AN EXISTENCE THEOREM FOR PERIODICALLY PERTURBED CONSERVATIVE SYSTEMS

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1. INTRODUCTION

This paper is the culmination of a series of investigations by several authors. W. S. Loud should be credited with originating these studies. In [5] he proved the following theorem.

THEOREM 1.1. *Let* $g(x)$ *be an odd function of class* C^1. *If there exist an integer* n *and a positive number* δ *satisfying the condition*

$$(n + \delta)^2 \leq g'(x) \leq (n + 1 - \delta)^2,$$

then for each number E *the differential equation*

$$x'' + g(x) = E \cos t$$

has a unique 2π*-periodic solution, which is even and odd-harmonic.*

In [3], D. E. Leach partially generalized this theorem by showing that if $g(x)$ satisfies the inequality stated in Loud's theorem and if $g(0) = 0$, then for each continuous 2π-periodic function $e(t)$ the differential equation

$$x'' + g(x) = e(t)$$

has a unique 2π-periodic solution. In [2], A. C. Lazer and D. A. Sánchez considered the vector differential equation

$$x'' + \operatorname{grad} G(x) = p(t) = p(t + 2\pi), \tag{1}$$

where $p \in C(R, R^n)$ and $G \in C^2(R^n, R)$. This equation represents the Newtonian equations of motion of a mechanical system subject to conservative internal forces and periodic external forces. Lazer and Sánchez were able to show that if there exist an integer N and numbers μ_N and μ_{N+1} such that

$$N^2 < \mu_N \leq \mu_{N+1} < (N + 1)^2,$$

and if for all a in R^n

$$\mu_N I \leq \left(\frac{\partial^2 G(a)}{\partial x_i \partial x_j}\right) \leq \mu_{N+1} I,$$

where I is the identity matrix, then (1) has at least one 2π-periodic solution. Later, Lazer [1] showed that under far less restrictive conditions, (1) has at most one 2π-periodic solution. In particular, Lazer's conditions assume the existence of two real,

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constant, n-by-n matrices A and B such that if $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ are the eigenvalues of A and B, respectively, then there exist integers $N_k \geq 0$ $(k = 1, 2, \cdots, n)$ satisfying the condition

$$N_k^2 < \lambda_k \leq \mu_k < (N_k + 1)^2,$$

and such that for all $a \in R^n$,

$$A \leq \left(\frac{\partial^2 G(a)}{\partial x_i \, \partial x_j}\right) \leq B.$$

The purpose of this paper is to establish the existence of a 2π-periodic solution of equation (1) under Lazer's conditions. Our techniques are different from those used by any of the authors mentioned above.

2. SOME PRELIMINARY RESULTS

We state two theorems and a lemma that we use to establish our result.

Throughout the remainder of this paper, we use the term "solution of a differential equation" in the sense of an absolutely continuous function that satisfies the equation almost everywhere.

THEOREM 2.1 (see A. C. Lazer [1]). *Let* Q *be a real* n-by-n *symmetric matrix whose elements are bounded, measurable, and* 2π*-periodic on the real line. Suppose there exist real constant symmetric matrices* A *and* B *such that*

$$A \leq Q(t) \leq B \qquad \text{on } (-\infty, \infty)$$

and with the property that if $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ *and* $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ *denote the eigenvalues of* A *and* B, *respectively, there exist integers* $N_k \geq 0$ $(k = 1, 2, \cdots, n)$ *such that*

$$N_k^2 < \lambda_k \leq \mu_k < (N_k + 1)^2.$$

Then there is no nontrivial 2π*-periodic solution of*

$$w'' + Q(t)\,w = 0.$$

Actually, Lazer assumed that Q(t) is continuous; but an examination of the proof in (1) shows that our stronger statement holds.

