LINK MANIFOLDS

Louis H. Kauffman

INTRODUCTION

A *link-manifold* M^{2n+1} is a smooth closed manifold admitting a smooth action of the orthogonal group $O(n)$ such that the isotropy subgroups are conjugate to $O(n)$, $O(n-1)$, or $O(n-2)$, and such that for $n > 1$ the orbit space is the 4-disk D^4. The set of fixed points in M corresponds to a link $L \subset S^3 = \partial D^4$. (For $n > 1$, one assumes that all three orbit types occur. For $n = 1$, the orbit space is taken to be S^3 and the orbits are 0-spheres and fixed points.)

These manifolds occur readily in nature. For example, let $M_{a,b}$ denote the Brieskorn manifold [2] $V(Z_0^a + Z_1^b + Z_2^2 + \cdots + Z_{n+1}^2) \cap S^{2n+3}$. Then $O(n)$ acts on $M_{a,b}$ via the last n coordinates, giving it the structure of a link manifold whose fixed-point set is a torus link of type (a, b).

In this paper, we generalize results of F. Hirzebruch and D. Erle [6] (see also [1] and [7] to [10]) to obtain a classification of link manifolds in terms of embedding invariants of links in S^3 (Theorems 10 and 11).

Link manifolds are a larger class than knot manifolds. We show that for $n = 2k - 1$ $(k \geq 2)$ *every* $(n - 1)$ connected $(2n + 1)$-manifold that bounds a parallelizable manifold is a link manifold (Theorem 7).

The results in this paper were announced in [11].

1. LINKING NUMBERS AND INVARIANTS OF LINKS

A. *Seifert Pairing*

Given a Link $L \subset S^3$ with preassigned orientations for the components, one may form a connected oriented surface $F \subset S^3$ with $\partial F = L$ such that F induces the chosen orientation for L (see [16, p. 572]). Define

$$\theta: H_1(F) \times H_1(F) \to \mathbb{Z}$$

via $\theta(a, b) = \ell(i_* a, b)$, where $\ell(\ ,\ )$ denotes linking numbers in S^3 and i_* denotes the operation of pushing away from F in the positive normal direction. This bilinear pairing is called the Seifert pairing. Symmetrizing, one obtains the mapping $f: H_1(F) \times H_1(F) \to \mathbb{Z}$ defined by the formula $f(x, y) = \theta(x, y) + \theta(y, x)$.

An argument due to J. Levine [13] shows that if F' is another surface in S^3 whose boundary is ambient-isotopic to L and if V and V' denote matrices for the Seifert pairings for F and F', respectively, then V and V' are *related*. This means that V' may be obtained from V by a chain of operations of the two types

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(i) $V \leftrightarrow PVP^t$, where P is unimodular over $\mathbb{Z}$ and P^t denotes the transposed matrix of P,

(ii)
$$V \leftrightarrow \left[\begin{array}{c|cc} V & 0 & \\ \hline \alpha & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \quad \text{or} \quad V \leftrightarrow \left[\begin{array}{c|cc} V & \beta & 0 \\ \hline & 0 & 1 \\ 0 & & \\ & 0 & 0 \end{array}\right],$$

where α is a row vector and β is a column vector.

Definition. The *signature* of the link L (notation: $\sigma(L)$) is defined by $\sigma(L) = \sigma(f)$, where $\sigma(f)$ denotes the signature of the bilinear form in the preceding paragraph.

Note that $\sigma(L)$ depends only on the embedding type of L, since the condition that V is related to V' implies that $V + V^t$ and $V' + V'^t$ have the same signature.

Remark. Our signature is the same as the Murasugi signature [14, pp. 389-394]. K. Murasugi defines a signature for links in terms of a symmetric matrix associated with a link projection. His matrix is the matrix of f for an appropriate choice of F and basis for $H_1(F)$.

B. *Pass-Equivalence*

This section is devoted to an equivalence relation on links that is useful for classifying link manifolds whose dimension is congruent to 1 modulo 4. Corresponding to each link $L \subset S^3$, there is a connected oriented surface F spanning L. In fact, F may be represented as a standardly embedded disk with attached (possibly knotted, twisted, and linked) bands.

Definition. A *band-pass* operation on F is the local replacement of an over-crossing of bands with an under-crossing, or vice versa. This may be performed between two different bands or upon the same band (see Figure 1).

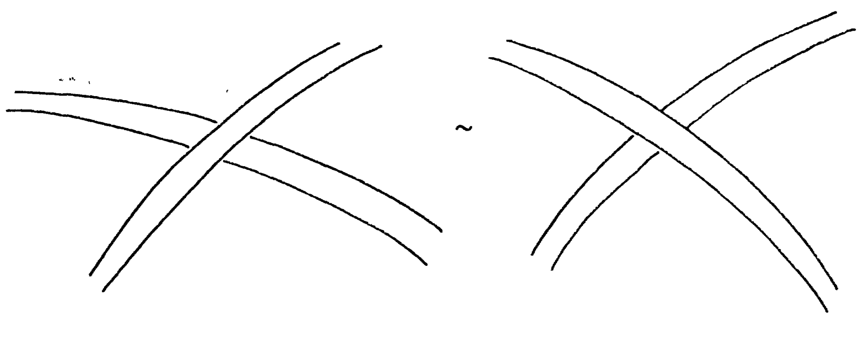


Figure 1.

