INTERPOLATION AND UNAVOIDABLE FAMILIES OF MEROMORPHIC FUNCTIONS

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The well-known Weierstrass factorization theorem says that given a sequence $\{a_n\}$ of complex numbers with no finite limit point, one can always construct an entire function f that vanishes precisely at the points a_n, where multiple occurrences of an a_n correspond to zeros of the corresponding multiplicity. It is natural to ask whether it is possible to construct an entire function whose zeros and one-points are prescribed. Now, by the well-known interpolation theorem for entire functions (see [6, p. 298]), given two sequences $\{a_n\}$ and $\{b_n\}$ that are disjoint and have no finite limit point, one can find an entire function f such that $f(a_n) = 0$ and $f(b_n) = 1$ for all n. But this does not answer the question, since it is conceivable that such an f must have other zeros or one-points. To see that this unpleasant possibility may actually arise, let $\{a_n\}$ be a finite nonempty set of cardinality A, and let $\{b_n\}$ be one of cardinality $B \neq A$. If a suitable f were to exist, it would omit 0, 1, and ∞ in a neighborhood of ∞, and would therefore have to be a polynomial, by Picard's Great Theorem; this is impossible, since $A \neq B$.

Our question is: For what pairs of disjoint sequences $\{a_n\}$ and $\{b_n\}$ without finite limit points can one construct an entire function f whose zero-sequence is exactly $\{a_n\}$ and whose one-sequence is exactly $\{b_n\}$? If this is possible, we call $(\{a_n\}, \{b_n\})$ the *zero-one set* of f. A more general form of this question was briefly studied by R. Nevanlinna in [5].

There are also *infinite* sequences $(\{a_n\}, \{b_n\})$ that are not zero-one sets. One way to see this was shown to us by J. Miles. By a result of A. Edrei [1, p. 277], an entire function with only real zeros and real ones has order at most 1. Since the exponent of convergence of the a-points of an entire function is no greater than the order of the function, we need only take $\{a_n\}$ and $\{b_n\}$ real and $\{a_n\}$, say, to have exponent of convergence greater than 1. By a slight variation of this argument, we can take the $\{a_n\}$ and $\{b_n\}$ arbitrarily sparse, so long as they lie on the real axis and each b_n is very close to some a_n; for this would force the derivative of an admissible entire function f to have order exceeding 1. This is impossible, since the order of f' equals the order of f. The same ideas show, for example, that there are three disjoint discrete sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ such that no pair of them forms a zero-one set.

In Theorem 1, we prove that to each sequence $\{a_n\}$ there corresponds a disjoint discrete sequence $\{b_n\}$ such that $(\{a_n\}, \{b_n\})$ is not the zero-one set of any entire function. We give two proofs of this result. The first proof is a bit complicated, but does not require deeper tools than Nevanlinna's First Fundamental Theorem. The second proof is due to J. Miles, whom we thank for allowing us to use it. It is simpler than the first proof, but does use Nevanlinna's Second Fundamental Theorem.

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Then we prove Theorem 2, which says that if an entire function has the same zero-one set as $\sin z$, then that function must be identically equal to $\sin z$.

Finally, we consider unavoidable families of meromorphic functions. A family F of meromorphic functions is *unavoidable* if the graph of any meromorphic function must somewhere intersect the graph of some function in F. We prove in Theorem 3 that the minimum cardinality of an unavoidable family of meromorphic functions is 3.

THEOREM 1. *Corresponding to each infinite sequence* $\{a_n\}$ *in the complex plane, there exists a discrete infinite sequence* $\{b_n\}$ *of complex numbers, disjoint from* $\{a_n\}$, *such that* $(\{a_n\}, \{b_n\})$ *is not a zero-one set of any entire function.*

First proof. We shall use the standard ideas and notation of the Nevanlinna Theory (see [2]). Let $\{c_n\}$ be any discrete sequence of complex numbers. We denote by $n(r, \{c_n\})$ the number of c_j in $|z| \le r$, with due count of multiplicity. We write

$$N(r, \{c_n\}) = \int_0^r \frac{n(t, \{c_n\})}{t}\, dt,$$

where we suppose that no c_n is 0. For simplicity, we suppose now that no a_n is 0, and that $\{a_n\}$ has no finite limit point. We let $\Pi_1(z)$ be the canonical product formed with the $\{a_n\}$ as zeros. Now let $\{b_n\}$ be an infinite discrete sequence of non-zero complex numbers distinct from all the a_n, and satisfying the conditions $b_2 = -b_3$, $b_4 = -b_5$, $\cdots$. Furthermore, we demand that

$$\text{(c)} \qquad T(r, \Pi_1(z)) = o(N(r, \{b_n\}) \quad \text{as } r \to \infty.$$

