SENSE-PRESERVING PL INVOLUTIONS OF SOME LENS SPACES

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1. INTRODUCTION

Let $L = L(p, q)$ be a 3-dimensional lens space. We say that L is *symmetric* if $q^2 \equiv \pm 1 \pmod p$. Recall that $L(p, q)$ and $L(p, q')$ are homeomorphic [3], [4] if and only if $q' \equiv \pm q$ or $qq' \equiv \pm 1 \pmod p$. Hence, symmetry of L is a topological property. A map $f: L \to L$ is called *sense-preserving* if f induces the identity of $H_1(L)$. For odd indices p ($p \geq 3$), we investigate all PL involutions of L that preserve sense and have nonempty fixed-point sets. It is known [1] that if $p \geq 3$, then L does not have an orientation-reversing involution. Also, simple examples show that an orientation-preserving involution need not be sense-preserving. If an involution can be extended to an effective circle action, it must clearly be sense-preserving. In our case, it turns out that the condition is also sufficient.

THEOREM. *Let* $L = L(p, q)$ (p *odd*, $p \geq 3$). *Let* h *be a* PL *involution of* L *with a nonempty fixed-point set. The* Z_2*-action generated by* h *can be extended to an effective* S^1*-action if and only if* h *is sense-perserving. Up to* PL *equivalences, there is exactly one such sense-preserving involution* h *if* L *is symmetric, and there are exactly two if* L *is not symmetric.*

Henceforth, we assume that $L = L(p, q)$ (p odd, $p \geq 3$) and that h is a sense-preserving PL involution of L with nonempty fixed-point set F. We shall simply call the orbit space of the Z_2-action generated by h the *orbit space* of h.

Remark. We can easily describe the orbit space of h as follows. If $q^2 \equiv \pm 1 \pmod p$, the orbit space is $L(p, q')$, where q' is any integer such that $2q' \equiv q \pmod p$. If $q^2 \not\equiv \pm 1$, we have two nonhomeomorphic orbit spaces $L(p, q')$ and $L(p, q'')$, where q' and q'' are any integers such that $2q' \equiv q$ and $2qq'' \equiv 1 \pmod p$.

2. THE FIXED-POINT SET F OF h

PROPOSITION 2.1. *F is a simple closed curve.*

Proof. Since L is a Z_2-homology sphere, F must be a sphere. Since $F \neq \emptyset$ and h preserves orientation, F is a simple closed curve, by the parity theorem.

PROPOSITION 2.2. *Let* $i: F \subset L$. *Then*

$$i_\#: \pi_1(F) \to \pi_1(L)$$

is an epimorphism.

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Proof. Let $f: S^3 \to L$ be the universal covering. Choose $x_0 \in F$ and $y_0 \in f^{-1}(x_0)$. By the lifting theorem and the unique-lifting theorem, there exists a PL involution h' of (S^3, y_0) such that $fh' = hf$ and the component F' of $f^{-1}(F)$ containing y_0 is pointwise fixed.

Let $i_\# \pi_1(F)$ be a subgroup of $\pi_1(L)$ of index k $(1 \leq k \leq p)$. Then $f^{-1}(F)$ is the union of k disjoint simple closed curves. Hence F' is the fixed-point set of h'. Let α be a path from y_0 to any $y_1 \in f^{-1}(x_0)$. Then $f\alpha$ represents an element u of $\pi_1(L)$, and $fh'\alpha = hf\alpha$ represents $h_\#(u) = u$. Hence $h'\alpha$, which starts at y_0, must end at y_1, or $h'(y_1) = y_1$. Hence $f^{-1}(x_0) \subset F'$, and therefore $F' = f^{-1}(F)$ and $k = 1$.

PROPOSITION 2.3. *Let* N *be a regular neighborhood of* F. *Then* $N' = \overline{L - N}$ *is a solid torus.*

Proof. Again, let $f: S^3 \to L$ be the universal covering. As in the proof of Proposition 2.2, there exists a PL involution h' of S^3 such that $F' = f^{-1}(F)$ is the fixed-point set. By a theorem of F. Waldhausen [5], F' is unknotted. The set $T = f^{-1}(N)$ is a regular neighborhood of F', and $T' = \overline{S^3 - T}$ is a solid torus. Also, $T' = f^{-1}(N')$, and $f \mid T': T' \to N'$ is a regular covering with the group of covering transformations isomorphic to Z_p. Hence, there exists an exact sequence

$$0 \to Z \to \pi_1(N') \to Z_p \to 0\,.$$

Let t generate Z, and let $\beta \in \pi_1(N')$ be a pullback of the generator of Z_p. Then t and β generate $\pi_1(N')$. Since Z is normal in $\pi_1(N')$, $\beta^{-1} t \beta = t$ or t^{-1}. However, it is easy to see that if $\beta^{-1} t \beta = t^{-1}$, then the abelianization $H_1(N')$ of $\pi_1(N')$ is finite. This is impossible, because N' is a rational homology circle, by Alexander duality. Hence $\beta^{-1} t \beta = t$, and $\pi_1(N')$ is an abelian group of rank 1. However, the universal covering space of N' (therefore, of T') is contractible, and $\pi_1(N')$ acts freely on it. Hence $\pi_1(N')$ cannot contain a nontrivial element of a finite order. Hence $\pi_1(N') \simeq Z$. On the other hand, N' is clearly irreducible, since it is covered by T'. Hence, by a theorem of J. Stallings [4], N' is fibered over S^1 with 1-connected fiber. Since N' is compact and orientable, N' is a solid torus.

