THE SEGMENTAL VARIATION OF HOLOMORPHIC FUNCTIONS

Walter Rudin

E. Lindelöf and P. Montel proved the following theorems about the class H^∞ of all bounded holomorphic functions in the open unit disc U:

(a) If $f \in H^\infty$ and f has a limit, say L, along some arc in U that terminates at the point 1, then the radial limit of f exists at the point 1 and equals L.

(b) If $f \in H^\infty$ and f has a radial limit at 1, then f actually has a nontangential limit at 1.

The union of these statements is often called the *sectorial-limit theorem*. For a proof we refer to [1, Theorem 6.7].

These theorems suggest two questions, obtained by replacing the property of having a limit by the stronger one of having finite total variation:

(A) If $f \in H^\infty$ and f has finite total variation on some arc in U with one end-point at 1, does it follow that f has finite total variation on the radius $[0, 1)$?

(B) If $f \in H^\infty$ and f has finite total variation on $[0, 1)$ must the same be true on other line segments in U that end at 1?

An affirmative answer to (A) would lead to a quick proof that every $f \in H^\infty$ has finite total variation on some radius. (This possibility was not ruled out in [2].) However, we shall see that both (A) and (B) have negative answers, even if H^∞ is replaced by the disc algebra A, that is, by the class of all continuous functions on the closed unit disc $\overline{U}$ that are holomorphic in U.

To state the result concisely, we associate with each $\alpha \in (-\pi/2, \pi/2)$ the segment

$$I(\alpha) = \{1 - te^{i\alpha}: 0 < t < \cos\alpha\}, \tag{1}$$

and we define $V(f, \alpha)$ to be the total variation of any $f \in H^\infty$ on $I(\alpha)$:

$$V(f, \alpha) = \int_0^{\cos\alpha} |f'(1 - te^{i\alpha})|\, dt. \tag{2}$$

Note that one end-point of $I(\alpha)$ is 1 and that the other lies in U. Also, $I(\alpha)$ lies above $I(\beta)$ if and only if $\alpha < \beta$.

THEOREM. *To every* $\beta \in (-\pi/2, \pi/2)$ *correspond functions* f, g, h *in the disc algebra* A *such that*

(i) $V(f, \alpha) < \infty$ *if and only if* $\alpha < \beta$,

(ii) $V(g, \alpha) < \infty$ *if and only if* $\alpha \le \beta$,

(iii) $V(h, \alpha) < \infty$ *if and only if* $\alpha = \beta$.

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Proof. In the complement of the set of all nonpositive real numbers, let $\log z$ denote the branch of the logarithm that is 0 when $z = 1$. Define ϕ in $\overline{U} - \{1\}$ by

$$\pi\phi(\lambda) = i(\pi + \beta) - \log(1 - \lambda). \tag{3}$$

Note that every $\lambda \in U$ has the form $\lambda = 1 - te^{i\alpha}$, where $-\pi/2 < \alpha < \pi/2$ and $0 < t < 2\cos\alpha$. Since

$$\pi\phi(1 - te^{i\alpha}) = -\log t + (\pi + \beta - \alpha)i, \tag{4}$$

we see that ϕ is a conformal map of U onto a region Ω lying in the half-strip defined by the inequalities

$$x > -\frac{1}{\pi}\log 2, \quad \frac{1}{2} + \frac{\beta}{\pi} < y < \frac{3}{2} + \frac{\beta}{\pi},$$

and that ϕ maps $I(\alpha)$ onto the half-line

$$\{x + iy(\alpha) : c(\alpha) < x < \infty\}, \tag{5}$$

where

$$y(\alpha) = 1 + \frac{\beta - \alpha}{\pi}, \qquad c(\alpha) = -\frac{1}{\pi}\log\cos\alpha. \tag{6}$$

Also, $\phi(\lambda) \to \infty$ in $\overline{\Omega}$ as $\lambda \to 1$ in $\overline{U}$.

Next, put

$$\psi(z) = \exp\{iz\log z\}, \tag{7}$$

$$\mu(z) = \psi(z)/(\log z)^3 \tag{8}$$

for z in the upper half-plane, and define

$$f(\lambda) = \psi(\phi(\lambda)), \qquad g(\lambda) = \mu(\phi(\lambda)) \tag{9}$$

in $\overline{U} - \{1\}$.

If $z = x + iy = re^{i\theta}$, then

$$|\psi(z)| = \exp\{-y\log r - x\theta\}. \tag{10}$$

Hence $\psi(z) \to 0$ as $z \to \infty$ within $\overline{\Omega}$. If $f(1)$ and $g(1)$ are defined to be 0, it follows that $f \in A$ and $g \in A$.

Since $\psi'(z) = i\psi(z)(1 + \log z)$, (10) implies that

$$|\psi'(x + iy)| \sim (ex)^{-y}\log x \tag{11}$$

in the sense that the ratio of the two sides tends to 1 as $x \to \infty$. Our construction shows that

$$V(f, \alpha) = \int_{c(\alpha)}^{\infty} |\psi'(x + iy(\alpha)| \, dx, \tag{12}$$

where the notation is as in (5) and (6). By (11) and (12), $V(f, \alpha) < \infty$ if and only if $y(\alpha) > 1$, which happens precisely when $\beta > \alpha$.

Thus part (i) of the theorem is proved.

Part (ii) is proved in the same manner; in place of (11), we use

$$|\mu'(x + iy)| \sim (ex)^{-y} (\log x)^{-2} . \tag{13}$$

Since (ii) holds, we can also find a function $\tilde{g} \in A$ for which $V(\tilde{g}, \alpha) < \infty$ if and only if $\beta \leq \alpha$. Then the function $h = g + \tilde{g}$ satisfies (iii).

REFERENCES

1. W. H. J. Fuchs, *Topics in the theory of functions of one complex variable.* Van Nostrand, Princeton, N.J., 1967.

2. W. Rudin, *The radial variation of analytic functions.* Duke Math. J. 22 (1955), 235-242.

University of Wisconsin
Madison, Wisconsin 53706