THE GREEN FUNCTION OF DOMAINS CONTAINING A FIXED ELLIPSE

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INTRODUCTION AND SUMMARY

Recently, E. Złotkiewicz and the present author [4] showed that domains of hyperbolic type have a property of "uniform local convexity." More precisely, if Ω is a domain of hyperbolic type, then any two points $w_1, w_2 \in \Omega$ whose hyperbolic distance $h(w_1, w_2; \Omega)$ with respect to Ω is less than $\tanh^{-1}(1/\sqrt{2})$ can be joined in Ω by a segment $[w_1, w_2]$. The constant $\tanh^{-1}(1/\sqrt{2})$ is the best possible.

The natural question arises whether the segment $[w_1, w_2]$ can be replaced by a larger set, after a suitable diminution of hyperbolic distance. In fact, if $0 < r < 1/\sqrt{2}$ and $h(w_1, w_2; \Omega) = \tanh^{-1} r$, then Ω contains an open ellipse with foci w_1 and w_2 and with eccentricity $\varepsilon(r) = 2r\sqrt{1 - r^2}$ (Theorem 3). In order to prove Theorem 3, we first solve an extremal problem involving the Green function $g(0, 1; \Omega)$ of domains Ω containing a fixed, maximal ellipse E with foci 0 and 1 (Theorem 1). Next, we consider a related problem for ring domains (Theorem 2). The well-known ring domain of A. Mori turns out to be extremal in this case. As corollaries of Theorem 3, estimates for the Green function $g(w_1, w_2; \Omega)$ are obtained under the assumption that Ω contains a fixed maximal ellipse with foci w_1 and w_2 (Theorem 4). As a consequence of Theorem 3 we also obtain a result that extends to arbitrary univalent majorants a theorem recently proved by Z. Lewandowski and J. Stankiewicz [6] for starlike majorants (Theorem 5).

1. TWO EXTREMAL PROBLEMS IN CONFORMAL MAPPING

We shall be concerned with the maximal value of the Green function $g(b, c; \Omega)$ for the class of simply connected domains Ω in the finite plane $\mathbb{C}$, each Ω containing a fixed ellipse E with foci b and c. Obviously, we may assume that $b = 0$ and $c = 1$, and that some boundary points of Ω actually lie on the boundary ∂E of E. We show that the extremal domain is the finite plane minus a ray on the prolongation of the minor axis of E.

THEOREM 1. *Let $\{\Omega\}$ be the class of simply connected domains Ω in the finite plane $\mathbb{C}$, each Ω containing the open ellipse* E *with foci* 0 *and* 1 *and with eccentricity ε. Let us also assume that the intersection $(\mathbb{C} \setminus \Omega) \cap \partial E$ is not empty. Then the Green function $g(0, 1; \Omega)$ is a maximum for $\Omega = \Omega_0 = \mathbb{C} \setminus \ell_0$, where ℓ_0 is one of the two vertical rays that lie outside of* E *and join the ends of the minor axis of* E *to the point at infinity. Moreover,*

$$(1.1) \qquad g(0, 1; \Omega_0) = -\frac{1}{2}\log\frac{1}{2}(1 - \sqrt{1 - \varepsilon^2}) = -\log\frac{1}{2}\sqrt{2 - \sqrt{4 - a^{-2}}},$$

where $2a = 1/\varepsilon$ *is the major axis of* E.

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Proof. Let Ω be a simply connected domain that contains the points b and c and omits the point $w = -b - c$. It was shown in [4] that for each w, the Green function $g(b, c; \Omega)$ is a maximum for $\Omega_1 = \mathbb{C} \setminus \Gamma_1$, where Γ_1 is the image of the segment $[0, 1/2]$ under the mapping $\wp(\,\cdot\,; 1, \tau_1)$. The period τ_1 satisfies the equation

$$\lambda(\tau_1) = \frac{b - c}{w - c} = \frac{1}{1 - w_1}. \tag{1.2}$$

We assume here that after a suitable similarity transformation the points b, c, w are carried into 0, 1, w_1; moreover, τ_1 lies in the fundamental region B of the modular function λ. By symmetry, we may assume that τ_1 lies in the right-hand half B^+ of B. Thus

$$w_1 = \frac{\lambda(\tau_1) - 1}{\lambda(\tau_1)} = \lambda\left(\frac{\tau_1 - 1}{\tau_1}\right).$$