It is clear that such sequences $\{b_n\}$ exist. We claim that $(\{a_n\}, \{b_n\})$ is not the zero-one set for any entire function. Proceeding by contradiction, suppose that there exists an entire function f having $\{a_n\}$ as its precise zero set and $\{b_n\}$ as its precise one set. Let $\Pi(z)$ be the canonical product with $\{b_n\}_{n=2}^{\infty}$ as its zero set. Then

$$\text{(1)} \qquad f(z) = \Pi_1(z)\, e^{\alpha(z)}$$

and

$$\text{(2)} \qquad f(z) - 1 = (z - b_1)\, \Pi(z)\, e^{\beta(z)},$$

where α and β are entire functions. Here we note that we may take $\Pi(z) = \Pi(-z)$, because $b_{2n} = -b_{2n+1}$ for $n = 1, 2, 3, \cdots$. By the basic properties of the Nevanlinna characteristic function,

$$\text{(3)} \qquad T(r, f) = T(r, \Pi_1 e^{\alpha}) \le T(r, \Pi_1) + T(r, e^{\alpha})$$

and

$$\text{(4)} \qquad N\left(r, \frac{1}{f-1}\right) \le T(r, f) + O(1).$$

It follows from condition (c) and equations (3) and (4) that

$$\text{(5)} \qquad T(r, \Pi_1) = o(T(r, e^{\alpha})) \quad \text{as } r \to \infty.$$

Before proceeding further, we need the following result, which is a very slight simplification of a lemma of G. Hiromi and M. Ozawa [3].

LEMMA A. *Let* $a_0(z), a_1(z), \cdots, a_n(z)$ *and* $g_1(z), g_2(z), \cdots, g_n(z)$ *be entire functions. Suppose that*

$$T(r, a_j(z)) = o\left(\sum_{\nu=1}^{n} T(r, e^{g_\nu})\right) \quad (j = 0, 1, \cdots, n). \tag{6}$$

If the identity

$$\sum_{\nu=1}^{n} a_\nu(z)\, e^{g_\nu(z)} = a_0(z)$$

holds, then there is an identity

$$\sum_{\nu=1}^{n} c_\nu\, a_\nu(z)\, e^{g_\nu(z)} = 0,$$

where the c_ν $(\nu = 1, 2, \cdots, n)$ *are constants that are not all zero.*

We now return to the proof of our theorem. By subtracting (2) from (1), we get the equation

$$\Pi_1(z)\, e^{a(z)} - (z - b_1)\, \Pi(z)\, e^{\beta(z)} = 1. \tag{7}$$

Hence

$$e^{-\beta(z)} - \Pi_1(z)\, e^{\alpha(z)-\beta(z)} = -(z - b_1)\, \Pi(z).$$

From this and the evenness of $\Pi(z)$ it follows that

$$(-z - b_1)\,[e^{-\beta(z)} - \Pi_1(z)\, e^{\alpha(z)-\beta(z)}] = (z - b_1)\,[e^{-\beta(-z)} - \Pi_1(-z)\, e^{\alpha(-z)-\beta(-z)}],$$

and hence

$$e^{-\beta(z)}[(-z - b_1) - (z - b_1)\,\Pi_1(z)\, e^{\alpha(z)} - (z - b_1)\, e^{-\beta(-z)+\beta(z)}$$
$$+ (z - b_1)\,\Pi_1(-z)\, e^{\alpha(-z)-\beta(-z)+\beta(z)}] = 0.$$

Consequently,

$$(z + b_1)\,\Pi_1(z)\, e^{\alpha(z)} - (z - b_1)\, e^{-\beta(-z)+\beta(z)}$$
$$+ (z - b_1)\,\Pi_1(-z)\, e^{\alpha(-z)+\beta(z)-\beta(-z)} = z + b_1.$$