Definition. Two links L and L' are said to be *pass-equivalent* (notation: $L \sim L'$) if we can obtain L' from L by a sequence of band passes, possibly choosing a new spanning surface at each stage.

Definition. Let $L = K_1 \cup K_2 \cup \cdots \cup K_r$, where the K_i are the components of the link L. We say that L is *proper* if $\ell(K_i, L - K_i)$ is even for $i = 1, 2, \cdots, r$.

LEMMA 1. *If* $L \sim L'$ *and* L *is proper, then* L' *is also proper.*

Proof. Simply note that a single band-pass preserves the parity of the linking numbers $\ell(K_i, L - K_i)$.

Let δ denote the trivial knot, τ the trefoil knot, λ_0 a trivial link of two components, λ_1 a link consisting of two unknotted circles with linking number one. When I write $L = L' \cup L''$, where L' and L'' are links, this means that there exist disjoint three-balls in S^3 enclosing L' and L'', respectively.

Given two links L_1 and L_2, let $L_1 \# L_2$ be the link obtained from $S^3 \# S^3$ with L_1 in one piece and L_2 in the other by choosing a three-ball B_i in each S^3 such that $B_i \cap L_i$ is a single strand. Form $S^3 \# S^3$ by removing these balls, and let the gluing homeomorphism $S^2 \to S^2$ take the two points in $\partial B_1 \cap L_1$ to the two in $\partial B_2 \cap L_2$. The link $L_1 \# L_2$ is not well-defined, but this symbol will denote any link obtained by this procedure.

Definition. Let $L_1, L_2 \subset S^3$ be two links. Suppose that L and L' denote two choices for a connected sum $L_1 \# L_2$. We say that L' is obtained from L by a *rearrangement.*

Given any two links $L, L' \subset S^3$, we say that L is *place-equivalent* to L' (notation: $L \leftrightarrow L'$) if L' may be obtained from L by a finite sequence of pass-equivalences and rearrangements.

Note that connected sum is well-defined on place classes of links. Also, if L is proper and $L \leftrightarrow L'$, then L' is proper.

LEMMA 2. $\tau \# \tau \sim \delta, \quad \tau \# \lambda_1 \sim \lambda_1, \quad \lambda_0 \# \lambda_1 \sim \lambda_1 \# \lambda_1.$

Proof. The first of these pass-equivalences is illustrated in Figure 2. The others follow similarly. Note that in Figure 2, the surfaces F_1, F_2, and F_3 are ambient-isotopic. The knot $\tau \# \tau$ is ambient-isotopic to the boundary of the surface F. The surface F' is ambient-isotopic to F, and the surface F'' is obtained from F' by band-passing. Since the boundary of F'' is the trivial knot, this shows that $\tau \# \tau \sim \delta$.

PROPOSITION 3. *Let* $L \subset S^3$ *be any link. Then*

(i) $L \sim \lambda_0 \# \cdots \# \lambda_0 \# \lambda_1 \# \cdots \# \lambda_1 \# \tau \# \cdots \# \tau$,

(ii) L *is place-equivalent to one of the following:*

$$A = \lambda_0 \# \lambda_0 \# \cdots \# \lambda_0,$$

$$B = \lambda_0 \# \lambda_0 \# \cdots \# \lambda_0 \# \tau,$$

$$C = \lambda_0 \# \lambda_0 \# \cdots \# \lambda_0 \# \lambda_1.$$

Proof. Choose a spanning surface F for L so that the cores of the bands represent a basis for $H_1(F, D^2) \simeq H_1(F)$ and so that, with respect to this basis, the matrix of the intersection pairing

$$\langle\ ,\ \rangle : H_1(F) \times H_1(F) \to Z$$

has the form $S(-1) \oplus S(-1) \oplus \cdots \oplus S(-1) \oplus [0]$. Here $S(-1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and $[0]$ is an s-by-s zero matrix. Thus F is a disk with $2r + s$ bands. We may list the bands as $B_1, \overline{B}_1, \cdots, B_r, \overline{B}_r, B_1', \cdots, B_s'$ so that the pairs correspond to copies of $S(-1)$ and the collection of singlets corresponds to the zero matrix. Applying band-passing

