ON THE EXISTENCE OF SIMPLE QUADRATURES

Seymour Haber

If S is a set of functions that are Riemann-integrable on [0, 1], then a formula

$$(1) \qquad \int_0^1 f(x)\,dx = \sum_{i=1}^{\infty} a_i f(x_i),$$

in which the x_i are distinct and the a_i and x_i are fixed independently of the function f, is called a *simple quadrature* for S if it holds for every function f in S. The functions may be complex-valued, and the a_i and x_i may be any complex numbers.

The notion of simple quadrature was introduced by Philip Davis, who investigated it in a series of papers ([1], [3], [4]; see also [2, pp. 357-358]). The term "simple" indicates the contrast with the usual numerical quadrature rules, which have the form

$$(2) \qquad \int_0^1 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} a_{i,n} f(x_{i,n}).$$

There are rules of the form (2) — for example the trapezoidal rule, with $x_{i,n} = (i - 1)/(n - 1)$ and $a_{i,n} = 1/(n - 1)$ for $i \neq 1, n$; $a_{1,n} = a_{n,n} = 1/2(n - 1)$ — that are valid for all (properly) Riemann-integrable functions. In contrast, Davis showed that no rule of the form (1), with distinct x_i, is valid for all continuous functions, and he asked what classes of functions have simple quadratures. He proved [1] that there is a simple quadrature for the class of polynomials, and later [3], [4], he showed that there are some regions R in the complex plane — R containing the integration interval — such that the set of functions analytic on R has a simple quadrature. In this paper I shall construct simple quadratures for some wide classes of continuous functions; it will follow, for example, that the class of all functions continuously differentiable on [0, 1] has a simple quadrature; this extends Davis's results.

The present construction is related to a theorem of Fritz John ([5] to [7]), who found formulas of the form (1), where the x_i are not distinct, that are valid for every Riemann-integrable f. One way to obtain such formulas is as follows: We first generate a rule of the form (2) in which the sums on the right are Riemann sums: For $n = 1, 2, \cdots$, let

$$\Pi_n = (w_{n,0}, w_{n,1}, \cdots, w_{n,n}), \qquad \text{where } 0 = w_{n,0} < w_{n,1} < \cdots < w_{n,n} = 1,$$

be a partition of [0, 1] into n subintervals; and for $i = 1, 2, \cdots, n$, let $x_{n,i}$ be a point in the ith subinterval. Writing

$$a_{n,i} = w_{n,i} - w_{n,i-1} \qquad \text{and} \qquad \Delta_n = \max_i \{a_{n,i}\},$$

Received June 4, 1971.

Michigan Math. J. 19 (1972).

we specify that the Π_n be such that $\Delta_n \to 0$ as $n \to \infty$. With these $x_{n,i}$ and $a_{n,i}$, (2) holds for all Riemann-integrable functions. Furthermore,

$$\int_0^t f(x)\,dx = \lim_{n\to\infty} \sum_{x_{n,i}\leq t} a_{n,i} f(x_{n,i}) \tag{3}$$

for every $t \in [0, 1]$, since (3) is simply equation (2) with $f \cdot \chi_{[0,t]}$ in place of f, where $\chi_{[0,t]}$ is the characteristic function of the interval $[0, t]$; and Riemann integrability of f implies Riemann integrability of $f \cdot \chi_{[0,t]}$. What is important, for the remainder of the construction, is that the convergence in (3) is uniform in t. To establish the uniformity, let $i^* = i^*(n, t)$ be the greatest index i such that $x_{n,i} \leq t$. Then

$$\left| \int_0^t f(x)\,dx - \sum_{x_{n,i}\leq t} a_{n,i} f(x_{n,i}) \right|$$

$$\leq \left| \int_0^t f(x)\,dx - \int_0^{w_{n,i^*}} f(x)\,dx \right| + \left| \int_0^{w_{n,i^*}} f(x)\,dx - \sum_{x_{n,i}\leq t} a_{n,i} f(x_{n,i}) \right|$$