Obviously, the function

$$w = \lambda_1(\tau) = \lambda\left(\frac{\tau - 1}{\tau}\right) \tag{1.3}$$

maps the region B^+ onto the upper half-plane H^+ so that the points 0, 1, ∞ remain invariant. On the other hand, the function

$$w = \frac{1}{2}\left[\, 1 + \sin\frac{\pi}{2}(2\zeta - 1)\,\right] \tag{1.4}$$

maps the upper half S^+ of the strip $0 < \Re\,\zeta < 1$ onto the upper half-plane H^+ so that the segments $\Im\,\zeta = \text{constant}$ correspond to arcs of ellipses in H^+ with foci 0 and 1. Consider now the compound transformation

$$\tau = \Phi(\zeta)\colon S^+ \to B^+ \tag{1.5}$$

defined by (1.3) and (1.4). Again, the points 0, 1, and ∞ remain invariant under Φ. By symmetry, the image of the ray $\Re\,\zeta = 1/2$, $\Im\,\zeta > 0$ is the ray $\Re\,\tau = 1/2$, $\Im\,\tau > 1/2$. As we showed in [4], the maximal value of $g(0, 1; \Omega)$ for domains Ω omitting the point w_1 is equal to

$$g(0, 1; \Omega_1) = -\log \nu^{-1}\left(\frac{1}{2}\,\Im\,\tau_1\right), \tag{1.6}$$

where w_1 and τ_1 satisfy (1.2), that is,

$$w_1 = \lambda_1(\tau_1), \tag{1.7}$$

and where $\nu(r) = \frac{1}{4} K(\sqrt{1 - r^2})/K(r)$ denotes the modulus of the ring domain $\Delta_1 \setminus [0, r]$. Because under (1.4) the points $w \in H^+$ on ellipses with foci 0 and 1 correspond to the points ζ on segments $\Im\,\zeta = \text{constant}$, it follows from (1.6) that the maximal value of $g(0, 1; \Omega)$ corresponds to the maximal value of $\Im\,\tau_1 = \Im\,\Phi(\zeta)$ for ζ moving on the segment $\Im\,\zeta = \text{constant}$ in S^+ that corresponds to ∂E.

We next prove that $\Im\,\Phi(\zeta)$ attains its maximal value at the center of the segment. To this end, consider the mapping $\tau = \Phi(\zeta)$ in the left-hand half S_1 of S^+, that is, in the domain $0 < \Re\,\zeta < 1/2$, $\Im\,\zeta > 0$. The function $u(\zeta) = \Im \log \Phi'(\zeta)$ is harmonic and bounded in S_1. Its boundary values are zero on vertical boundary rays of S_1, and they do not surpass $\pi/2$ on $(0, 1/2)$. Hence $0 < \arg \phi'(\zeta) < \pi/2$ in S_1. This implies that the local rotation of infinitesimal segments in S_1 under the mapping ϕ is contained between 0 and $\pi/2$. Consequently, $\Im\,\Phi(t + i\eta_0)$ is a strictly increasing function of t in $(0, 1/2)$, for each fixed $\eta_0 > 0$. By symmetry, $\Im\,\Phi(t + i\eta_0)$ is a strictly decreasing function of t in $(1/2, 1)$, and therefore $\Im\,\Phi(t + i\eta_0)$ has an absolute maximum for $t = 1/2$.