3. PROOF OF THE THEOREM

Let D^2 and S^1 denote the unit disk and its boundary in the complex plane. Then $D^2 \times S^1$ is a solid torus whose points we shall denote by $(\rho z_1, z_2)$ $(z_1, z_2 \in S^1, 0 \leq \rho \leq 1)$. A similar statement applies to $S^1 \times D^2$.

Consider $X = D^2 \times S^1 \cup_g S^1 \times D^2$, where $g: S^1 \times S^1 \to S^1 \times S^1$ is an attaching homeomorphism. Groups involved are abelian. We shall ignore base points. Let α be the element of $\pi_1(S^1 \times S^1)$ represented by the path $(e^{2\pi i t}, 1)$ $(0 \leq t \leq 1)$. Let β be represented by $(1, e^{2\pi i t})$. Suppose

$$g_\#(\alpha) = \alpha^a \beta^c \qquad \text{and} \qquad g_\#(\beta) = \alpha^b \beta^d\,.$$

By suitable choice of orientations, we may assume that $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = 1$ and $a \geq 0$. The integers a, b, c, d completely determine the isotopy class of g, and therefore the homeomorphism type of X. If $a = 0$, then X is homeomorphic to $S^1 \times S^2$. If $a = 1$, then $X \approx S^3$. If $a > 1$, X is homeomorphic to the lens space $L(a, b)$.

Suppose h_0 is a PL involution of $D^2 \times S^1$ given by the rule

$$h_0(\rho z_1, z_2) = (-\rho z_1, z_2).$$

It can be shown that if a PL involution h of a solid torus T fixes pointwise a set $F \subset T$ such that $(T, F) \approx (D^2 \times S^1, 0 \times S^1)$, then h is PL equivalent to h_0. That is, there exists a PL homeomorphism $t: T \to D^2 \times S^1$ such that $h = t^{-1} h_0 t$. An explicit proof of this is given in a yet unpublished paper of P. Kim. Also it is easy to see that each free PL involution of a solid torus is PL equivalent to the involution h_1 of $S^1 \times D^2$ with $h_1(z_1, \rho z_2) = (-z_1, \rho z_2)$.

Hence we may assume that

$$L = D^2 \times S^1 \cup_g S^1 \times D^2$$

and that h is given by the formula

$$h(\rho z_1, z_2) = (-\rho z_1, z_2) \quad \text{for } (\rho z_1, z_2) \in D^2 \times S^1,$$

$$h(z_1, \rho z_2) = (-z_1, \rho z_2) \quad \text{for } (z_1, \rho z_2) \in S^1 \times D^2,$$

where $g: S^1 \times S^1 \to S^1 \times S^1$ is an appropriate equivariant attaching map. The isotopy class of g is determined by integers p, b, c, d.

PROPOSITION 3.1. *The integer* b *is even.*

Proof. The orbit space L' of h can be given as $L' = D^2 \times S^1 \cup_{g'} S^1 \times D^2$. A simple computation shows that the isotopy class of g' is determined by the integers p, b/2, 2c, d.

PROPOSITION 3.2. *Let* $L_i = D^2 \times S^1 \cup_{g_i} S^1 \times D^2$ $(i = 1, 2)$, *where the* g_i *are isotopic. Let a* PL *involution* h_i *of* L_i *be given by the formulas*

$$h_i(\rho z_1, z_2) = (-\rho z_1, z_2) \quad \text{and} \quad h_i(z_1, \rho z_2) = (-z_1, \rho z_2).$$

Then there exists an equivariant homeomorphism $t: L_1 \to L_2$ *such that* $t(D^2 \times S^1) = D^2 \times S^1$.

Proof. Let the orbit space L_i' be given by

$$L_i' = D^2 \times S^1 \cup_{g_i'} S^1 \times D^2.$$

Then, by the proof of Proposition 3.1, g_1' and g_2' are isotopic. Hence there exists a PL homeomorphism $t': L_1' \to L_2'$ such that $t'(D^2 \times S^1) = D^2 \times S^1$. We obtain t by lifting t'. (This can be done, though the orbit map is not a covering projection.)