$$\leq \left| \int_t^{w_{n,i^*}} f(x)\,dx \right| + \sum_{x_{n,i}\leq t} a_{n,i}(M_{n,i} - m_{n,i}),$$

where $M_{n,i}$ is the supremum, and $m_{n,i}$ the infimum, of f on $[w_{n,i-1}, w_{n,i}]$. If M is an upper bound for $|f|$ on $[0, 1]$, then

$$\left| \int_0^t f(x)\,dx - \sum_{x_{n,i}\leq t} a_{n,i} f(x_{n,i}) \right| \leq M\Delta_n + \sum_{i=1}^{n} a_{n,i}(M_{n,i} - m_{n,i}),$$

and the right-hand side, which is independent of t, tends to zero as $n \to \infty$.

Now form a series (1), block by block, as follows: The first block contains only the term $a_{1,1} f(x_{1,1})$. For $m \geq 2$, the mth block consists of the terms $a_{m,i} f(x_{m,i})$ $(i = 1, \cdots, m)$, together with the terms $-a_{m-1,i} f(x_{m-1,i})$ $(i = 1, \cdots, m-1)$, arranged in the natural order of the abscissas x occurring in them. The sum of the first m blocks is thus

$$\sum_{i=1}^{m} a_{m,i} f(x_{m,i}),$$

and this converges to the integral as $m \to \infty$; to show that the series converges, it is now sufficient to show that the maximum of the partial sums of terms in the mth block is $o(1)$ as $m \to \infty$. But any such partial sum consists of precisely those terms in the mth block whose abscissas do not exceed t, for some particular t in $[0, 1]$. Thus the partial sum is

$$\sum_{x_{m,i}\le t} a_{m,i} f(x_{m,i}) - \sum_{x_{m-1,i}\le t} a_{m-1,i} f(x_{m-1,i})$$

$$= \left(\int_0^t f(x)\,dx - \sum_{x_{m,i}\le t} a_{m,i} f(x_{m,i}) \right) \tag{4}$$

$$- \left(\int_0^t f(x)\,dx - \sum_{x_{m-1,i}\le t} a_{m-1,i} f(x_{m-1,i}) \right),$$

and each of these two quantities is $o(1)$ uniformly in t.

The series constructed does not define a simple quadrature, since each of the abscissas occurs in it twice.

The construction above is based on the idea of converting a rule of the form (2) into one of the form (1) by use of the standard identity

$$\lim_{n\to\infty} S_n = S_1 + (S_2 - S_1) + (S_3 - S_2) + \cdots .$$

Each S_n is a Riemann sum, and each $S_{n+1} - S_n$ is a difference of two sums; we arranged the terms of $S_{n+1} - S_n$ so as to ensure that the partial sums of $S_{n+1} - S_n$ tend to zero uniformly as n increases. Interleaving the terms of S_{n+1} and those of $-S_n$ as described accomplished this.

We can form simple quadrature formulas by modifying the construction to avoid the double occurrence of abscissas. In proving the theorem below, when the Riemann sum S_n is to be cancelled by introduction of its negative $-S_n$, we shift the abscissas occurring in the terms of $-S_n$ by a small amount. Let us temporarily call the sum actually introduced $-S'_{n,0}$. We shall have cancelled S_n only approximately, introducing an error. To cancel this error—again only approximately, since that is all that we can do without duplicating abscissas—we introduce the negative of $-S'_{n,0}$ at a later stage, with abscissas shifted once more, but by a smaller amount. Call this last sum introduced $S_{n,1}$. At the same time, we must cancel the original sum S_n anew, by a new sum $-S'_{n,1}$ whose abscissas differ from those of S_n —but differ by a much smaller amount than did the abscissas in $-S'_{n,0}$. The errors introduced at this stage are reduced by introducing, at a later stage, a sum $-S'_{n,2}$ to cancel (very nearly) both $S_{n,1}$ and S_n, and a sum $S_{n,2}$ to do the same for $-S'_{n,1}$ and $-S'_{n,0}$; and so the process continues.