Since the line of symmetry $\Re\,\zeta = 1/2$ in S^+ remains unchanged under (1.4), we see that the extremal continuum emanates from a point w_1 with $\Re\, w_1 = 1/2$ on ∂E. Moreover, the value τ_1 associated with w_1 satisfies the conditions $\Re\,\tau_1 = 1/2$ and $\Im\,\tau_1 > 1/2$. In order to obtain the extremal domain Ω_0, note that in our case ($\Re\,\tau_1 = 1/2$) the pair 1 and τ_1 of periods of $\wp$ may be replaced by another pair τ_1 and $\overline{\tau}_1$ of periods. Hence the image line of $[0, 1/2]$ under $\wp(\,\cdot\,; 1, \tau_1)$ and also under $\wp(\,\cdot\,; \tau_1; \overline{\tau}_1)$ is a half-line on the real axis. Moreover, $\wp(1/2) = e_1$ is real, while $e_2 = \wp(\tau_1/2) = \overline{e}_3$. Since the points e_1, e_2, and e_3 become w_1, 0, and 1 after a suitable similarity transformation, the extremal domain is the finite plane $\mathbb{C}$ minus a ray ℓ_0 on the perpendicular bisector of the segment [0, 1]. By Lindelöf's principle, ℓ_0 does not intersect the segment; hence it must lie on the prolongation of the minor axis of the ellipse E. In order to evaluate $g(0, 1; \mathbb{C} \setminus \ell_0)$, we map $\mathbb{C} \setminus \ell_0$ conformally onto the unit disc Δ_1 so that 0 and 1 correspond to 0 and r ($0 < r < 1$), respectively. Then $g(0, 1; \mathbb{C} \setminus \ell_0) = -\log r$, by virtue of the conformal invariance of the Green function. After elementary calculations, we obtain (1.1), and this completes the proof.

Theorem 1 has a counterpart involving ring domains. The extremal ring domain is the well-known ring domain of Mori (see for example [5, p. 61]). Thus we have the following result.

THEOREM 2. *Let* $\{R\}$ *be the class of ring domains* R *such that the bounded component* Γ_0 *of the complement of* R *contains the points* 0 *and* 1, *while the unbounded component* Γ_∞ *lies outside a fixed ellipse* $E = \{w: |w| + |w - 1| < 2a\}$ *and has a nonempty intersection with the boundary of* E. *Then the modulus* mod R *is a maximum in case* Γ_∞ *is one of the two vertical rays in Theorem* 1 *while* Γ_0 *is a circular arc (disjoint from* Γ_∞*) whose endpoints are the foci* 0 *and* 1 *and whose center is the finite endpoint of* Γ_∞.

Proof. Let R^* be an extremal ring domain, and let Γ_0^* and Γ_∞^* be the components of $\hat{\mathbb{C}} \setminus R^*$ ($\hat{\mathbb{C}}$ being the extended plane). Consider the family $\{\gamma^*\}$ of closed, rectifiable Jordan curves γ^* in R^* that separate Γ_0^* from Γ_∞^*, and let $\{\gamma\}$ be the family of closed, rectifiable Jordan curves in $R^* \cup \Gamma_0^*$ that separate 0 and 1 from Γ_∞^*. It follows from the extremal-length characterization of the Green function [2] and from Theorem 1 that

$$\begin{aligned} \operatorname{mod} R^* &= \operatorname{mod}\{\gamma^*\} \le \operatorname{mod}\{\gamma\} = \nu(\exp[-g(0, 1; \mathbb{C} \setminus \Gamma_\infty^*)]) \\ &\le \nu(\exp[-g(0, 1; \mathbb{C} \setminus \ell_0)]) \\ &= \nu\left(\frac{1}{2}\sqrt{2 - \sqrt{4 - a^{-2}}}\right) = \nu\left(\sqrt{\frac{1 - \sqrt{1 - \varepsilon^2}}{2}}\right). \end{aligned}$$

On the other hand, a direct calculation shows that the last expression represents the modulus of the ring domain $\mathbb{C} \setminus (\ell_0 \cup \gamma_0)$ (see [5, page 61], for example). This proves Theorem 2.

2. A COVERING THEOREM FOR DOMAINS OF HYPERBOLIC TYPE

THEOREM 3. *Let* Ω *be a simply connected domain of hyperbolic type, and let* w_1 *and* w_2 *be points of* Ω *whose hyperbolic distance* $h(w_1, w_2; \Omega)$ *with respect to* Ω *is equal to* $\tanh^{-1} r$, *where* $0 < r < 1/\sqrt{2}$. *Then the domain* Ω *contains the ellipse*

$$E_r = \{w: |w - w_1| + |w - w_2| < |w_1 - w_2|/(2r\sqrt{1 - r^2})\} \tag{2.1}$$

with foci w_1 *and* w_2 *and eccentricity* $\varepsilon(r) = 2r\sqrt{1 - r^2}$. *The lower estimate* $|w_1 - w_2|/(2r\sqrt{1 - r^2})$ *of the major axis is sharp.*

