A MINIMAL EXTENSION THAT IS NOT CONSERVATIVE

Robert G. Phillips

1. INTRODUCTION

Let L denote a countable first-order language that has relation symbols for addition, multiplication, and order, and that has 0 and 1 as constant symbols. P denotes Peano's axioms, either for the natural numbers or for the integers, formulated in L. When the context does not distinguish which set of axioms is being denoted by P, then the results or definitions involved are to be interpreted as being valid for either set.

When M is a model for P, then L_M will denote the extension of L obtained by adding the elements of $M - \{0, 1\}$ to L as constant symbols. It is to be understood that when formulas of L_M are interpreted in M, constant symbols will always denote themselves. A formula of L_M is called an *M-formula,* and a relation on M is called *M-definable* if it can be represented in M by an M-formula. If M^* is an extension of M, and R is an n-ary relation on M^*, then

$$\{(x_1, \cdots, x_n): x_1 \in M \wedge \cdots \wedge x_n \in M \wedge (x_1, \cdots, x_n) \in R\}$$

will be called the *restriction* of R to M.

Assume then that M models P and that M^* is a proper elementary extension of M with respect to L_M. If no proper elementary substructure of M^* properly extends M, then M^* is called a *minimal extension* of M. M^* is called a *conservative extension* of M if the restriction of each M^*-definable relation to M is also M-definable.

In [3], H. Gaifman formulated the concept of a minimal extension and proved that each model of P has a minimal extension. In [5], conservative extensions were introduced, and it was proved that each model of P has a conservative extension. It is the primary purpose of this paper to prove the following theorem.

THEOREM 1. *There exists a minimal extension of the standard model of* P *that is not a conservative extension.*

The theorem, and a related result, will be proved in the last section; this section is concluded with a brief account of the reasons that motivated the theorem.

First of all, it is clear that the concepts of minimal and conservative extension do not coincide. A conservative extension of a conservative extension of M is still a conservative extension of M, but it is obviously not a minimal extension of M. Nevertheless, there are certain similarities between the two concepts; a few of these are now listed:

(a) Gaifman's construction of minimal extension and that of conservative extension both depend upon the same basic principle. This principle is that for each unbounded M-definable subset X of M and for each M-definable relation Q on M, there exists an M-definable function f in 2^M such that

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$$M \models \forall x\ \forall y\ \exists z > y\ \forall u \leq x\ [Q^{f(u)}(u, z) \wedge z \in X],$$

where Q^f is the formula "$f = 0 \to Q \wedge f = 1 \to \neg Q$". In other words, the following inductive process can be formalized within L_M: X_0 is $\{x \in X\colon M \models Q(0, x)\}$ or X_0 is $\{x \in X\colon M \models \neg Q(0, x)\}$, subject only to the condition that X_0 is unbounded. X_{u+1} is $\{x \in X_u\colon M \models Q(u + 1, x)\}$ or X_{u+1} is $\{x \in X_u\colon M \models \neg Q(u + 1, x)\}$, again subject only to the condition that X_{u+1} is unbounded.

(b) It is a direct consequence of (a) that the concepts of minimal and conservative extension are compatible. In [5], this was used to show that each model of P has an extension that is both minimal and conservative.

(c) In [5], it was proved that each conservative extension is an end-extension. In [1], A. Blass proves that if a minimal extension M^* of M is not an end-extension of M, then M is cofinal in M^*. Restated in terms of types (see [1] and [3]), minimal types and conservative types are both end-extension types.

(d) If P denotes Peano's axioms for the integers, and consequently the models of P are rings, the additive group structure of conservative extensions is canonical in form and minimal in extent. In order to explain this statement, we let A denote all functions f that map $\{x \in M\colon x > 1\}$ into M, satisfy $0 \leq f(x) < x$ for each x, and have the property that for each i and j in M the greatest common divisor of i and j divides f(i) - f(j). R. MacDowell and E. Specker [4] show that the additive group of each proper elementary extension M^* of M has the form $K \oplus G_{M^*}$. Here, K denotes the vector-space over the rational numbers whose dimension is the cardinality of M^*, and G_{M^*} is some subset of A. If B denotes the set of M-definable functions in A, then B is always a subset of G_{M^*}, and furthermore, the additive group of a conservative extension is always of the form $K \oplus B$; K varies only to the extent that its dimension is the cardinality of the conservative extension in question. The reader is referred to [5] and [7] for the proofs of these assertions.