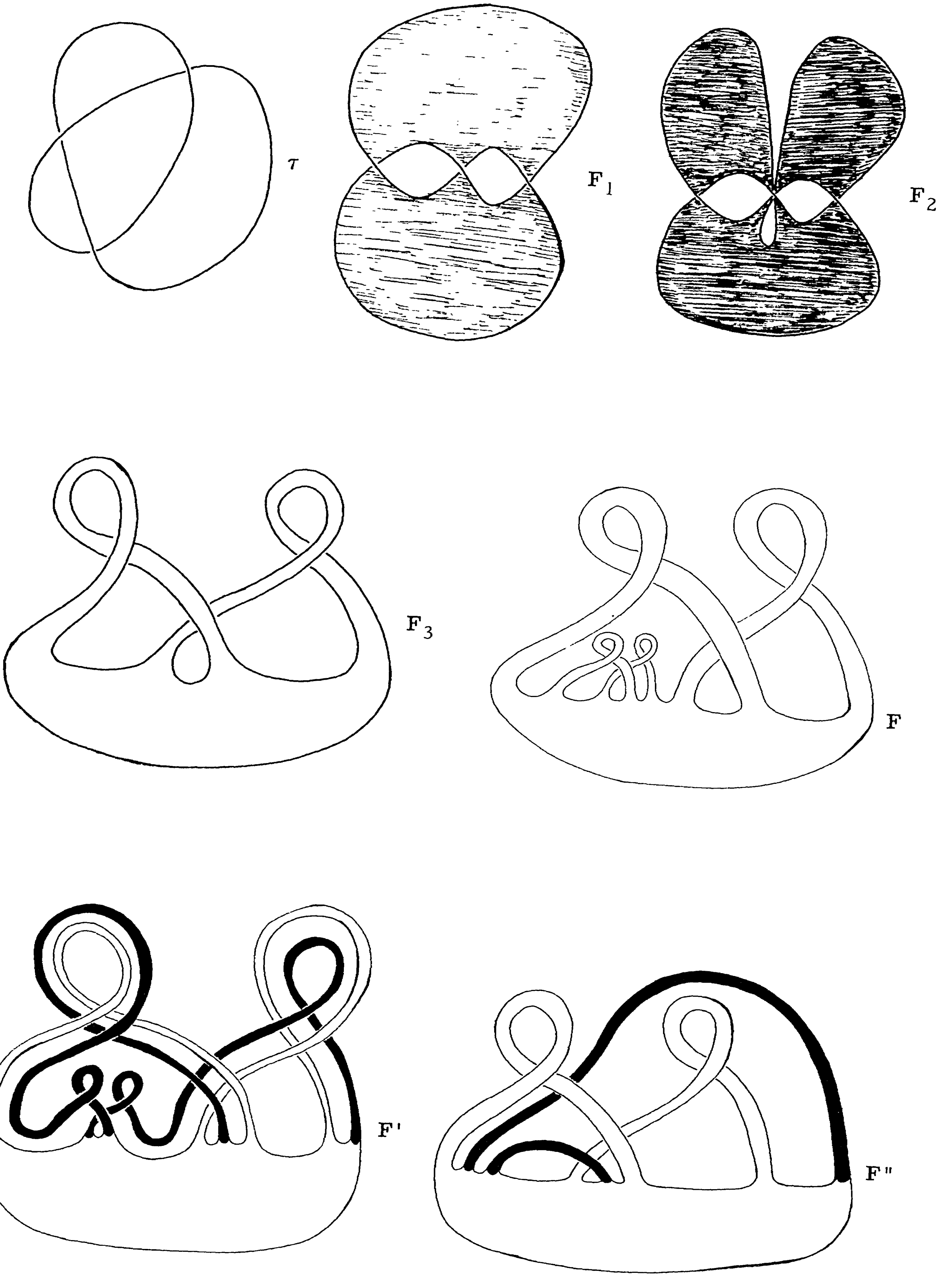


Figure 2.

operations to F, separate the bands so that B_i and $\overline{B}_i$ are not linked with B_j, $\overline{B}_j$, or B'_k for $i \neq j$ and any k. Pass the bands of form B'_k so that they are not linked with one another. Letting $\overline{F}$ denote this new surface, we see that $\overline{F}$ decomposes into a boundary-connected sum of surfaces, each carrying at most two bands.

The proposition now follows from repeated application of Lemma 2.

To strengthen Proposition 3, we would like to show that A is not pass- or place-equivalent to B. In order to do this, we introduce some algebra.

Let $L \subset S^3$ be a link with spanning surface F, and let $W = H_1(F)$. Note that the intersection and Seifert pairings are related by the formula

$$\langle x, y \rangle = \theta(x, y) - \theta(y, x)$$

(see [6, p. 178]).

Definition. Let $\overline{W} = W \otimes Z_2$, and define $\psi: \overline{W} \to Z_2$ by the equation $\psi(x) = \theta(x, x) \pmod 2$. Note that $\psi(x + y) = \psi(x) + \psi(y) + \langle x, y \rangle \pmod 2$ and thus ψ is a Z_2-quadratic form associated with the skew form $\langle\ ,\ \rangle$.