Since the hypothesis (6) of Lemma A is satisfied, we conclude that

$$\begin{aligned} &c_1(z + b_1)\,\Pi_1(z)\, e^{\alpha(z)} + c_2(z - b_1)\, e^{-\beta(-z)+\beta(z)} \\ &\qquad + c_3(z - b_1)\,\Pi_1(-z)\, e^{\alpha(-z)+\beta(z)-\beta(-z)} = 0 \end{aligned} \tag{8}$$

for some constants c_1, c_2, and c_3 that are not all 0. Clearly, $c_2 \neq 0$; for otherwise, on setting $z = 0$, we would see that $c_1 = c_3$ and consequently

$$(z + b_1)\,\Pi_1(z)\, e^{\alpha(z)-\beta(z)} = (z - b_1)\,\Pi_1(-z)\, e^{\alpha(-z)-\beta(-z)};$$

this implies that

$$b_1 \Pi_1(0) = -b_1 \Pi_1(0),$$

which is impossible since $\Pi_1(0) \neq 0$ and $b_1 \neq 0$. Now, if $c_1 \neq 0$, then we see from (8) that

$$c_1(z + b_1)\Pi_1(z) = -c_2(z - b_1)e^{\beta(z)-\beta(-z)-\alpha(z)} - c_3(z - b_1)\Pi_1(-z)e^{\alpha(-z)-\alpha(z)+\beta(z)-\beta(-z)}.$$

Again applying Lemma A, we have the relation

$$c_4(z - b_1)e^{\beta(z)-\beta(-z)-\alpha(z)} + c_5(z - b_1)\Pi_1(-z)e^{\beta(z)-\beta(-z)-\alpha(z)+\alpha(-z)} = 0 \tag{9}$$

for some constants c_4 and c_5 with $c_4 c_5 \neq 0$. But (9) says that

$$c_4 + c_5 \Pi_1(-z)e^{\alpha(-z)} = 0,$$

which is impossible.

Thus, we conclude that $c_1 = 0$. Consequently,

$$c_2 + c_3 \Pi_1(-z)e^{\alpha(-z)} = 0,$$

which is impossible since $c_2 \neq 0$. It follows that the identity (7) cannot hold, and our theorem is proved by contradiction.

Second proof of Theorem 1. For an entire function f, the Cauchy integral formula implies that $r M(r, f') \leq M(2r, f)$, so that $M(r, f') < 2M(2r, f)$ for $r > 1$. Since $\log|f(z)|$ is subharmonic, we have for $R > |z| = r$ the inequality

$$\log|f(z)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(Re^{it})| \,\Re\left(\frac{Re^{it} + z}{Re^{it} - z}\right) dt.$$

We take $R = 2r$ to get the relation

$$\log M(r, f) \leq \frac{3}{2\pi}\int_{-\pi}^{\pi} |\log|f(2re^{it})||\,dt = 3\left[m(2r, f) + m\left(2r, \frac{1}{f}\right)\right].$$

Since Nevanlinna's First Fundamental Theorem implies that

$$m\left(2r, \frac{1}{f}\right) + N\left(2r, \frac{1}{f}\right) = m(2r, f) + O(1),$$

we get the estimate

$$\log M(r, f) \leq 6m(2r, f) + O(1) < 7m(2r, f) \qquad \text{for } r > r_0,$$

supposing that f is not a constant. We see then that for $r > r_0$,

$$\log M(r, f') < \log M(2r, f) < 7m(4r, f).$$

Now let $\phi(r)$ be an arbitrary increasing function on $[0, \infty)$. We are given $\{a_n\}$. Let $r_n = |a_n|$, and let $b_n = a_n e^{i\varepsilon_n}$, where $\varepsilon_n > 0$ is chosen so that if f has a zero at a_n and a one at b_n, then there exists a z on the short circular arc γ of $\{|z| = r_n\}$ joining a_n to b_n, for which $|f'(z)| > \exp(7\phi(4r_{n+1}))$. This is clearly possible, since $1 = f(b_n) - f(a_n) = \int_\gamma f'(z)\,dz$. Suppose now that there exists an f whose zero-set is $\{a_n\}$ and whose one-set is $\{b_n\}$. Then

$$\log M(r_n, f') > 7\phi(4r_n + 1).$$

Let us now take $r \in [r_n, r_{n+1}]$, where n is large. Then

$$7m(4r, f) > \log M(r, f') \geq \log M(r_n, f') > 7\phi(4r_{n+1}) \geq 7\phi(4r).$$

Thus, for large r, we have the inequality $m(r, f) \geq \phi(r)$. Now we suppose, in addition, that $N(r, \{a_n\})/\phi(r) \to 0$ as $r \to \infty$. Then $N(r, \{a_n\})/m(r, f) \to 0$. Hence $\delta(0, f) = 1$, where $\delta(0, f)$ is the deficiency of zero for f. But

$$N(r, \{b_n\}) = N(r, \{a_n\}),$$

so that $\delta(1, f) = 1$ also. But $\delta(\infty, f) = 1$, since f is entire, and therefore the sum of the deficiencies of f is at least 3, contrary to Nevanlinna's Second Fundamental Theorem.

Our next result is related to material in the second part of Chapter V of [4].

