 2 \cdot 2^{\frac{\log 2}{\log(1+1/\rho(u))}} - 1.$$

7. THE COMPOSITION OF QS FUNCTIONS

THEOREM 7. *Let* $u_1, u_2, \cdots, u_n$ *be* QS *functions such that the domain of each* u_{i+1} *is contained in the range of the preceding* u_i. *Then the composed function* u *defined as* $u(x) = u_n \circ u_{n-1} \circ \cdots \circ u_1(x)$ *is also* QS, *and*

$$\rho(u) \leq \frac{P_n(1/2)}{1 - P_n(1/2)},$$

where $P_n = P_{\lambda(u_n)} \circ P_{\lambda(u_{n-1})} \circ \cdots \circ P_{\lambda(u_1)}$. *This result is sharp.*

Proof. The proof is by induction on n.

(i) Assume $n = 2$, and let $x_1, x_2 \in (a, b)$, where (a, b) is the domain of u_1. Let $c = P_{\lambda(u_2)}(\lambda(u_1))$. Then, by Theorems 1 and 2,

$$u\left(\frac{x_1 + x_2}{2}\right) = u_2\left(u_1\left(\frac{x_1 + x_2}{2}\right)\right) \leq u_2((1 - \lambda(u_1))\, u_1(x_1) + \lambda(u_1) u_1(x_2))$$

$$\leq (1 - P_{\lambda(u_2)}(\lambda(u_1)))\, u(x_1) + P_{\lambda(u_2)}(\lambda(u_1))\, u(x_2) = (1 - c)\, u(x_1) + c\, u(x_2).$$

Similarly,

$$u\left(\frac{x_1+x_2}{2}\right) = u_2\left(u_1\left(\frac{x_1+x_2}{2}\right)\right)$$

$$\geq [1 - p_{\lambda(u_2)}(1 - \lambda(u_1))]u(x_1) + p_{\lambda(u_2)}(1 - \lambda(u_1))\, u(x_2)$$

$$= P_{\lambda(u_2)}(\lambda(u_1))\, u(x_1) + [1 - P_{\lambda(u_2)}(\lambda(u_1))]u(x_2) = c\, u(x_1) + (1 - c)\, u(x_2).$$

Hence

$$c\, u(x_1) + (1 - c)\, u(x_2) \leq u\left(\frac{x_1+x_2}{2}\right) \leq (1 - c)\, u(x_1) + c\, u(x_2).$$

By Theorem 1, u is QS on (a, b) and

$$\lambda(u) \leq c = P_{\lambda(u_2)}(\lambda(u_1)) = P_{\lambda(u_2)}(P_{\lambda(u_1)}(1/2)) = P_2(1/2).$$

If we change to the QS dilatation, this becomes

$$\rho(u) \leq \frac{P_2(1/2)}{1 - P_2(1/2)}.$$

(ii) Assume the theorem true for $n = N_0$, and let $\hat{u} = u_{N_0} \circ u_{N_0-1} \circ \cdots \circ u_1$. Then $u = u_{N_0+1} \circ \hat{u}$. But u_{N_0+1} is assumed to be QS, and $\hat{u}$ is QS by the induction hypothesis, with $\lambda(\hat{u}) \leq P_{N_0}(1/2)$. Hence, by part (i), $u = u_{N_0+1} \circ \hat{u}$ is also QS, and

$$\lambda(u) = \lambda(u_{N_0+1} \circ \hat{u}) \leq P_{\lambda(u_{N_0+1})}(P_{\lambda(\hat{u})}(1/2)) = P_{\lambda(u_{N_0+1})}(\lambda(\hat{u})) = P_{N_0+1}(1/2).$$

By Definition 1, this is equivalent to the inequality

$$\rho(u) \leq \frac{P_{N_0+1}(1/2)}{1 - P_{N_0+1}(1/2)}.$$

Equality holds when $u_i(x) = x^{\alpha_i}$ $(i = 1, 2, \cdots, n)$ on $[0, \infty)$, for any choice of $\alpha_1, \alpha_2, \cdots, \alpha_n$ with all $\alpha_i \geq 1$. Obviously $u(x) = x^{\Pi \alpha_i}$ on $[0, \infty)$, so that [1, p. 133] gives

$$\lambda(u_i) = \frac{\rho(u_i)}{1 + \rho(u_i)} = \frac{2^{\alpha_i} - 1}{2^{\alpha_i}} = 1 - 1/2^{\alpha_i} \quad \text{for each } i.$$