Remark. Certainly ψ depends upon the choice of spanning surface. However, it follows as in Part A of Section 1 that ψ is determined up to direct sums with a form which we denote by ϕ_0

$$(\phi_0: Z_2 \oplus Z_2 \to Z_2,\ \phi_0(a) = \phi_0(b) = 0,\ \langle a, b \rangle = 1, \text{ where } a = (1, 0), b = (0, 1)).$$

Since the Arf invariant $c(\phi_0)$ is 0, we conclude as follows.

LEMMA 4. *When defined, the* Arf *invariant* $c(\psi)$ *is an invariant of link type.*

Definition. Let ψ_1 and ψ_2 be two Z_2-quadratic forms. We say ψ_1 is *s-equivalent* to ψ_2 (notation: $\psi_1 \sim \psi_2$) if and only if

$$\psi_1 \oplus \phi_0 \oplus \cdots \oplus \phi_0 \simeq \psi_2 \oplus \phi_0 \oplus \cdots \oplus \phi_0$$

with ϕ_0 as above.

LEMMA 5. *If* L *and* L' *are links with* Z_2*-quadratic forms* ψ *and* ψ' *corresponding to spanning surfaces* F *and* F', *then* $L \longleftrightarrow L'$ *implies* $\psi \sim \psi'$.

Proof. It is clear that a band-pass operation leaves the Z_2-form alone. Change of spanning surface induces s-equivalence, as we remarked above. Since any connected sum of links gives rise to the (well-defined) connected sum of forms, rearrangements have no effect.

COROLLARY 6. *If* L *is a link with* Z_2*-form* ψ, *then the* Arf *invariant* $c(\psi)$ *is defined exactly when* L *is proper. Furthermore, if* L *is proper, then*

$$c(\psi) = 0 \iff L \longleftrightarrow A, \qquad c(\psi) = 1 \iff L \longleftrightarrow B.$$

In particular, A *is not place-equivalent to* B.

Proof. This follows from Proposition 3, Lemmas 4 and 5, and the definition of ψ, since the Arf invariant is defined exactly when $\psi \mid \text{Rad}\ \psi \equiv 0$ (see [3, p. 56]).

Definition. If L is a proper link, let

$$\phi(L) = \begin{cases} 0 & \text{if } L \leftrightarrow A, \\ 1 & \text{if } L \leftrightarrow B. \end{cases}$$

We call $\phi(L)$ the Robertello-Arf invariant of L. This generalizes a definition due to Robertello for knots (see [15]). One can show that $\phi(L)$ is a concordance invariant [12].

Remark. Letting P denote the set of place-classes of links and C the set of s-equivalence classes of Z_2-quadratic forms, we may define $T: P \to C$ by saying that $T(L)$ is the s-equivalence class of $\psi(L)$. It is an easy corollary of the discussion above and the classical theory of quadratic forms that T is one-to-one and surjective, and that it takes # into $\oplus$.

C. *Linking Invariant*

Let $f: V \times V \to \mathbb{Z}$ be a symmetric bilinear form, where V is a finitely generated free $\mathbb{Z}$-module. Letting $V^* = \mathrm{Hom}(V, \mathbb{Z})$, one has the adjoint

$$Af: V \to V^*, \qquad Af(x)(y) = f(x, y).$$

Let G denote the cokernel of Af and τG the torsion subgroup of G.

Definition. Define the linking pairing $b(f): \tau G \times \tau G \to Q/\mathbb{Z}$ by the equation $b(f)(\bar{x}, \bar{y}) = \frac{1}{rs} f(X, Y) \pmod 1$. Here $x, y \in V^*$ are representatives of $\bar{x}, \bar{y}$, and $rx = Af(X)$, $sy = Af(Y)$, $r, s \in \mathbb{Z}$.

If f is even, that is, if $f(x, x)$ is even for all $x \in V$, then one also defines $q(f): \tau G \to Q/\mathbb{Z}$ by

$$q(f)(\bar{x}) = \frac{1}{r^2}\left(\frac{1}{2} f(X, X)\right) \pmod 1.$$

This is a quadratic form associated with $b(f)$.

Definition. Using the notation of Part A of Section 1, let $V = H_1(F)$, where F is a spanning surface for L, and let f be the pairing defined on V by the equation $f(x, y) = \theta(x, y) + \theta(y, x)$. Define

$$G(L) = \text{cokernel } (Af), \qquad b(L) = b(f), \qquad q(L) = q(f) \quad (f \text{ is even}).$$

The group $G(L)$ and the forms $b(L)$ and $q(L)$ are then invariants of link type. In fact, $G(L)$ is isomorphic to the first homology group of the double branched cover of S^3 with branch set L; the pairing $b(L)$ is the linking invariant for this manifold. (This last fact follows from the remarks at the beginning of the next section.)

2. CLASSIFICATION OF LINK MANIFOLDS

Definition. We denote by B_{2n} the set of $O(n-1)$-link manifolds; by BP_{2n} the set of $(n-2)$-connected $(2n-1)$-manifolds that bound parallelizable manifolds.