Proof. Without loss of generality, we may assume that $w_1 = 0$ and $w_2 = 1$. Let f be the univalent function that maps the unit disc Δ_1 onto Ω so that $f(0) = 0$ and the inverse image r of $w_2 = 1$ lies on the radius $(0, 1)$. By the conformal invariance of hyperbolic distance, $f(r) = 1$. As was shown in [4], the domain $f(\Delta_1)$ contains the closed segment $[0, 1]$, if $r < 1/\sqrt{2}$. Hence $\Omega = f(\Delta_1)$ also contains a maximal ellipse E with major axis $2a > 1$ and foci 0 and 1. By the conformal invariance of the Green function, $g(0, 1; \Omega) = -\log r$. Thus the complementary set of Ω has a non-empty intersection with ∂E, while $E \subset \Omega$. Consequently, we can apply Theorem 1 and the formula (1.1) to obtain the inequalities

$$g(0, 1; \Omega) = -\log r \leq -\log \frac{1}{2}\sqrt{2 - \sqrt{4 - a^{-2}}}\ .$$

It follows that

$$2a \geq (2r\sqrt{1 - r^2})^{-1}\ . \tag{2.2}$$

Hence the major axis of E is at least $(2r\sqrt{1 - r^2})^{-1}$, or $|w_1 - w_2|(2r\sqrt{1 - r^2})^{-1}$ in the general case. In the case of the extremal domain considered in Theorem 1, the major axis of the maximal ellipse E_r is actually equal to

$$|w_1 - w_2|(2r\sqrt{1 - r^2})^{-1},$$

so that the estimate (2.2) is sharp. We can restate Theorem 3 in terms of so-called Koebe sets. For $0 < r < 1$, let S^r be the class of functions regular and univalent in Δ_1 that are normalized by the conditions $f(0)$ and $f(r) = 1$. The intersection $\bigcap_{f \in S^r} f(\Delta_1)$ is called the *Koebe set* $\mathscr{K}(S^r)$ for the class S^r (see [3], [4]). Although the exact form of $\mathscr{K}(S^r)$ has been determined in [4], it is still desirable to determine a large subset of $\mathscr{K}(S^r)$ with a fairly simple characterization. From Theorem 3 we obtain at once the following result.

COROLLARY 1. *If* $0 < r < 1/\sqrt{2}$, *then the Koebe set* $\mathscr{K}(S^r)$ *contains the open ellipse* E_r *with foci* 0 *and* 1 *and eccentricity* $\varepsilon(r) = 2r\sqrt{1 - r^2}$.

Another subset of $\mathscr{K}(S^r)$ can be obtained in an elementary way. With each $f \in S^r$ we can associate a constant λ and a univalent function $F = \lambda f$ subject to the standard normalization $F(0) = 0$, $F'(0) = 1$. The equation $F(r) = \lambda f(r) = \lambda$ implies that

$$f(z) = F(z)/F(r) . \tag{2.3}$$

From (2.3) and Koebe's 1/4-theorem we readily deduce that $\mathscr{K}(S^r)$ also contains the disc $\Delta_r' = \{w: |w| < (1 - r)^2/(4r)\}$. By symmetry, $\mathscr{K}(S^r)$ also contains the disc Δ_r'' of the same radius and center 1. Hence we have the following proposition.

COROLLARY 2. *If* $0 < r < 1/\sqrt{2}$, *then* $\Delta_r' \cup \Delta_r'' \cup E_r \subset \mathscr{K}(S^r)$.

3. ESTIMATES OF THE GREEN FUNCTION

From Theorem 1, we shall now obtain sharp lower and upper estimates of the Green function $g(w_1, w_2; \Omega)$, under the assumption that Ω contains a maximal ellipse E with foci w_1 and w_2 and eccentricity ε.

THEOREM 4. *If* Ω *is a simply connected domain that contains a maximal ellipse* E *with foci* w_1 *and* w_2 *and eccentricity* ε, *then*

$$-\log \nu^{-1}\left(\frac{1}{2\pi}\log\frac{1+\sqrt{1-\varepsilon^2}}{\varepsilon}\right) \leq g(w_1, w_2; \Omega) \leq -\frac{1}{2}\log\frac{1}{2}(1 - \sqrt{1-\varepsilon^2}) . \tag{3.1}$$

Both estimates are sharp.