Care must be taken to interleave the terms of the various sums introduced at each stage so that in the final simple series, all blocks of consecutive terms will have appropriately small sums. For this purpose we will find it useful, each time a previously introduced sum is to be approximately cancelled, to have the number of terms in the cancelling sum greater than that in the sum being cancelled, by a factor that is a power of 2—the power being related to the number of stages between the sum being cancelled and the cancelling sum.

By Davis's theorem, no such construction can succeed for all continuous functions. We must restrict the rapidity of variation of the function to ensure the convergence to zero of the errors introduced at the successive cancellation stages.

Finally, it is necessary to arrange that all the various shifted abscissas, arising from different sums S_n at different stages, be distinct. A number-theoretic device accomplishes this.

THEOREM. *Let* $\omega(x)$ *be a continuous, strictly increasing, real function defined on* $[0, 1]$, *with* $\omega(0) = 0$. *Let* $C(\omega)$ *be the class of all functions* f *defined on* $[0, 1]$ *and satisfying the condition*

$$|f(x_1) - f(x_2)| \leq A\,\omega(|x_1 - x_2|)$$

for some constant $A = A(f)$ *and for all* $x_1, x_2 \in [0, 1]$. *The class* $C(\omega)$ *has a simple quadrature* (1), *in which the* a_i *are real and the* x_i *lie in* (0, 1).

Proof. For $n = 1, 2, \cdots$, let p_n be the nth odd prime, and let $q_n = (p_n - 1)/2$. Set

$$t_{n,r} = \frac{2r - 1}{p_n} \qquad (r = 1, 2, \cdots, q_n). \tag{5}$$

Let $\{\varepsilon_1, \varepsilon_2, \cdots\}$ be a sequence of irrational numbers such that, for each n, the numbers $1, \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ are linearly independent over the field of rationals. Let the ε_n satisfy the further conditions

$$\begin{cases} \omega(2^n \varepsilon_n) = O(2^{-n} n^{-2}) & \text{as } n \to \infty, \\ \varepsilon_n \leq \dfrac{1}{2p_n} & (n = 1, 2, \cdots), \\ \varepsilon_n > 2\varepsilon_{n+1} + 4\varepsilon_{n+2} + 8\varepsilon_{n+3} + \cdots & (n = 1, 2, \cdots). \end{cases} \tag{6}$$

(One way to construct such a sequence is to set $\varepsilon_n = \{\sqrt{p_n}\}/m_n$, where $\{a\}$ denotes the fractional part of a and where $\{m_1, m_2, \cdots\}$ is a sequence of positive integers defined recursively as follows: m_1 is chosen sufficiently large to make $\omega(2\varepsilon_1) < 1/2$ and $\varepsilon_1 < 1/6$. For $n > 1$, m_n is chosen sufficiently large to make $\varepsilon_n < \min\{2^{-n}\omega^{-1}(2^{-n}n^{-2}), 1/2p_n, \varepsilon_{n-1}/4\}$.)

I now define a collection of sets $C_{n,r,+}$ and $C_{n,r,-}$ whose union will be the set of abscissas of the simple quadrature formula. I shall use the following notation: If S is a set of real numbers, x is a real number, and m is a positive integer, then

$$S + x = \{s + x \mid s \in S\},$$

$$S + m * x = \{s + hx \mid s \in S,\ h \text{ is an integer},\ h \neq 0,\ |h| \leq 2^{m-1}\}.$$

Set $C_{n,0,+} = \{t_{n,\ell} \mid 1 \leq \ell \leq q_n\}$ and $C_{n,0,-} = C_{n,0,+} + \varepsilon_n$, for $n = 1, 2, \cdots$, and

$$C_{n,r,\pm} = \bigcup_{\ell=0}^{r-1} (C_{n,\ell,\mp} + (r - \ell) * \varepsilon_{n+r}) \tag{7}$$

for n, r, = 1, 2, $\cdots$. Each number in the set $C_{n,r,\pm}$ is of the form

$$t_{n,s} + \alpha_0\varepsilon_n + \alpha_1\varepsilon_{n+1} + \cdots + \alpha_r\varepsilon_{n+r}, \tag{8}$$

where $\alpha_0 = 0$ or 1 and $\alpha_1, \alpha_2, \cdots, \alpha_r$ are integers such that $|\alpha_i| \leq 2^{i-1}$.