Because of these considerations, it was not immediately clear whether all minimal extensions are also conservative extensions. It did not seem unreasonable to expect minimal extensions to have at least the minimal additive structure just described in (d); but it will be shown that this is not the case. After this paper was first written, Blass [1] proved that each nonstandard countable model of P has a minimal nonconservative extension. This was done by proving that for each such M there exists a minimal extension M^* in which M is cofinal. However, this method is not applicable to the standard model, since all of its proper elementary extensions are end-extensions.

2. BASIC RESULTS

Let N denote the standard model of P, and let F denote all M-definable functions in N^N. If D is an ultra-filter on N, then D-Prod N denotes the usual ultrapower on N with respect to D, and D-Prod N | F denotes the substructure obtained by restricting D-Prod N to the functions in F. This is essentially T. Skolem's construction [6], and it is well known that D-Prod N | F is a proper elementary extension of N with respect to L_N if D is nonprincipal.

When N^* is an elementary extension of N, N^*_a will denote the elementary substructure of all elements of N^* of the form f(a) as f varies throughout F. Clearly, N^* is minimal if and only if $N^* = N^*_a$ for each a in N^* - N.

We present Gaifman's construction of a minimal extension of N, since we must refer to it later. Let X be an infinite N-definable subset of N, and let f be in F. If there are infinite subsets of X on which f is constant, one of these subsets will be denoted by $D_f(X)$; otherwise, there is an infinite N-definable subset of X on which f is one-to-one, and this subset will be denoted by $D_f(X)$. Let $\{f_n\}$ be an enumeration of the functions in F, and define the sets D_n inductively as follows:

$$D_1 = D_{f_1}(N), \quad \cdots, \quad D_{n+1} = D_{f_{n+1}}(D_n), \quad \cdots.$$

Then $D_1 \supset \cdots \supset D_n \supset \cdots$ is a decreasing sequence of infinite N-definable subsets of N, and therefore, these sets are all included in some nonprincipal ultra-filter D on N. Furthermore, for each n, f_n is either constant on D_n and thus constant modulo D, or else there is a function f_n^* in F such that $f_n^*(f_n(x)) = x$ on D_n. Hence, if N^* is D-Prod $N \mid F$ and π is the identity function on N, then for each f in F that belongs to $N^* - N$ (in other words, for each f that is not constant modulo D) there is an f^* in F such that $N^* \models f^*(f) = \pi$. This implies that N_π^* is a substructure of N_f^*, and since $N_\pi^* = N^*$, $N^* = N_f^*$. Consequently, N^* is a minimal extension of N.

A typical property of conservative extensions is the following: the set of standard prime divisors (that is, the prime divisors in N) of an element in a conservative extension N^* of N is an N-definable subset of N. This is so because such a set is the restriction to N of all prime divisors (standard and nonstandard) of the element, which is of course an N^*-definable subset of N^*.

It has been pointed out by the referee that the property just described is in fact equivalent to N^* being conservative. We add that the minimal additive group structure, discussed under section (d) in the introduction, is also a property equivalent to conservativity. G. Zahn [7] first proved that there are elementary extensions of N with the minimal additive group structure, and A. Cantor [2] first proved the existence of conservative extensions of N. However, it was not immediately recognized that Zahn's extensions are also conservative.

To prove Theorem 1, we shall construct a minimal extension of N of the form D-Prod $N \mid F$ so that the set of standard prime divisors of the identity function π is not N-definable. For this purpose, we introduce the notion of an N-like set.

3. N-LIKE SETS

N_n will denote the set of multiples of n in N. A subset X of N is called N-like with respect to an infinite increasing sequence of prime numbers $\{p_i\}$ if for each n

$$X \cap T_1 \cap \cdots \cap T_n$$

is infinite for each choice of $T_i = N_{p_i}$ or $T_i = N - N_{p_i}$. The next lemma follows directly from this definition.

LEMMA 1. *If* X *is* N-*like with respect to* $\{p_i\}$ *and*

$$A = X \cap T_1 \cap \cdots \cap T_n,$$

where each T_i *is either* N_{p_i} *or* $N - N_{p_i}$, *then* A *is* N-*like with respect to* $\{p_{i+n}\}$.