THEOREM 2.2 (see D. E. Leach [3]). *Let* $f(t, \bar{x}, \mu)$ *be continuous and have continuous partial derivatives with respect to the components of* $\bar{x}$ *for* $(t, \bar{x}, \mu)$ *in* $(-\infty, \infty) \times R^n \times [0, 1]$. *Suppose that* $f(t + T, \bar{x}, \mu) \equiv f(t, \bar{x}, \mu)$, *where* $T > 0$, *and suppose that all solutions of*

$$(D_\mu) \qquad \bar{x}' = f(t, \bar{x}, \mu)$$

are defined for all t. *Assume that if* $\mu_0 \in [0, 1]$ *and if* $\bar{p}(t)$ *is a* T*-periodic solution of*

$$(D_{\mu_0}) \qquad \bar{x}' = f(t, \bar{x}, \mu_0),$$

then $\|\bar{p}(0)\| \leq A$, *where* A *is independent of* μ_0, *and there exists no nontrivial* T-*periodic solution of the linear* T-*periodic differential system*

$$\bar{\omega}' = f_x(t, \bar{p}(t), \mu_0)\,\bar{\omega}\,.$$

If for $\mu = 1$, (D_μ) *has a* T-*periodic solution, then* (D_μ) *has a* T-*periodic solution for all* $\mu \in [0, 1]$.

Next we restate Lemma 1A of [4] in a form convenient for our use.

LEMMA 2.1 (p. 157 of [4]). *Let* A *be a compact convex subset of* R^p. *Let* $[a, b]$ *be some interval, and let* $\mathfrak{F}$ *be the set of functions* $f\colon [a, b] \to R^p$ *such that* $f(t) \in A$ a.e. *and the components of* f *are measurable. Let* $\{f_n\}$ *be a sequence in* $\mathfrak{F}$ *such that if* $f_n = (f_n^1, \cdots, f_n^p)$, *then* f_n^k *converges weakly to a function* g^k $(k = 1, \cdots, p)$. *Then the function* $g = (g^1, \cdots, g^p)$ *belongs to* $\mathfrak{F}$.

3. AN EXISTENCE THEOREM

LEMMA 3.1. *Let the matrices* Q, A, *and* B *satisfy the conditions in Theorem* 2.1. *Let* $K > 0$ *be fixed. Then there exists a number* $r > 0$ *such that for every real vector-valued,* 2π-*periodic, bounded and continuous function* f *with* $\|f(t)\| \leq K$, *each* 2π-*periodic solution* u *of the equation*

$$u'' + Qu = f$$

satisfies the inequality $\|u(t)\|^2 + \|u'(t)\|^2 \leq r^2$ *for all* t.

Proof. Suppose that the conclusion of our lemma is false. Then for every natural number m there exist a 2π-periodic matrix function $Q_m(t)$, 2π-periodic vector functions $f_m(t)$ and $u_m(t)$, and a number t_m such that

$$A \leq Q_m(t) \leq B, \tag{2}$$

$$\|f_m(t)\| \leq K, \tag{3}$$

$$u_m''(t) + Q_m(t)\,u_m(t) = f_m(t), \tag{4}$$

$$\|u_m(t_m)\|^2 + \|u_m'(t_m)\|^2 \geq m^2 \qquad (t_m \in [0, 2\pi]), \tag{5}$$

and

$$\|u_m(t_m)\|^2 + \|u_m'(t_m)\|^2 = \max_{0 \leq t \leq 2\pi} (\|u_m(t)\|^2 + \|u_m'(t)\|^2) = \omega_m^2. \tag{6}$$

If we let $z_m = (1/\omega_m)\,u_m$ and $g_m = (1/\omega_m)\,f_m$, then

$$z_m'' + Q_m(t)\,z_m = g_m(t) \qquad \text{and} \qquad z_m(t + 2\pi) = z_m(t)\,.$$