Let us describe the sequences $(\alpha_0, \alpha_1, \cdots, \alpha_r)$ that actually occur in writing elements of $C_{n,r,+}$ or $C_{n,r,-}$ in this form (call those *admissible* sequences): An r-*symbol* σ will be an ordered $(r + 1)$-tuple of nonnegative integers, some of them starred (for example, (1, 0, 1, 0) and (0, 1*, 2*, 1) are 3-symbols). For each r-symbol σ, $S(\sigma)$ will be the set of all ordered $(r + 1)$-tuples of integers that we can obtain from σ by replacing each starred integer n^* by nonzero integers between -2^{n-1} and 2^{n-1}. The set of admissible sequences for $C_{n,0,+}$ is $S((0))$; that for $C_{n,0,-}$ is $S((1))$; that for $C_{n,1,+}$ is $S((1, 1^*))$, that for $C_{n,2,+}$ is

$$S((1, 0, 2^*)) \cup S((0, 1^*, 1^*)),$$

and so forth. It is easy to see, by induction, that the set of admissible sequences for $C_{n,r,+}$ or $C_{n,r,-}$ is of the form

$$\bigcup_{i=1}^{2^{r-1}} S(\sigma_i),$$

where each σ_i is an r-symbol having the property that the sum of the starred integers in it is r, and where the 2^{r-1} sets $S(\sigma_i)$ are disjoint. Since each $S(\sigma_i)$ has 2^r members, there are exactly 2^{2r-1} admissible sequences for $C_{n,r,+}$ or for $C_{n,r,-}$.

If $(\alpha_0, \alpha_1, \cdots, \alpha_r)$ and $(\alpha_0', \alpha_1', \cdots, \alpha_r')$ are two admissible sequences, then, by the rational independence of the ε_n the equation

$$t_{n,s} + \alpha_0 \varepsilon_n + \cdots + \alpha_r \varepsilon_{n+r} = t_{n,s'} + \alpha_0' \varepsilon_n + \cdots + \alpha_r' \varepsilon_{n+r}$$

can hold only when $s = s'$ and $\alpha_0 = \alpha_0'$, $\alpha_1 = \alpha_1'$, $\cdots$, $\alpha_r = \alpha_r'$. Thus each of $C_{n,r,+}$ and $C_{n,r,-}$ has $2^{2r-1} q_n$ members for $r > 0$, and q_n for $r = 0$. The various sets $C_{n,0,+}$ are disjoint, and it follows from the rational independence of the ε_n that all the sets $C_{n,r,\pm}$ are disjoint. Since $1/p_n \leq t_{n,s} \leq 1 - 1/p_n$, it follows from (8) and from the second and third conditions of (6) that each point in each $C_{n,r,\pm}$ is in (0, 1).

It remains to define the coefficients a_i and the order of appearance of the terms involving the various abscissas. I shall refer to terms of the series by way of the abscissas used in them, in phrases such as "the $C_{n,0,+}$-terms."

For each n and r, the coefficients (or weights) of the $C_{n,r,+}$-terms will all be $(2^r q_n)^{-1}$, and those of the $C_{n,r,-}$-terms will be $-(2^r q_n)^{-1}$. It follows that the series (1) will not converge absolutely, because for each n the sum of the absolute values of the $C_{n,r,+}$-terms and the $C_{n,r,-}$-terms will be 2^r when $f \equiv 1$.