LEMMA 2. *If* X *is* N-*like with respect to* $\{p_i\}$ *and* $X = A \cup B$, *then there is an* m *such that either* A *or* B *is* N-*like with respect to* $\{p_{i+m}\}$.

Proof. $X \cap T_1 \cap \cdots \cap T_n = (A \cap T_1 \cap \cdots \cap T_n) \cup (B \cap T_1 \cap \cdots \cap T_n)$. If for all n and all choices of $T_i = N_{p_i}$ or $T_i = N - N_{p_i}$ it is the case that $A \cap T_1 \cap \cdots \cap T_n$ is infinite, then A is N-like with respect to $\{p_i\}$. Otherwise, there exist an m and a choice of $T_1, \cdots, T_m$ such that $A \cap T_1 \cap \cdots \cap T_m$ is finite. This implies that B is N-like with respect to $\{p_{i+m}\}$.

LEMMA 3. *For each* N-*like set* X *with respect to* $\{p_i\}$, *and for each* f *in* F, *there exist a subset* $D_f(X)$ *of* X *and an* m *such that*

(a) $D_f(X)$ *is* N-*like with respect to* $\{p_{i+m}\}$,

(b) f *is either constant or one-to-one on* $D_f(X)$,

(c) *if both* X *and* $\{p_i\}$ *are* N-*definable, then* $D_f(X)$ *is* N-*definable.*

Proof. Let $\{S_k\}$ be an enumeration of all sets of the form $X \cap T_1 \cap \cdots \cap T_n$, where as usual each T_i is either N_{p_i} or $N - N_{p_i}$ and n ranges over N. Let f be in F.

Choose x_1 arbitrarily in S_1. If $f(x) = f(x_1)$ for all $x > x_1$ such that $x \in S_2$, let $D_f(X) = \{x \in S_2 : x > x_1\}$. Then $D_f(X)$ satisfies part (a) of the lemma, because of Lemmas 1 and 2. It is trivial to see that $D_f(X)$ also satisfies parts (b) and (c) of the lemma.

Otherwise, choose x_2 in S_2 so that $x_2 > x_1$ and $f(x_1) \neq f(x_2)$. If $x_1 < x_2 < \cdots < x_n$ have been chosen so that $x_i \in S_i$ and $f(x_i) \neq f(x_j)$ for $0 \leq i < j \leq n$, suppose that for each x such that $x > x_n$ and $x \in S_{n+1}$, there is an i such that $f(x) = f(x_i)$. Then an application of Lemmas 1 and 2 shows that there exists an i $(1 \leq i \leq n)$ such that if $D_f(X)$ is set equal to $\{x \in S_{n+1} : x > x_n \wedge f(x) = f(x_i)\}$, then $D_f(X)$ satisfies part (a) of the lemma. In this case, part (b) of the lemma is trivially satisfied, and part (c) is satisfied because only a finite number of the sets S_k are involved in the definition of $D_f(X)$. Otherwise, choose $x_{n+1} > x_n$ so that $x_{n+1} \in S_{n+1}$ and $f(x_{n+1}) \neq f(x_i)$ for $i = 1, \cdots, n$.

The process just described will yield one of two possible outcomes: either a subset $D_f(X)$ of X will be defined that satisfies the lemma, and on which f is constant, because the process terminates at some stage; or the process does not terminate, and therefore an infinite sequence $x_1 < x_2 < \cdots < x_n < \cdots$ will be defined such that $x_n \in S_n$ and $f(x_i) \neq f(x_j)$ for $1 \leq i < j$. If the latter outcome occurs, $D_f(X)$ will be defined to be the infinite sequence $\{x_n\}$. It is now shown that this choice for $D_f(X)$ also satisfies the lemma.

Clearly, part (b) is satisfied because f is one-to-one on $D_f(X)$. The sequence $D_f(X)$ is N-like with respect to $\{p_i\}$ because $D_f(X) \cap S_n$ is infinite for each n. This is so because for each n there exists an infinite sequence $\{n_i\}$ such that $S_n \supset S_{n_1} \supset \cdots \supset S_{n_i} \supset \cdots$. For instance, n_1 can be taken to be the first $j > n$ such that S_j is $S_n \cap N_{p_{i+1}} \cap \cdots \cap N_{p_j}$ when $S_n = X \cap T_1 \cap \cdots \cap T_i$. Then $\{x_n, x_{n_1}, \cdots\} \subset D_f(X) \cap S_n$.