By directly carrying over the equivariant surgery technique of [6, pp. 201-207], we can easily prove the following facts.

(1) $B_{2n} \subset BP_{2n}$. Indeed, if $L \subset S^3$ is an oriented link and F is a connected orientable spanning surface for L, then by O(n - 1)-equivariant surgery on S^{2n-1} one can construct $M^{2n-1}(L) = \partial N^{2n}(F) \in B_{2n}$. The manifold $N^{2n}(F)$ is (n - 1)-connected and parallelizable. Its boundary $M^{2n-1}(L)$ is a link manifold corresponding to L, and it is independent of the choice of Seifert surface F.

(2) $N^{2n}(F)$ has intersection matrix $V + (-1)^{n+1} V^t$, where V is the matrix of the Seifert pairing $\theta: H_1(F) \times H_1(F) \to \mathbb{Z}$.

(3) If n is odd and $n \geq 3$, and if F' is obtained from F by a band-pass operation (see Part B of Section 1), then $N^{2n}(F) \simeq N^{2n}(F')$.

As we noted above, $B_{2n} \subset BP_{2n}$. We shall see shortly that if n is odd then $B_{2n} \neq BP_{2n}$. However, we have the following result.

THEOREM 7. $BP_{4k} = B_{4k}$ *for* $k > 1$.

The proof of this theorem will proceed by way of a lemma about Seifert matrices for links.

Definition. Let A be a symmetric square matrix with even entries on the diagonal. We say that A is of *link-type* if no row contains more than one odd entry.

LEMMA 8. *If* A *is of link-type, then* $A = V + V^t$, *where* V *is a Seifert matrix for some link in* S^3.

Proof. We shall prove the lemma by constructing a disk with attached bands such that $A = V + V^t$, where V is the Seifert matrix for this surface. If A is an $n \times n$ matrix, we want a surface F with n bands. Let $\alpha_i \in H_1(F)$ $(i = 1, \cdots, n)$ be the homology classes corresponding to the bands. Thus, if $A = (a_{ij})$, then F must be constructed so that $a_{ij} = \theta(\alpha_i, \alpha_j) + \theta(\alpha_j, \alpha_i)$. The following observations are in order.

(1) The matrix element $a_{ii} = 2\theta(\alpha_i, \alpha_i)$ specifies the twisting of the band corresponding to α_i.

(2) For $i \neq j$, the linking number $\theta(\alpha_i, \alpha_j)$ is independent of the twisting of the ith and jth bands. It is specified by the embeddings of their cores.

(3) Consider the two points of intersection of a band core with D^2. Call these the *feet* of the band. Choose an orientation for the disk and therefore for its boundary. Given two points $p, q \in S^1 = \partial D^2$ that divide S^1 into unequal segments, let [p, q] be the smaller segment. We say $p < q$ if, when this segment is oriented from p to q, the orientation agrees with the orientation of S^1.

Assume that the feet of each band divide S^1 into unequal segments. If p and q are the feet of a band with $p < q$, we say that a point x is between p and q if $x \in [p, q]$.

(4) Letting $\psi(\alpha, \alpha') = \theta(\alpha, \alpha') + \theta(\alpha', \alpha)$, note that we can ensure that $\psi(\alpha, \alpha')$ is odd by planting one foot of α' between the feet of α and adjusting the linking accordingly.

(5) We can ensure that $\psi(\alpha, \alpha')$ is even by keeping both feet of α' out from between the feet of α.

Induction Hypothesis. The lemma is true for all matrices A of link type and size $r \times r$ for $r \leq n$. Assume that if α and α' are band cores and $\psi(\alpha, \alpha')$ is even, then α has no feet between the feet of α', and vice versa; if $\psi(\alpha, \alpha')$ is odd, then each band has one foot between the feet of the other.

Case 1. If A is a 1-by-1 matrix (a), take a disk with one band having a half-twists.

Case 2. Suppose $A = (a_{ij})$ is $(n+1)$-by-$(n+1)$ and of link type. Let $\overline{A} = (a_{ij})$ $(i \leq n,\ j \leq n)$. Since $\overline{A}$ is also of link type, we may apply the induction hypothesis. Since $a_{n+1,j}$ is odd for at most one j $(1 \leq j \leq n)$, we can choose the feet of α_{n+1} as follows: If $a_{n+1,j}$ is odd, let p be a point between the feet of α_j and choose $q > p$ so that q does not lie between the feet of any band. Then p and q are the feet of α_{n+1}. Note that such p and q can be chosen because $a_{n+1,j}$ is odd and, hence, $a_{j,k}$ is even for $1 \leq k \leq n$ (since A is of link type). Therefore no other feet stand between the feet of α_j, nor do the feet of α_j stand between any feet (by the induction hypothesis).

If $a_{n+1,j}$ is even for $1 \leq j \leq n$, choose $p < q$ so that p and q stand outside the feet of all the bands. Again, p and q will be the feet of α_{n+1}.