We note that

$$\|z_m(t)\|^2 + \|z_m'(t)\|^2 \leq 1, \tag{7}$$

$$\|z_m(t_m)\|^2 + \|z_m'(t_m)\|^2 = 1, \tag{8}$$

and

$$\|g_m\| \leq K/m.$$

Now we wish to show that the elements q_{ijm} of Q_m are bounded by a number independent of m. We note that $0 \leq \langle x, Q_m(t)x \rangle \leq \langle x, Bx \rangle$. Hence, if we let x be the ith unit vector e_i, it follows that $0 \leq q_{iim}(t) \leq b_{ii}$. For $i \neq j$, let $x = e_i + e_j$. Then $\langle e_i, Q_m e_j \rangle = \langle e_j, Q_m e_i \rangle = q_{ijm}$. Now,

$$q_{ijm}(t) = \frac{1}{2}(\langle e_i + e_j, Q_m(t)(e_i + e_j)\rangle - \langle e_i, Q_m e_i \rangle - \langle e_j, Q_m e_j \rangle).$$

Therefore,

$$q_{ijm}(t) \leq \frac{1}{2}\langle (e_i + e_j), Q_m(e_i + e_j)\rangle \leq \frac{1}{2}\langle e_i + e_j, B(e_i + e_j)\rangle .$$

Similarly, the elements $q_{ijm}(t)$ of Q_m are bounded below by

$$\frac{1}{2}(\langle -e_i, Be_i \rangle - \langle e_j, Be_j \rangle).$$

It follows that $\{z_m\}$ is a uniformly bounded and equicontinuous sequence of 2π-periodic functions. In fact, since $\|z_m(t)\| \leq 1$ and $\|z'_m(t)\| \leq 1$ by (7), for $t' \leq t''$ we have the relations

$$\|z_m(t'') - z_m(t')\| = \left\|\int_{t'}^{t''} z'_m(s)\,ds\right\| \leq \int_{t'}^{t''} \|z'_m(s)\|\,ds \leq |t'' - t'| .$$

We note that $\|z''_m\| \leq \|-Q_m z_m + f_m/\omega_m\| \leq L$ for some constant L. Thus, a similar argument shows that $\{z'_m\}$ is a uniformly bounded equicontinuous sequence. Therefore, there exist a subsequence $\{z_{m_k}\}$ of $\{z_m\}$ and functions z and w such that $z_{m_k} \to z$ and $z'_{m_k} \to w$. We see that $w = z'$, since

$$z_m(t) = z_m(0) + \int_0^t z'_m(s)\,ds$$

and hence

$$z(t) = z(0) + \int_0^t w(s)\,ds.$$

Letting $Q_m(t) = (q_{ijm}(t))$, we may assume without loss of generality that $\{q_{ijm_k}\}$ converges weakly to q_{ij} as $m_k \to \infty$, since the elements of $Q_m(t)$ are bounded.

Now, we wish to show that if $Q = (q_{ij})$, then

$$A \leq Q(t) \leq B \text{ a.e.} \tag{9}$$

With each symmetric matrix $S = (s_{ij})$ we associate the point

$$(s_{11}, \cdots, s_{1n}, s_{22}, \cdots, s_{2n}, \cdots, s_{nn})$$

in R^p, where $p = n(n+1)/2$. With this identification, the set H of symmetric matrices S satisfying the condition $A \leq S \leq B$ forms a compact convex subset of R^p. The convexity is obvious, since the inequalities $\langle x, S_1 x \rangle \leq \langle x, Bx \rangle$ and $\langle x, S_2 x \rangle \leq \langle x, Bx \rangle$ imply that

$$\begin{aligned}\langle x, [(1-t)S_1 + tS_2]x \rangle &= (1-t)\langle x, S_1 x \rangle + t\langle x, S_2 x \rangle \\ &\leq (1-t)\langle x, Bx \rangle + t\langle x, Bx \rangle = \langle x, Bx \rangle .\end{aligned}$$