I shall describe first the order of certain blocks of terms, and then the order within those blocks: The first block, B_1, consists of the $C_{1,0,+}$-terms; B_2 consists of the $C_{2,0,+}$-terms and the $C_{1,0,-}$-terms; B_3 of the $C_{3,0,+}$-, $C_{2,0,-}$-, $C_{1,1,+}$- and $C_{1,1,-}$-terms; in general, B_n consists of the $C_{n,0,+}$- and $C_{n-1,0,-}$-terms, and all the $C_{n-i,i-1,+}$- and $C_{n-i,i-1,-}$-terms for all $i = 2, 3, 4, \cdots, n - 1$.

For $n > 1$, B_n consists of $n - 1$ consecutive subblocks; $B_{n,1}$ contains the $C_{n,0,+}$- and the $C_{n-1,0,-}$-terms; for $i = 2, 3, \cdots, n - 1$, $B_{n,i}$ contains the $C_{n-i,i-1,+}$- and the $C_{n-i,i-1,-}$-terms. I shall write $B_{1,1} = B_1$.

For each n, the terms in $B_{n,1}$ are arranged in the natural order of the abscissas occurring in them. For $i > 1$, $B_{n,i}$ consists of consecutive subblocks

$B_{n,i,j}$ $(j = 1, 2, \cdots, q_{n-i})$. $B_{n,i,j}$ contains the terms of $B_{n,i}$ whose abscissas have the form

$$t_{n-i,j} + \alpha_0 \varepsilon_{n-i} + \alpha_1 \varepsilon_{n-i+1} + \cdots + \alpha_{i-1} \varepsilon_{n-1},$$

where $(\alpha_0, \alpha_1, \cdots, \alpha_{i-1})$ is an admissible sequence for $C_{n-1,i-1,+}$ or $C_{n-1,i-1,-}$. Now consider all the numbers of the form

$$\alpha_0 \varepsilon_{n-i} + \alpha_1 \varepsilon_{n-i+1} + \cdots + \alpha_{i-3} \varepsilon_{n-3},$$

where $(\alpha_0, \alpha_1, \cdots, \alpha_{i-3}, \alpha_{i-2}, \alpha_{i-1})$ is an admissible sequence for $C_{n-1,i-1,+}$ or $C_{n-1,i-1,-}$; denote these, in their natural order, by $y_1, y_2, \cdots, y_M$. Divide $B_{n,i,j}$ into consecutive subblocks $B_{n,i,j,k}$ $(k = 1, 2, \cdots, M)$, where $B_{n,i,j,k}$ contains terms of $B_{n,i,j}$ whose abscissas have the form $t_{n-i,j} + y_k + \alpha_{i-2}\varepsilon_{n-2} + \alpha_{i-1}\varepsilon_{n-1}$.

LEMMA 1. *In each* $B_{n,i,j,k}$ *there are exactly as many terms with positive coefficients as with negative coefficients.*

Proof. $B_{n,i}$ consists of all the $C_{n-i,i-1,+}$- and the $C_{n-i,i-1,-}$-terms; each $B_{n,i,j,k}$ consists of all the terms with abscissas in $C_{n-i,i-1,+}$ or $C_{n-i,i-1,-}$ that have the form

$$t_{n-i,j} + \alpha'_0 \varepsilon_{n-i} + \alpha'_1 \varepsilon_{n-i+1} + \cdots + \alpha'_{i-3} \varepsilon_{n-3} + \alpha_{i-2}\varepsilon_{n-2} + \alpha_{i-1}\varepsilon_{n-1} \tag{9}$$

with certain fixed $j, \alpha'_0, \alpha'_1, \cdots, \alpha'_{i-3}$. Thus we need to show that there are the same number of abscissas of the form (9) in $C_{n-i,i-1,+}$ as in $C_{n-i,i-1,-}$. Now

$$C_{n-i,i-1,\pm} = (C_{n-i,i-2,\mp} + 1^* \varepsilon_{n-1}) \cup \left[\bigcup_{\ell=0}^{i-3} (C_{n-i,\ell,\mp} + (i - 1 - \ell)^* \varepsilon_{n-1}) \right] \tag{10}$$

and

$$C_{n-i,i-2,\mp} = \bigcup_{\ell=0}^{i-3} (C_{n-i,\ell,\mp} + (i - 2 - \ell)^* \varepsilon_{n-2}). \tag{11}$$