Finally, change an arc from p to q by cutting small segments from it and replacing these by segments linking the α_i so that $a_{n+1,i} = \psi(\alpha_{n+1}, \alpha_i)$. This constructs the core of the $(n+1)$st band. Now thicken this core into a band and introduce $a_{n+1,n+1}$ half-twists. The result is a new surface satisfying the induction hypothesis and such that $a_{ij} = \psi(\alpha_i, \alpha_j)$ for $i, j \leq n+1$.

Hence the lemma is proved by induction.

LEMMA 9. *Let* A *be a symmetric square matrix with even entries on the diagonal. Then there exists a unimodular matrix* P *such that* PAP^t *is of link type.*

Proof. A is a matrix over $\mathbb{Z}$. Let $\overline{A} = (\bar{a}_{ij})$ be the matrix of residue classes modulo 2 over $\mathbb{Z}_2$. A matrix over $\mathbb{Z}_2$ will be said to be of *link type* if no row contains more than one nonzero entry. Thus, if A is of link type, then $\overline{A}$ is of link type, and conversely. However, over $\mathbb{Z}_2$, the symmetric matrix $\overline{A}$ is congruent to a matrix $\overline{B}$ of the form

$$\overline{B} = \begin{bmatrix} 0 & & & 1 \\ & & \cdot & \\ & \cdot & & \\ 1 & & & 0 \end{bmatrix} \oplus [0].$$

In fact, $\overline{B}$ is obtained from $\overline{A}$ by a sequence of simultaneous row and column operations. Each operation is represented by conjugation with an elementary matrix $\overline{E}$, where E is invertible over $\mathbb{Z}$. Hence $\overline{B} = \overline{P}\,\overline{A}\,\overline{P}^t$, where P is invertible over $\mathbb{Z}$. Now $\overline{PAP^t} = \overline{P}\,\overline{A}\,\overline{P}^t = \overline{B}$. Hence PAP^t is of link type. This proves the lemma.

Proof of Theorem 7. Given $M \in BP_{4k}$ $(k > 1)$, we know that there exists a manifold N^{4k}, parallelizable and $(2k-1)$-connected, such that $\partial N = M$. Also, by [17] and [18], N is determined up to diffeomorphism by its intersection form on $H_{2k}(N)$. If A is the matrix of this form with respect to some basis, then A is symmetric, with even diagonal entries. By Lemma 9, we see that PAP^t is of link type, and hence there is a link $L \subset S^3$ with spanning surface F such that $N^{4k}(F)$ has intersection form PAP^t. Thus $N(F) \simeq N$, and thus $M^{4k-1}(L) \simeq M^{4k-1}$; this proves the theorem.

The next two theorems give a more detailed picture of the diffeomorphism classification.

THEOREM 10. *Let* $k > 1$ *and* $M(L), M(L') \in B_{4k}$, *and suppose that*

$$G(L) \simeq G(L'), \qquad q(L) \simeq q(L'), \qquad \sigma(L) \geq \sigma(L')$$

(see Section 1); *then*

$$M(L) \simeq M(L') \# \frac{1}{8}(\sigma(L) - \sigma(L')) \cdot \Sigma,$$

where Σ *is the Milnor sphere.* (If $G(L)$ and $G(L')$ have no summands of order 2 or 4, we may replace the hypothesis $q(L) \simeq q(L')$ by $b(L) \simeq b(L')$.)

Proof. The remarks at the beginning of this section show that

$$G(L) \simeq H_{2k-1}(M(L)), \; q(L), \text{ and } b(L)$$

may be identified with the linking quadratic and bilinear forms on the torsion part of $H_{2k-1}(M(L))$. Also, $\sigma(L)$ is the signature of $N(L)$. The theorem now follows from the fact that $B_{4k} \subset BP_{4k}$ and from a theorem of A. Durfee (see [5]), classifying manifolds in BP_{4k}.

THEOREM 11. *Let* $M = M(L) \in B_{2n}$ *for* n *odd,* $n > 2$. *Let the link* $L \subset S^3$ *have* $(r + 1)$ *components.*

(1) *If* L *is proper, then*

$$M \simeq \begin{cases} (S^{n-1} \times S^n) \# (S^{n-1} \times S^n) \# \cdots \# (S^{n-1} \times S^n) & \text{if } \phi(L) = 0, \\ (S^{n-1} \times S^n) \# (S^{n-1} \times S^n) \# \cdots \# (S^{n-1} \times S^n) \# \Sigma_1 & \text{if } \phi(L) = 1. \end{cases}$$

There are r *copies of* $S^{n-1} \times S^n$ *in each connected sum. The symbol* Σ_1 *denotes the Kervaire sphere, and* $\phi(L)$ *is the Robertello-Arf invariant as defined in* Section 1.