THEOREM 2. *If* f *is an entire function with the same zero-one set as* sin z, *then* $f(z) \equiv \sin z$.

Proof. By the theorem of Edrei mentioned earlier, f has order at most 1, since it has real zeros and real ones. Alternately, we could use a well-known theorem of E. Borel, since the exponent of convergence of the ones and of the zeros of sin z are both 1. We may write

$$f(z) = e^{A(z)} \sin z,$$

$$f(z) - 1 = e^{B(z)}(\sin z - 1),$$

where A(z) and B(z) are entire functions. It follows that

$$\text{(i)} \qquad f(z) = a e^{bz} \sin z$$

and

$$\text{(ii)} \qquad f(z) - 1 = c e^{dz}(\sin z - 1),$$

with constants a, b, c, d and $ac \neq 0$. We shall show that $a = c = 1$ and $b = d = 0$, so that $f(z) \equiv \sin z$. Suppose that this is not the case. Then, unless $a = c$ and $b = d$, we must have the relation $ae^{bz} - ce^{dz} \not\equiv 0$. But if $a = c$ and $b = d$, then it is easy to show that $b = d = 0$ and $a = c = 1$. Thus, we may assume that $ae^{bz} - ce^{dz} \not\equiv 0$. Eliminating f from the equations (i) and (ii) above, we get the equations

$$\frac{e^{iz} - e^{-iz}}{2i} \equiv \sin z = \frac{1 - ce^{dz}}{ae^{bz} - ce^{dz}} = \frac{e^{-dz} - c}{ae^{(b-d)z} - c}.$$

Letting $z = 0$, we see that $c = 1$. We then have the relation

(10) $$ae^{(b-d+i)z} - ae^{(b-d-i)z} - e^{iz} + e^{-iz} - 2ie^{-dz} + 2i \equiv 0.$$

Now an exponential polynomial vanishes identically if and only if all of its terms are zero. Therefore, the last four terms on the left side of equation (10) involve at most two distinct frequencies (where the frequency of $Ae^{\lambda z}$ is λ by definition). An elementary analysis shows that this is impossible, so that the proof by contradiction is complete.

On the other hand, there exist entire functions f and g, with $f \not\equiv g$, that have infinitely many zeros and infinitely many ones, such that f and g have the same zero-one sets. For g to have the same zeros and ones as f, we need the two relations

$$g(z) = e^{\alpha(z)} f(z)$$

and

$$g(z) - 1 = e^{\beta(z)}(f(z) - 1),$$

or

$$f(z) = \frac{1 - e^{\beta(z)}}{e^{\alpha(z)} - e^{\beta(z)}} = \frac{e^{a(z)} - 1}{e^{b(z)} - 1},$$

say. Therefore we now choose

$$f(z) = \frac{e^{\sin \pi z} - 1}{e^{2\pi i z} - 1},$$

and we are assured that some g ($g \not\equiv f$) has the same zero-one set as f. Clearly, f has infinitely many zeros, since the numerator vanishes whenever $(\sin \pi z)/2\pi i$ is an integer, while the denominator vanishes only when z is an integer. Also, $f(z) = 1$ whenever

$$\sin \pi z = 2\pi i z + 2k\pi i$$

for some integer k, and this happens infinitely often by the three-functions theorem (see [2, p. 47]). The assertion is proved.

Finally, we consider a loosely related question. We say that a family F of meromorphic functions in the plane is *unavoidable* if for every meromorphic function g there exist an $f \in F$ and a complex number z such that $g(z) = f(z)$.

THEOREM 3. *The minimum cardinality of an unavoidable family of meromorphic functions in the plane is* 3.

Proof. First we show that there exists an unavoidable family of three functions. Let $a_1(z)$, $a_2(z)$, and $a_3(z)$ be three different polynomials such that $a_1(z) - a_2(z)$ and $a_2(z) - a_3(z)$ are not both constants. Suppose now that $f(z) - a_i(z)$ has no zeros for $i = 1, 2, 3$, and consider the function

$$F(z) = \frac{f(z) - a_1(z)}{f(z) - a_2(z)} \, \frac{a_3(z) - a_2(z)}{a_3(z) - a_1(z)}.$$

It can take the values 1, ∞, and 0 only where $a_1(z) = a_2(z)$ or $a_1(z) = a_3(z)$ or $a_2(z) = a_3(z)$, and this accounts for at most finitely many points.

Therefore, by Picard's Great Theorem, F has a nonessential singularity at ∞, and hence F must be a rational function. Hence f must be a rational function that avoids $a_1(z)$, $a_2(z)$, and $a_3(z)$, which is clearly impossible.