Therefore an abscissa of the form (9) is in $C_{n-i,i-1,+}$ if

$$t_{n-i,j} + \alpha'_0 \varepsilon_{n-i} + \cdots + \alpha'_{i-3} \varepsilon_{n-3}$$

is either in $C_{n-i,\ell,+}$ or in $C_{n-i,\ell,-}$, for some ℓ between 0 and $i - 3$. If it is in $C_{n-i,\ell,-}$, then by (10), there are $2^{i-1-\ell}$ abscissas of the form (9) in $C_{n-1,i-1,+}$, and by (11) and (10), there are $2^{i-2-\ell} \cdot 2^1 = 2^{i-1-\ell}$ such terms in $C_{n-1,i-1,-}$. Repeating the last 2 sentences with subscripts + and - interchanged, we see that the lemma holds.

I now complete the definition of the formula (1) by specifying that in each $B_{n,i,j,k}$ the terms are arranged in any manner that results in their coefficients alternating in sign, with the first one positive.

LEMMA 2. *For each* $f \in C(\omega)$,

$$\lim_{n \to \infty} (B_1 + B_2 + \cdots + B_n) = \int_0^1 f. \tag{12}$$

This lemma says that the series (1) constructed above converges "by blocks" to the desired sum. To prove it, we first define T(C), for any set C of abscissas, to be the sum of the terms of (1) involving the abscissas of C. Then

$$(13)\qquad B_1 + B_2 + \cdots + B_n = T(C_{n,0,+}) + \sum_{i=1}^{n-1} \sum_{r=0}^{i-1} \{T(C_{n-i,r,+}) + T(C_{n-i,r,-})\}.$$

Clearly,

$$T(C_{n,0,+}) = \frac{1}{q_n} \sum_{r=1}^{q_n} f\left(\frac{2r-1}{p_n}\right) \to \int_0^1 f \quad \text{as } n \to \infty,$$

and the lemma will follow if we show that the double sum in (13) is o(1). Now

$$\sum_{r=0}^{i-1} \{T(C_{n-i,r,+}) + T(C_{n-i,r,-})\}$$

$$= T(C_{n-i,i-1,+}) + T(C_{n-i,i-1,-}) + \sum_{r=0}^{i-2} \{T(C_{n-i,r,+}) + T(C_{n-1,r,-})\}$$

$$= \sum_{r=0}^{i-2} \{T(C_{n-i,r,-} + (i-1-r)^* \varepsilon_{n-1}) + T(C_{n-i,r,-})\}$$

$$+ \sum_{r=0}^{i-2} \{T(C_{n-i,r,+} + (i-1-r)^* \varepsilon_{n-1}) + T(C_{n-i,r,+})\}.$$

But

$$T(C_{n-i,r,-}) + T(C_{n-i,r,-} + (i-1-r)^* \varepsilon_{n-1})$$

$$= \frac{1}{2^r q_{n-i}} \sum_{x \in C_{n-i,r,-}} \left(f(x) - \frac{1}{2^{i-1-r}} \sum f(x + m\varepsilon_{n-1})\right),$$

where the inner sum extends over all integers m between -2^{i-2-r} and $+2^{i-2-r}$, except 0. For each m,

$$f(x + m\varepsilon_{n-1}) = f(x) + \theta A\omega(m\varepsilon_{n-1}) = f(x) + \theta A\omega(2^{i-2-r}\varepsilon_{n-1})$$

$$= f(x) + \theta A\omega(2^{n-3}\varepsilon_{n-1}) = f(x) + \theta A\omega(2^{n-1}\varepsilon_{n-1}),$$

where each θ is some number in [-1, 1]. Since the number of abscissas in $C_{n-i,r,\pm}$ is no more than $2^{2r} q_{n-i}$, it follows that

$$|T(C_{n-i,r,-}) + T(C_{n-i,r,-} + (i-1-r)^* \varepsilon_{n-1})| \le 2^r A\omega(2^{n-1}\varepsilon_{n-1});$$

a similar statement holds for $C_{n-i,r,+}$. Thus the inner sum in (13) has absolute value less than $2^i A\omega(2^{n-1}\varepsilon_{n-1})$; the double sum is therefore bounded by $2^n A\omega(2^{n-1}\varepsilon_{n-1})$, and the lemma follows from the first condition in (6).