(2) *If* L *is improper, then*

$$M \simeq (S^{n-1} \times S^n) \# \cdots \# (S^{n-1} \times S^n) \# T.$$

This is a connected sum of r *manifolds. The symbol* T *denotes the tangent sphere bundle to* S^n.

(Note that for $n = 3$ or 7, the theorem degenerates to the statement

$$M \simeq (S^{n-1} \times S^n) \# \cdots \# (S^{n-1} \times S^n).)$$

Proof. It follows from fact (3) at the beginning of this section that the relation $L \sim L'$ implies $M(L) \simeq M(L')$ ($\sim$ denotes pass-equivalence). The theorem now follows from Proposition 3, the definition of $\psi(L)$, and identification of $M(A)$, $M(B)$, and $M(C)$, where A, B, and C are the three link types discussed in Section 1.

Since a connected sum of links corresponds to a connected sum of the corresponding manifolds, it suffices to identify $M(\lambda_0)$, $M(\lambda_1)$, and $M(\tau)$. But

$$M(\lambda_0) \simeq S^{n-1} \times S^n, \qquad M(\lambda_1) \simeq T, \qquad M(\tau) \simeq \Sigma_1.$$

This may be seen by direct calculation of the corresponding quadratic forms.

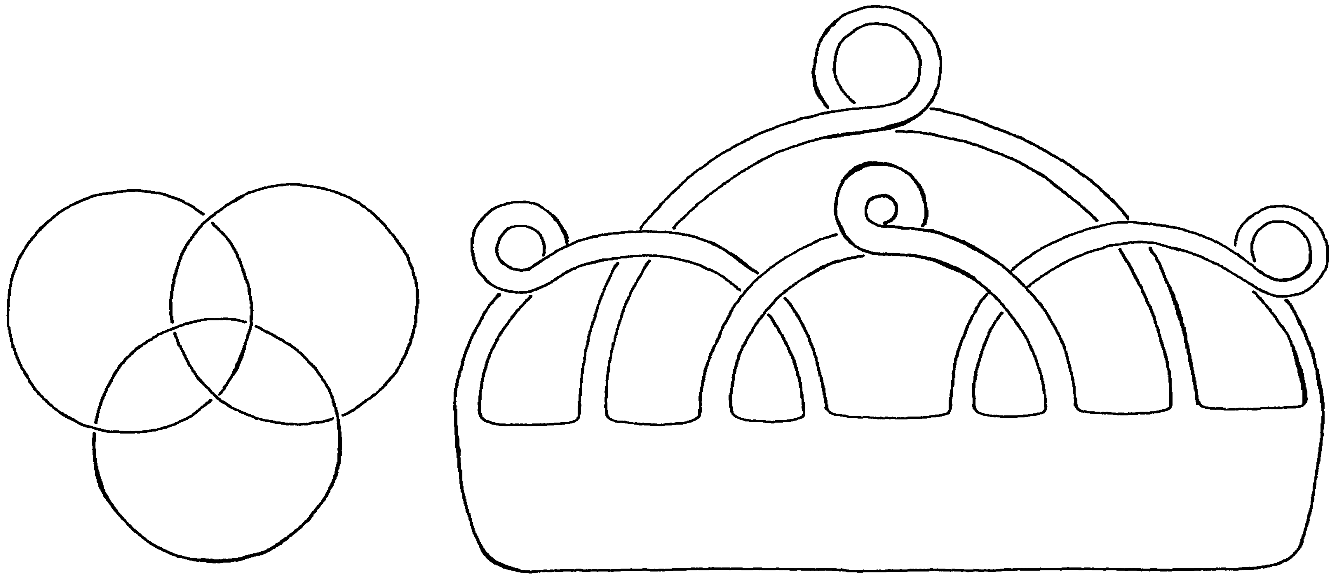
3. APPLICATIONS AND EXAMPLES

Throughout this section, n will denote an odd integer greater than 2.

(a) If L_k denotes a torus link of type $(2, k)$, then $L_k \sim L_{k+8}$, and thus one can deduce the 8-fold periodicity in the list $M^{2n-1}(L_k)$ $(k = 1, 2, \cdots)$. This was explained in [11].

(b) Let L be the Borommean rings. Figure 3 illustrates a surface F with $\partial F = L$ from which it is easy to deduce by band-passing that

$$M^{2n-1}(L) \simeq (S^{n-1} \times S^n) \# (S^{n-1} \times S^n) \# \Sigma_1 .$$



L F

Figure 3.

(c) Suppose L is any link with two components. Say $L = K_1 \cup K_2$, and let $\ell = \ell(K_1, K_2)$ be the linking number in S^3. Theorem 11 implies that $M^{2n-1}(L)$ is then diffeomorphic to $S^n \times S^{n-1}$ or to $(S^{n-1} \times S^n) \# \Sigma_1$ if ℓ is even and to T if ℓ is odd. This was a conjecture of Michael Davis (see [4, p. 311]).