To conclude, we prove that no two meromorphic functions form an unavoidable family. Let $g = g_1/g_2$ and $h = h_1/h_2$ denote two distinct meromorphic functions, g_1, g_2, h_1, h_2 being entire, and neither g_2 nor h_2 being identically zero. Obviously, we can assume that g_1 and g_2 have no common zeros, and that the same is true of h_1 and h_2.

To construct a meromorphic function f that avoids both g and h, we construct entire functions Φ and Ψ to satisfy certain conditions that will be imposed soon, and let $\phi = \exp \Phi$ and $\psi = \exp \Psi$. Then we let

$$f_1 = \frac{h_1\phi - g_1\psi}{h_1 g_2 - g_1 h_2}, \quad f_2 = \frac{h_2\phi - g_2\psi}{h_1 g_2 - g_1 h_2}.$$

It follows that

$$\frac{f_1}{f_2} - \frac{g_1}{g_2} = \frac{f_1 g_2 - f_2 g_1}{f_2 g_2} = \frac{\phi}{f_2 g_2} \tag{11}$$

and

$$\frac{f_1}{f_2} - \frac{h_1}{h_2} = \frac{f_1 h_2 - f_2 h_1}{f_2 h_2} = \frac{\psi}{f_2 h_2}. \tag{12}$$

We shall construct Φ and Ψ so that, with the notation $f = f_1/f_2$,

(i) f_1 and f_2 are entire functions,

(ii) f and g have no common pole,

(iii) f and h have no common pole.

If this is done, then f avoids g and h since by (11), f and g are equal at no point where they are both finite, and by (12), f and h are equal at no point where they are both finite, and (ii) and (iii) exclude common poles as well. Let $k = h_1 g_2 - g_1 h_2$, and let the distinct zeros of k be at a_n, with multiplicity q_n. For (i) to hold, it is enough that $A_1 \equiv h_1\phi - g_1\psi$ and $A_2 \equiv h_2\phi - g_2\psi$ each have a zero of order at least q_n at a_n, for each n. Now, by the definition of f_2, f_2 and g_2 cannot have a common zero at a point a unless $h_2(a) = 0$ also, in which case $a = a_n$ for some n. We need only guarantee, therefore, that at each a_n, A_2 has a zero of order q_n and A_1 has a zero of order at least q_n. Now our conditions on ϕ and ψ depend only on the first $q_n + 1$ Taylor coefficients of ϕ and ψ at a_n. Discarding powers of $(z - a_n)$ higher than the q_n-th amounts to working in the ring $R = \mathbb{C}[z]/(z - a_n)^{q_n+1}$. We remark that an element p of this ring is invertible if and only if $p(a)_n \neq 0$. The same condition, $p(a_n) \neq 0$, guarantees that there is an element P of R such that $p = \exp P$. If we can satisfy the requirements on ϕ and ψ at a_n with polynomials ϕ_n and ψ_n such that $\phi_n(a_n) \neq 0$ and $\psi_n(a_n) \neq 0$, then we can find polynomials Φ_n and Ψ_n for which $\phi_n = \exp \Phi_n$ and $\psi_n = \exp \Psi_n$. Then, by the interpolation theorem for entire functions [6, p. 298], we can find entire functions Φ and Ψ for which Φ_n and Ψ_n are the truncated Taylor series of Φ and Ψ, respectively, at the point a_n.

We let

$$k(z) \equiv h_1(z) g_2(z) - g_1(z) h_2(z) = (z - a_n)^{q_n} K(z), \tag{13}$$

so that $K(a_n) \neq 0$. Remembering that we are now working in the ring R, we then let

$$\phi_n(z) = \frac{g_2(z)\,P(z) - g_1(z)\,Q(z)}{K(z)} \quad \text{and} \quad \psi_n(z) = \frac{h_2(z)\,P(z) - h_1(z)\,Q(z)}{K(z)},$$

where P and Q will be chosen in a moment. It follows that

$$A_1(z) \equiv h_1(z)\phi_n(z) - g_1(z)\psi_n(z) = (z - a_n)^{q_n} P(z)$$

and

$$A_2(z) \equiv h_2(z)\phi_n(z) - g_2(z)\psi_n(z) = (z - a_n)^{q_n} Q(z).$$

A case-by-case analysis shows that we may choose $P(z)$ and $Q(z)$ to be constants (recall that $g_1(a_n)$ and $g_2(a_n)$ are not both zero, and that $h_1(a_n)$ and $h_2(a_n)$ are not both zero), so that $\phi_n(a_n) \neq 0$ and $\psi_n(a_n) \neq 0$. That does it.

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