Proof. The second inequality is a consequence of formula (1.1). In order to obtain the first inequality, note that $E \subset \Omega$, and use the Lindelöf principle. The lower bound $g(w_1, w_2; E)$ thus obtained can be evaluated as follows. Use the extremal-length characterization of the Green function [2] and assume that $w_1 = -1$ and $w_2 = 1$. Let $\{\gamma\}$ be the family of rectifiable Jordan curves separating -1 and 1 from ∂E. Obviously,

$$M = \operatorname{mod}\{\gamma\} = \operatorname{mod}(E \setminus [-1; 1]),$$

and the value of the latter expression is readily found by means of the transformation $w = (z + z^{-1})/2$. Thus we have the relation

$$M = \frac{1}{2\pi}\log(a + \sqrt{a^2 - 1}) = \frac{1}{2\pi}\log\frac{1+\sqrt{1-\varepsilon^2}}{\varepsilon} . \tag{3.2}$$

On the other hand,

$$g(-1, 1; E) = -\log \nu^{-1}(M) . \tag{3.3}$$

The desired inequality now follows from (3.2) and (3.3).

4. FURTHER APPLICATIONS

Let F be a function regular and univalent in Δ_1, subject to the standard normalization $F(0) = 0$, $F'(0) = 1$. Suppose that f is regular in Δ_1 and that $f'(0) \geq 0$. If F is a modular majorant of f in Δ_1 (that is, if $|f(z)| \leq |F(z)|$ for each $z \in \Delta_1$), then there exists a positive number ρ such that $f(\Delta_r) \subset F(\Delta_1)$ for each $r < \rho$ and each pair of admissible functions f and F satisfying the conditions stated above. If

F is starlike with respect to the origin, then $\rho = 1/3$, and this value is best possible (see [6]). We shall now extend this result. To this end, we need two lemmas. The first is essentially due to Rogosinski (see [1, p. 327]).

LEMMA 1. *Let* B *be the class of functions* ω *regular in the unit disc that satisfy the conditions* $\omega(0) \geq 0$ *and* $|\omega(z)| \leq 1$ *for all* z *in the unit disc. The set* H_{z_0} *of all possible values* $\omega(z_0)$ *for a fixed* z_0 $(0 < |z_0| < 1)$ *and for* ω *ranging over* B *depends only on* $r = |z_0|$, *and it is a closed convex domain* H_r *whose boundary consists of the semicircle* $|z| = r$, $\Re z \leq 0$, *together with two circular arcs through* $z = 1$ *tangent to* $|z| = r$ *at* $z = \mp ir$.

LEMMA 2. *For each pair of admissible functions* f *and* F *and each* r $(0 < r < 1)$, *the relation* $f(\overline{\Delta}_r) \subset F(\Delta_1)$ *holds if and only if*

$$H_r \subset \mathscr{K}(S^r) . \tag{4.1}$$

(Here Δ_r denotes the disc $|z| < r$, and $\overline{\Delta}_r$ is its closure.)

Proof. With each pair of admissible functions f and F, we can associate a function $\omega \in B$ such that $f(z) \equiv \omega(z) F(z)$. The assertion that $f(\overline{\Delta}_r) \subset F(\Delta_1)$ holds for each pair of admissible functions can also be stated as follows. For each $z_0 \in \overline{\Delta}_r$ and each pair of admissible functions f and F, we can find $z_1 \in \Delta_1$ such that $f(z_0) = \omega(z_0) F(z_0) = F(z_1)$, in other words,

$$\omega(z_0) = F(z_1)/F(z_0) = \phi(z_1) , \tag{4.2}$$

where ϕ is univalent in Δ_1 and normalized by the conditions $\phi(0) = 0$ and $\phi(z_0) = 1$. By Lemma 1, the point $\omega(z_0)$ can be an arbitrary point of H_r. Obviously, we can find a point z_1 satisfying (4.2), for each ϕ, if and only if $\omega(z_0)$ belongs to the intersection $\bigcap_\phi \phi(\Delta_1)$; the latter set is readily identified as $\mathscr{K}(S^r)$ $(r = |z_0|)$. Because $\mathscr{K}(S^r)$ shrinks as r increases, (4.2) has a solution $z_1 \in \Delta_1$, for each admissible ϕ and each $\omega(z_0) = w_0 \in H_r$, if and only if $w_0 \in H_r$ implies $w_0 \in \mathscr{K}(S^r)$. This condition is equivalent to (4.1), and Lemma 2 is proved.