(d) PROPOSITION 12. *Suppose that* L *and* L' *are oriented links sharing the same unoriented link. Take* $M(L), M(L') \in B_{4k}$ *with* $k > 1$. *Then* $G(L) \simeq G(L')$, $b(L) \simeq b(L')$, *and hence, if* $G(L)$ *has no summands of order* 2 *or* 4, *then*

$$M(L) \simeq M(L') \# \frac{1}{8} (\sigma(L) - \sigma(L')) \cdot \Sigma$$

(we assume that $\sigma(L) \geq \sigma(L')$).

Proof. The double branched cover of S^3 with branching set L is independent of the choice of orientation for L. Hence $G(L) \simeq G(L')$ and $b(L) \simeq b(L')$. The Proposition now follows from Theorem 10.

For example, let L and L' be as in Figure 4. Then a calculation reveals that $\sigma(L) = 8$, $\sigma(L') = 0$, $G(L) = \mathbb{Z}$. Hence

$$M(L) \simeq S^{2k-1} \times S^{2k} \# \Sigma, \qquad M(L') \simeq S^{2k-1} \times S^{2k} .$$

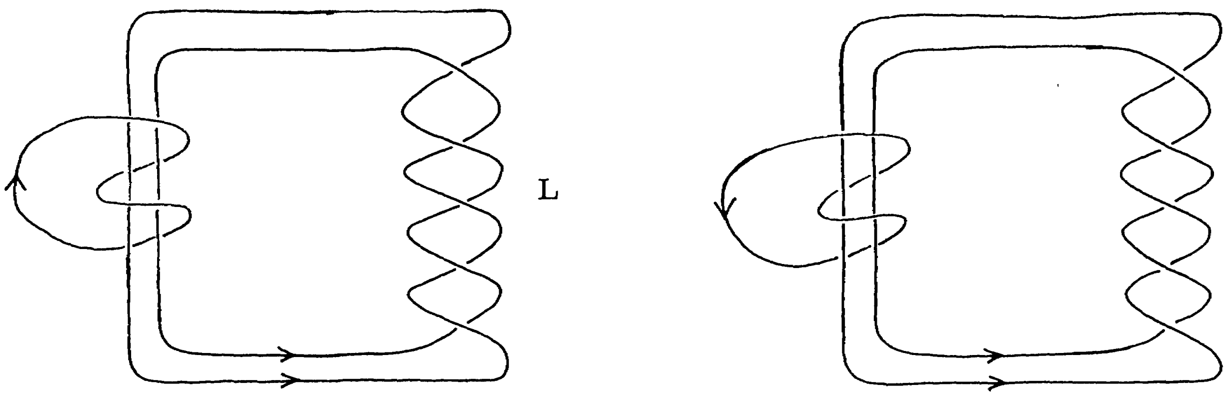


Figure 4.

(e) Let $L_{a,b}$ denote a torus link of type (a, b). If $d = \gcd(a, b)$, so that $a = d\alpha$ and $b = d\beta$, where α and β are relatively prime positive integers, then $L_{a,b}$ consists of d torus knots of type (α, β). (A torus knot is a knot that lies on a standardly embedded torus in S^3 and winds α times in the meridian direction and β times in the longitudinal direction on this torus.)

Torus links may also be described as follows: Let $f(x, y) = X^a + Y^b$ be a polynomial in two complex variables. Let

$$V(f) = \{(x, y) \in C^2 \mid f(x, y) = 0\} .$$

Then $L_{a,b} = V(f) \cap S^3$. If $L_{a,b} = K_1 \cup K_2 \cup \cdots \cup K_d$, where each component K_i is an (α, β)-torus knot, then one can verify that $\ell(K_i, K_j) = \alpha\beta$ for $i \neq j$.

PROPOSITION 13. *Let* $\approx$ *denote homeomorphism. For a fixed positive integer* b, *let* $M_a^{2n-1} = M^{2n-1}(L_{a,b})$. *Then the list of manifolds* M_a^{2n-1} $(a = 1, 2, \cdots)$ *has a homeomorphism periodicity* 2b; *that is*, $M_a^{2n-1} \approx M_{a+2b}^{2n-1}$.

Proof. Let $d(a) = \gcd(a, b)$ and $\ell(a) = ab/d(a)^2$. By Theorem 11 we know that $M_a^{2n-1} \approx M_{a'}^{2n-1}$ if and only if $d(a) = d(a')$ and $L_{a,b}$ and $L_{a',b}$ are either both proper or both improper. Now $\ell(a) \equiv \ell(a') \pmod 2$ certainly implies that $L_{a,b}$ and $L_{a',b}$ share propriety or impropriety. Thus the proposition follows from the easily verified fact that $d(a + 2b) = d(a)$ and $\ell(a + 2b) \equiv \ell(a) \pmod 2$.

Remark. This result is best possible. For example, the homeomorphism periodicity in Example (a) is exactly 4. We conjecture that Proposition 13 has a differentiable counterpart.

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University of Illinois at Chicago Circle
Chicago, Illinois 60680