THE EXISTENCE OF AN EVERYWHERE
DENSE INDEPENDENT SET

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Suppose that to every real number x there corresponds a set S(x) of real numbers such that $x \notin S(x)$. Two real numbers x and y are said to be *independent* provided that $x \notin S(y)$ and $y \notin S(x)$. A set of real numbers is said to be independent provided every pair of numbers in this set is independent.

P. Erdös has shown [2, Theorem 6] that if S(x) is nowhere dense for every x, then there exists an enumerable independent set. I shall prove the following stronger theorem.

THEOREM 1. *If* S(x) *is nowhere dense for every* x, *then there exists an everywhere dense independent set.*

It is interesting to note that if instead of assuming that each set S(x) is nowhere dense, one makes the assumption that no point x is a limit point of the corresponding set S(x), then there does not necessarily exist an everywhere dense independent set [1, Theorem 2].

The proof of the theorem uses facts about Baire category that can be found in [3]. Let $S^{-1}(y)$ denote the set of real numbers x for which $y \in S(x)$.

LEMMA. *Let* M *be a set that is everywhere of second category. Then there exists a residual set* R *such that, for every* $y \in R$, *the set* $M - S^{-1}(y)$ *is everywhere of second category.*

Assume that the conclusion of the lemma is false. Then there exists a set T of second category such that, for every $y \in T$, the set $M - S^{-1}(y)$ is not everywhere of second category. Hence, to every $y \in T$ there corresponds an open interval with rational endpoints such that the intersection of this interval with $M - S^{-1}(y)$ is a set of first category. Since there are only enumerably many such rational intervals and T is of second category, there exists a subset T_0 of T of second category such that to every $y \in T_0$ there corresponds the same rational interval J. There exists a subset T_1 of T_0 that is everywhere of second category in some open interval K. Hence, there exists an enumerable subset T_2 of T_1 that is everywhere dense in K. The set

$$J \cap \bigcup_{y \in T_2} [M - S^{-1}(y)]$$

is of first category, while $J \cap M$ is of second category. Therefore there exists an $x \in J \cap M$ for which $T_2 \subseteq S(x)$, which contradicts the fact that S(x) is nowhere dense. Thus our assumption is untenable, and the lemma is true.

Now to prove Theorem 1, let $\{J_1, J_2, \cdots, J_n, \cdots\}$ be the set of rational intervals. Denote the set of real numbers by M_1. According to the lemma, there exists a residual set R_1 such that for every $y \in R_1$, the set $M_1 - S^{-1}(y)$ is everywhere of

second category. Since $J_1 \cap M_1$ is of second category and R_1 is residual, $J_1 \cap M_1 \cap R_1 \neq \emptyset$; let $y_1 \in J_1 \cap M_1 \cap R_1$. Then the set

$$M_2 = M_1 - [S^{-1}(y_1) \cup S(y_1)]$$

is everywhere of second category; and for every $y \in M_2$, the points y and y_1 are independent. Again by the lemma, there exists a residual set R_2 such that for every $y \in R_2$, the set $M_2 - S^{-1}(y)$ is everywhere of second category, and again, $J_2 \cap M_2 \cap R_2 \neq \emptyset$; let $y_2 \in J_2 \cap M_2 \cap R_2$. Then the set

$$M_3 = M_2 - [S^{-1}(y_2) \cup S(y_2)] = M_1 - [S^{-1}(y_1) \cup S(y_1) \cup S^{-1}(y_2) \cup S(y_2)]$$

is everywhere of second category; and for every $y \in M_3$, the set $\{y_1, y_2, y\}$ is independent. It is evident that by continuing in this fashion we obtain an enumerable independent set $\{y_1, y_2, \cdots, y_n, \cdots\}$, which is everywhere dense because $y_n \in J_n$ $(n = 1, 2, 3, \cdots)$. This completes the proof of the theorem.

COROLLARY. *If* $S(x)$ *is nowhere dense for every* x, *then there exists an independent set of order type* η.

This is stronger than the result in [4].

Remark. The proof of the theorem has shown actually that if we start with a set M_1 that is everywhere of second category, then there exists an everywhere dense independent subset of M_1. Now (see [5, p. 176]) every set that is everywhere of second category is the union of enumerably many mutually exclusive sets, each of which is everywhere of second category; and [5, p. 115] under the assumption of the continuum hypothesis, every set that is everywhere of second category is the union of $2^{\aleph_0}$ mutually exclusive sets, each of which is everywhere of second category. Combining these results, we obtain the following proposition.

THEOREM 2. *If* $S(x)$ *is nowhere dense for every* x *and if* M *is everywhere of second category, then there exist at least enumerably many mutually exclusive, everywhere dense, independent subsets of* M. *Under the added assumption of the continuum hypothesis,* M *has* $2^{\aleph_0}$ *mutually exclusive, everywhere dense, independent subsets.*

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