Similarly, $(1-t)S_1 + tS_2 \geq A$. To verify the compactness of H, we note that the boundedness of H follows from the argument, given above, for the uniform boundedness of $\{Q_m\}$. The closedness of H follows from the fact that $S_m \to S$ implies $\langle x, S_m x \rangle \to \langle x, Sx \rangle$, and hence $\langle x, Ax \rangle \leq \langle x, S_m x \rangle \leq \langle x, Bx \rangle$ implies $\langle x, Ax \rangle \leq \langle x, Sx \rangle \leq \langle x, Bx \rangle$. In view of Lemma 2.1, it follows that Q(t) is a symmetric matrix and satisfies (9).

We note that $z \not\equiv 0$, since the condition

$$\|z_m(t_m)\|^2 + \|z_m'(t_m)\|^2 = 1 \qquad (t_m \in [0, 2\pi])$$

implies that if $t_{m_k} \to t^*$, then $\|z(t^*)\|^2 + \|z'(t^*)\|^2 = 1$. Consider the column vectors

$$f_m = \operatorname{col}(f_m^1, \cdots, f_m^n), \qquad z_m = \operatorname{col}(z_m^1, \cdots, z_m^n), \qquad z = \operatorname{col}(z^1, \cdots, z^n).$$

It follows from (4) that for each t in $[0, 2\pi]$,

$$(z_{m_k}^{\ell})'(t) = (z_{m_k}^{\ell})'(0) - \int_0^t \sum_{j=1}^n q_{\ell j m_k}(s)\, z_{m_k}^j(s)\, ds + \int_0^t \frac{f_{m_k}^{\ell}(s)}{\omega_{m_k}}\, ds. \tag{10}$$

Clearly, $\int_0^t \frac{f_{m_k}^{\ell}(s)}{\omega_{m_k}}\, ds \to 0$ as $m_k \to \infty$. Now, for $j = 1, \cdots, n$,

$$\int_0^t q_{\ell j m_k}(s)\, z_{m_k}^j(s)\, ds \to \int_0^t q_{\ell j}(s)\, z^j(s)\, ds$$

as $m_k \to \infty$, because

$$\begin{aligned}&\int_0^t q_{\ell j m_k}(s)\, z_{m_k}^j(s)\, ds - \int_0^t q_{\ell j}(s)\, z^j(s)\, ds \\ &\qquad = \int_0^t q_{\ell j m_k}(z_{m_k}^j - z^j)\, ds + \int_0^t (q_{\ell j m_k} z^j - q_{\ell j} z^j)\, ds .\end{aligned}$$

Clearly, the first integral approaches 0 as $m_k \to \infty$. The second integral approaches 0, by weak convergence of $q_{\ell j m_k}$ to $q_{\ell j}$, for if g is defined by

$$g(s) = \begin{cases} z^j(s) & (0 \leq s \leq t), \\ 0 & (t < s \leq 2\pi), \end{cases}$$

then $g \in L^2[0, 2\pi]$ for a fixed t, and

$$\int_0^t (q_{\ell j m_k} z^j - q_{\ell j} z^j)\, ds = \int_0^{2\pi} (q_{\ell j m_k} g - q_{\ell j} g)\, ds .$$

Thus, taking the limit of both sides of (10), we see that

$$(z^\ell)'(t) = (z^\ell)'(0) - \int_0^t \sum_{j=i}^n g_{\ell j} z^j\, ds ;$$

in matrix form, we can now write

$$z'(t) = z'(0) - \int_0^t Q(s) z(s)\, ds ,$$

and hence z' is absolutely continuous. Consequently,

$$z'' + Qz = 0 .$$

But since z is 2π-periodic and $z \not\equiv 0$, this contradicts Theorem 2.1.