It is worthwhile to mention that Lemma 2 remains true if we allow F to range over a subclass of S^r and take the Koebe set for the corresponding subclass.

THEOREM 5. *Let* $F(z) = z + A_2 z^2 + \cdots$ *be regular and univalent in the unit disc* Δ_1. *Suppose that the function* $f(z) = a_1 z + a_2 z^2 + \cdots$ $(a_1 \geq 0)$ *is regular in* Δ_1 *and that* $|f(z)| \leq |F(z)|$ *for all* $z \in \Delta_1$. *Then* $f(\Delta_{1/3}) \subset F(\Delta_1)$. *The constant* 1/3 *is best possible.*

Proof. Suppose that $0 < r < 1/3$. We show that then $H_r \subset \mathscr{K}(S^r)$. By Corollary 2, it is sufficient to verify that $H_r \subset \Delta_r' \cup \Delta_r'' \cup E_r$ for $r \in (0, 1/3)$. Since $(1 - r)^2/(4r) > r$ if $0 < r < 1/3$, the boundary arc of H_r situated on $|z| = r$ is contained in Δ_r'. On the other hand, two remaining boundary arcs of H_r are contained in the rectangle $\{w: |\Im w| \leq r, 0 \leq \Re w \leq 1\}$, which is a proper subset of the ellipse E_r. In fact, ∂E_r intersects the imaginary axis at the points

$$\mp i(1 - 2r^2)^2/(4r\sqrt{1 - r^2}) ,$$

and $(1 - 2r^2)^2/(4r\sqrt{1 - r^2}) > r$, since obviously $1 - 2r^2 > 2r$ for $r \in (0, 1/3)$. Hence $H_r \subset \Delta_r' \cup \Delta_r'' \cup E_r$ for $r \in (0, 1/3)$, and by Lemma 2, $f(\overline{\Delta}_r) \subset F(\Delta_1)$ for each $r \in (0, 1/3)$ and each pair of admissible functions f and F. On the other hand, the pair

$$f(z) = -z^2(1 - z)^{-2}, \qquad F(z) = z(1 - z)^{-2}$$

is obviously admissible; however, the value $f(1/3) = -1/4$ is omitted by F. This shows that the radius $1/3$ is sharp.

In [4], the ellipse $E_r^c = \{w: |w| + |w - 1| < r^{-1}\}$ was indentified with the Koebe set $\mathscr{K}(S_c^r)$ for the subclass of S^r consisting of convex functions. The following result is analogous to Theorem 5.

THEOREM 6. *Let* $F(z) = z + A_2 z^2 + \cdots$ *be a convex, univalent function in* Δ_1. *Suppose that the function* $f(z) = a_1 z + a_2 z^2 + \cdots$ $(a_1 \geq 0)$ *is regular in* Δ_1, *and that* $|f(z)| \leq |F(z)|$ *in* Δ_1. *Then* $f(\Delta_{1/2}) \subset F(\Delta_1)$. *The constant* $1/2$ *is best possible.*

Proof. Obviously, $H_r \subset E_r^c$ for $0 < r < 1/2$. Hence $f(\overline{\Delta}_r) \subset F(\Delta_1)$ for each $r \in (0, 1/2)$ and each pair of admissible functions f and F. For $F(z) = z(1 - z)^{-1}$ and $f(z) = -z F(z)$, the value $f(1/2) = -1/2$ is omitted by F. This ends the proof.

Theorem 5 suggests the following problem. *Find the largest value* r_0 *such that for each* $r \in (0, r_0)$ *the inclusion* $f(\Delta_r) \subset F(\Delta_1)$ *holds for each pair of univalent functions*

$$f(z) = a_1 z + a_2 z^2 + \cdots \qquad (a_1 > 0),$$

$$F(z) = z + A_2 z^2 + \cdots$$

satisfying the inequality $|f(z)| \leq |F(z)|$ *in* Δ_1.

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