In order for $D_f(X)$ to be N-definable, it is sufficient to show that the sequence $\{S_k\}$ can be chosen in an N-definable manner. Therefore, assuming both X and $\{p_i\}$ are N-definable, for each n we order the finite number of sets of the form $X \cap T_1 \cap \cdots \cap T_n$ in some definite fashion, taking care to use the same procedure for each n. Order the groups of sets of the form $X \cap T_1 \cap \cdots \cap T_n$ according to n;

then the lexicographic order will recursively order $\{S_k\}$, and hence $\{S_k\}$ will be N-definable.

4. A MINIMAL EXTENSION THAT IS NOT CONSERVATIVE

The proof of Theorem 1 can now be presented. Let $\{Q_n\}$ be an enumeration of all N-definable subsets of N that consist of only prime numbers, and let $\{p_i\}$ be an enumeration of all prime numbers in increasing order. N is N-like with respect to $\{p_i\}$. If p_1 belongs to Q_1, set $T_1 = N - N_{p_1}$; otherwise, set $T_1 = N_{p_1}$. Set $n_1 = 1$ and $U_1 = T_{n_1}$, and let $\{f_n\}$ denote an enumeration of F. Thus, $D_1 = D_{f_1}(U_1)$ is N-like with respect to $\{p_{i+m}\}$ for some m. Set $n_2 = 1 + m$. If p_{n_2} belongs to Q_2, set $T_2 = N - N_{p_{n_2}}$; otherwise, set $T_2 = N_{p_{n_2}}$. Let $U_2 = D_1 \cap T_2$ and $D_2 = D_{f_2}(U_2)$. In this manner, we construct three sequences of N-definable infinite sets $\{D_n\}$, $\{T_n\}$, and $\{U_n\}$ having the following properties:

(i) $N \supset U_1 \supset D_1 \supset \cdots \supset U_n \supset D_n \supset \cdots$.

(ii) For all i, $T_i = N - N_{p_{n_i}}$ if p_{n_i} belongs to Q_i; otherwise, $T_i = N_{p_{n_i}}$.

(iii) For all n, $U_n \subset T_n$.

(iv) For all n, f_n is either constant on D_n or else one-to-one on D_n.

Therefore the sequence $\{D_n\}$ is a decreasing sequence of infinite N-definable subsets of N. Referring to Gaifman's construction of a minimal extension and statement (iv), we see that there exists a nonprincipal ultra-filter D, containing all of the sets D_n, such that $N^* = D \text{ Prod } N \mid F$ is a minimal extension of N.

Let π denote the identity function on N, and let S be the set of standard prime divisors of π in N^*. We assert that $S \neq Q_k$ for each k. For suppose p_{n_k} belongs to Q_k; then $T_k = N - N_{p_{n_k}}$. But since $D_k \subset U_k \subset T_k$, T_k is in D, and hence p_{n_k} does not divide π in N^*. Therefore, p_{n_k} is not in S. On the other hand, if p_{n_k} is not in Q_k, then $T_k = N_{p_{n_k}}$ is in D; thus, p_{n_k} divides π in N^*, and therefore p_{n_k} belongs to S. This completes the proof of Theorem 1, because S is not N-definable and therefore N^* is not conservative.

If P denotes Peano's axioms for the integers, then, referring to part (d) of the introduction, we see that the extension N^* just constructed also has the following property.

THEOREM 2. *The additive group of* N^* *is not isomorphic to the additive group of any conservative extension of* N. *Therefore, there exists a minimal extension of* N *that does not have the minimal additive group structure.*

Proof. If one accepts the results mentioned in the next-to-the-last paragraph of Section 2, then Theorem 2 is evident because N^* is not conservative. However, it is easy to give a direct proof. Suppose that α is an additive isomorphism of N^* onto a conservative extension M of N. If a is in N^* and n is in N, then n divides a in N^* if and only if n divides $\alpha(a)$ in M. Therefore the set of standard prime divisors of $\alpha(\pi)$ in M is the set S of standard prime divisors of π in N^*, which is impossible, since M is conservative.

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University of South Carolina
Aiken, South Carolina 29801