It is now trivial to show by induction that $P_n(1/2) = 1 - 1/2^{\Pi \alpha_i}$. Hence

$$\rho(u) = 2^{\Pi \alpha_i} - 1 = \frac{1 - 1/2^{\Pi \alpha_i}}{1/2^{\Pi \alpha_i}} = \frac{P_n(1/2)}{1 - P_n(1/2)}.$$

Remark. It would be interesting, in Theorem 7, to see how $\rho(u)$ depends on the individual dilatations $\rho(u_i)$. For simplicity, we restrict the investigation to the case $n = 2$.

If $n = 2$, then by Theorem 7

$$\rho(u) = \rho(u_2 \circ u_1) \leq \frac{P_2(1/2)}{1 - P_2(1/2)} = \frac{P_{\lambda(u_2)}(\lambda(u_1))}{1 - P_{\lambda(u_2)}(\lambda(u_1))}. \tag{12}$$

Since $1/2 \leq \lambda(u_1) < 1$, there exists an integer $k \geq 1$ with

$$1 - 1/2^k \leq \lambda(u_1) \leq 1 - 1/2^{k+1}.$$

Solving for k and using Definition 1, we find that

$$k \leq \frac{\log(1 + \rho(u_1))}{\log 2} \leq k + 1.$$

Thus

$$P_{\lambda(u_2)}(\lambda(u_1)) \leq P_{\lambda(u_2)}(1 - 1/2^{k+1}) = 1 - (1 - \lambda(u_2))^{k+1}$$

and using this in (12), we obtain the bound

$$\rho(u_2 \circ u_1) \leq (1 + \rho(u_2))^{k+1} - 1 \leq (1 + \rho(u_2))^{\frac{\log(2+2\rho(u_1))}{\log 2}} - 1. \tag{13}$$

Now suppose the function u_1 is fixed, and let $\alpha = \log(2 + 2\rho(u_1))/\log 2$. Then $2 \leq \alpha < \infty$, and (13) shows that

$$\rho(u_2 \circ u_1) \leq (1 + \rho(u_2))^{\alpha} - 1. \tag{14}$$

A simple expansion of the right-hand side in (14) shows that

$$\rho(u_2 \circ u_1) \leq \rho(u_2)^{\alpha} + O[\rho(u_2)^{\alpha-1}] \text{ as } \rho(u_2) \text{ approaches } \infty. \tag{15}$$

Suppose now that u_2 is fixed instead of u_1. Then, using (13) and the identity $A^{\log B} = B^{\log A}$, with $A = 1 + \rho(u_2)$ and $B = 2 + 2\rho(u_1)$, we obtain the relation

$$\rho(u_2 \circ u_1) \leq (1 + \rho(u_2))^{\frac{\log(2+2\rho(u_1))}{\log 2}} - 1 = (2 + 2\rho(u_1))^{\frac{\log(1+\rho(u_2))}{\log 2}} - 1. \tag{16}$$

Let $\beta = \log(1 + \rho(u_2))/\log 2$. Then $1 \leq \beta < \infty$, and (16) implies that

$$\rho(u_2 \circ u_1) \leq (2 + 2\rho(u_1))^{\beta} - 1. \tag{17}$$

A simple expansion of the right-hand side in (17) shows that

$$\rho(u_2 \circ u_1) \leq (2\rho(u_1))^{\beta} + O[\rho(u_1)^{\beta-1}] \quad \text{as } \rho(u_1) \text{ approaches } \infty.$$

The inequalities (14) and (17) show that if u_1 (or u_2) is fixed, then $\rho(u_2 \circ u_1)$ is bounded by a power function of $\rho(u_2)$ (or $\rho(u_1)$) as $\rho(u_2)$ (or $\rho(u_1)$) tends to infinity, the power depending on the finite constant $\rho(u_1)$ (or $\rho(u_2)$).

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