THEOREM 3.1 (existence theorem). *Consider the differential equation*

$$(*) \qquad x'' + \operatorname{grad} G(x) = p(t) = p(t + 2\pi),$$

where $p \in C(R, R^n)$ *and* $G \in C^2(R^n, R)$. *Assume that* A *and* B *are constant symmetric* n-*by*-n *matrices such that if* $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ *and* $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ *are the eigenvalues of* A *and* B, *respectively, then there exist integers* $N_k \geq 0$ $(k = 1, \cdots, n)$ *satisfying the condition*

$$N_k^2 < \lambda_k \leq \mu_k < (N_k + 1)^2 .$$

Further, assume that for all $a \in R^n$,

$$A \leq \left(\frac{\partial^2 G(a)}{\partial x_i \partial x_j} \right) \leq B .$$

Then (*) *has a* 2π-*periodic solution.*

Proof. Let

$$\operatorname{grad} G(x) = \operatorname{col}\left(\frac{\partial G}{\partial x_1}, \cdots, \frac{\partial G}{\partial x_n} \right),$$

and let

$$H(x) = \left(\frac{\partial^2 G(x)}{\partial x_i \partial x_j} \right).$$

We note that $H(x)$ is symmetric and $A \leq H(x) \leq B$. We can rewrite (*) as

$$x'' + M(x)x = g(t), \tag{**}$$

where

$$M(x) = \int_0^1 H(tx)\,dt$$

and $g(t) = p(t) - \text{grad } G(0)$. Let S be a fixed n-by-n symmetric matrix satisfying $A \leq S \leq B$. Consider the system

$$x' = y,$$

$$y' = -\mu Sx + (1 - \mu)(p(t) - \text{grad } G(x)),$$

which can be written in the form

$$z' = F(t, z, \mu), \tag{***}$$

where $z = \text{col}(x, y)$. We note that for $\mu = 0$, (***) reduces to (*). For $\mu = 1$, $z \equiv 0$ is a 2π-periodic solution. Thus the proof of our theorem will be complete if we can show that (***) satisfies the hypothesis of Theorem 2.2.

Suppose that $z(t) = \text{col}(u(t), v(t))$ is a 2π-periodic solution of (***) corresponding to $\mu = \mu_0$. Then

$$u''(t) + \mu_0 Su(t) + (1 - \mu_0)\text{grad } G(u(t)) = (1 - \mu_0)p(t).$$

Since $\text{grad } G(u(t)) = M(u(t))u(t) + \text{grad } G(0)$,

$$u''(t) + \mu_0 Su(t) + (1 - \mu_0)M(u(t))u(t) = (1 - \mu_0)(p(t) - \text{grad } G(0)).$$

Since $A \leq M(x) \leq B$, it follows that $A \leq \mu_0 S + (1 - \mu_0)M(u(t)) \leq B$. By Lemma 3.1, all 2π-periodic solutions of (***) are bounded by some number independent of μ.

Next we consider the equation of first variation. Since

$$F(t, (x, y), \mu) = \text{col}(y, -\mu Sx + (1 - \mu)(p(t) - \text{grad } G(x))),$$

$$F_z(t, (u(t), v(t)), \mu) = \begin{pmatrix} 0_n & I_n \\ -\mu S - (1 - \mu)H(u(t)) & 0_n \end{pmatrix}.$$

It follows that the equation of first variation

$$w' = F_z(t, (u(t), v(t)), \mu)w$$

has no nontrivial 2π-periodic solution. For if we partition $w = \text{col}(\alpha, \beta)$, where α and β are in R^n, the last equation is equivalent to the system

$$\alpha' = \beta,$$

$$\beta' = -\mu S\alpha - (1 - \mu)H(u(t))\alpha.$$

Thus the existence of a nontrivial 2π-period solution of this system implies the existence of a nontrivial 2π-periodic solution of the equation

$$\alpha'' + [\mu S + (1 - \mu) H(u(t))]\alpha = 0 .$$

Since

$$A \leq \mu S + (1 - \mu) H(u(t)) \leq B,$$

we would have a contradiction to Theorem 2.2. Thus the proof is complete.

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