Now we are in position to assume that

$$L = D^2 \times S^1 \cup_g S^1 \times D^2$$

and h is given as before but g is actually given by $g(z_1, z_2) = (z_1^p z_2^c, z_1^b z_2^d)$. This g is equivariant, because by Proposition 3.1 b is even.

4. THE CONCLUSION OF THE PROOF OF THE THEOREM

We assume that the situation is as at the end of Section 3, and that g is determined by p, b, c, d. The only information we have is that $b \equiv \pm q$ or $bq \equiv \pm 1 \pmod p$ and that b is even. We shall compare various cases arising from different b, c, d. Recall that a set of integers b, c, d with b even such that $pd - bc = 1$ determines an equivalence class of PL involutions. We can describe the representative element $h = h(b, c, d)$ by regarding L as $D^2 \times S^1 \cup_g S^1 \times D^2$, where $g(z_1, z_2) = (z_1^p z_2^c, z_1^b z_2^d)$, and by setting

$$h(\rho z_1, z_2) = (-\rho z_1, z_2) \quad \text{for } (\rho z_1, z_2) \in D^2 \times S^1,$$

$$h(z_1, \rho z_2) = (-z_1, \rho z_2) \quad \text{for } (z_1, \rho z_2) \in S^1 \times D^2 .$$

Case 1: b is fixed.

Let c' and d' be integers such that $pd' - bc' = 1 = pd - bc$. Then there exists an integer m such that $c' = c + mp$ and $d' = d + mb$. Let $g': S^1 \times S^1 \to S^1 \times S^1$ be given by the formula $g'(z_1, z_2) = (z_1^p z_2^{c'}, z_1^b z_2^{d'})$. Then an equivariant homeomorphism

$$t: D^2 \times S^1 \cup_g S^1 \times D^2 \to D^2 \times S^1 \cup_{g'} S^1 \times D^2$$

is given by the formulas $t(\rho z_1, z_2) = (\rho z_1 z_2^{-m}, z_2)$ and $t(z_1, \rho z_2) = (z_1, \rho z_2)$. Hence the equivalence class of h depends only on b.

Case 2: b is replaced by $b + mp$.

Here the resulting attaching map g' is determined by integers p, $b + mp$, c, and $d + mc$. (By Case 1, the values of c and d are irrelevant.) Assume that $g'(z_1, z_2) = (z_1^p z_2^c, z_1^{b+mp} z_2^{d+mc})$. An equivariant homeomorphism

$$t: D^2 \times S^1 \cup_g S^1 \times D^2 \to D^2 \times S^1 \cup_{g'} S^1 \times D^2$$

is given by $t(\rho z_1, z_2) = (\rho z_1, z_2)$ and $t(z_1, \rho z_2) = (z_1, \rho z_2 z_1^m)$. This t is actually equivariant, because m is required to be even if $b + mp$ is even.

Case 3: b is replaced by -b.

Let g' be determined by p, -b, -c, and d.

An equivariant homeomorphism t is given by the equations

$$t(\rho z_1, z_2) = (\rho z_1, \bar{z}_2) \quad \text{and} \quad t(z_1, \rho z_2) = (z_1, \rho \bar{z}_2).$$

Before we deal with the last case, observe that when g is determined by b, the orbit space of h is L(p, b/2), by the proof of Proposition 3.1.

Case 4: b is replaced by r, where $br \equiv 1 \pmod p$ and r is even.

Cases 1, 2, 3, and 4 and their combinations cover all possibilities. If $q^2 \equiv \pm 1$, then $b^2 \equiv \pm 1$ and $b \equiv \pm r$. Hence, Case 4 has already been covered. Therefore, the proof of the theorem will be complete when we have shown that (1) if $b^2 \not\equiv \pm 1$ and $br \equiv 1$, then L(p, b/2) and L(r/2) are not homeomorphic and (2) the Z_2-action of h can be extended to a circle action.

For (1), suppose $L(p, b/2)$ and $L(p, r/2)$ are homeomorphic. If $b/2 \equiv \pm r/2$, then $b \equiv \pm r$, and hence $b^2 \equiv \pm br \equiv \pm 1$. If $(b/2)(r/2) \equiv \pm 1$, then $1 \equiv br \equiv \pm 4$ and $p = 3$ or 5. If $p = 3$ or 5, then for any $x \not\equiv 0$, $x^2 \equiv \pm 1$. In particular, $b^2 \equiv \pm 1$.

For (2), suppose that $g(z_1, z_2) = (z_1^p z_2^c, z_1^b z_2^d)$. For each $z \in S^1$, define the S^1-action by the equations

$$z \cdot (\rho z_1, z_2) = (\rho z_1 z, z_2) \quad \text{and} \quad z \cdot (z_1, \rho z_2) = (z_1 z^p, \rho z_2 z^b).$$

Before closing, we remark that every $L(p, q)$ $(p \geq 3)$ admits a PL involution, with nonempty fixed-point set, that is not sense-preserving.

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