Now define M_n to be the maximum, over all integers s, of the absolute value of the sum of the first s terms in B_n. To prove the main theorem, it is sufficient, in view of Lemma 2, to show that $M_n \to 0$ as $n \to \infty$. This will be done in Lemmas 3, 4, and 5.

LEMMA 3. *Let* M'_n *be the maximum, over all integers* s, *of the absolute value of the sum of the first* s *terms in* $B_{n,1}$. *Then* $M'_n \to 0$ *as* $n \to \infty$.

Proof. Since the terms in $B_{n,1}$ are arranged in natural order of the abscissas occurring in them, we may write

$$M'_n = \left| \sum_{\substack{x \leq x_0 \\ x \in C_{n,0,+}}} \frac{1}{q_n} f(x) - \sum_{\substack{x \leq x_0 \\ x \in C_{n-1,0,-}}} \frac{1}{q_{n-1}} f(x) \right|$$

for some $x_0 = x_0(n)$. Furthermore,

$$\begin{aligned} -\frac{1}{q_{n-1}} \sum_{\substack{x \leq x_0 \\ x \in C_{n-1,0,-}}} f(x) &= -\frac{1}{q_{n-1}} \sum_{\substack{x \leq x_0 - \varepsilon_n \\ x \in C_{n-1,0,+}}} f(x + \varepsilon_n) \\ &= -\frac{1}{q_{n-1}} \sum_{\substack{x \leq x_0 \\ x \in C_{n-1,0,+}}} (f(x) + \theta A \omega(\varepsilon_n)) + \frac{\theta K}{q_{n-1}}, \end{aligned} \tag{14}$$

where each θ lies in [-1, 1] and K is an upper bound on $|f|$ in [0, 1]. (The last term in (14) is included to provide for the possibility that the last sum written in (14) contains one more term than the previous sum.) The number of points in $C_{n-1,0,+}$ is q_{n-1} ; therefore we may write

$$M'_n = \left| \sum_{\substack{x \leq x_0 \\ x \in C_{n,0,+}}} \frac{1}{q_n} f(x) - \sum_{\substack{x \leq x_0 \\ x \in C_{n-1,0,+}}} \frac{1}{q_{n-1}} f(x) + \theta A \omega(\varepsilon_n) + \frac{\theta K}{q_{n-1}} \right| .$$

The last two terms clearly tend to 0 as $n \to \infty$; that the sum of the first terms is o(1) uniformly in x_0, as $n \to \infty$, follows from the argument used to prove the same fact for (4).

LEMMA 4. *If* $i \geq 2$ *and* Σ *is the sum of the first* s *terms of* $B_{n,i}$, *where* s *is an even integer, then* $|\Sigma| < A' n^{-2}$, *where* $A' = A'(f)$ *is independent of* i *and of* s.

Proof. Denote the terms in Σ, in order, by $\beta_1, \beta_2, \cdots, \beta_s$. The set $B_{n,i}$ consists of blocks $B_{n,i,j,k}$, each of which contains, by Lemma 1, an even number of terms. Therefore we can write

$$\Sigma = (\beta_1 + \beta_2) + (\beta_3 + \beta_4) + \cdots + (\beta_{s-1} + \beta_s),$$

and each pair of terms $(\beta_{2r-1}, \beta_{2r})$ belongs to a particular $B_{n,i,j,k}$. The coefficient of β_{2r-1} is $(2^{i-1} q_{n-i})^{-1}$, and that of β_{2r} is $-(2^{i-1} q_{n-i})^{-1}$; the abscissas occurring in the two terms differ by no more than $2^{i-3}\varepsilon_{n-2} + 2^{i-2}\varepsilon_{n-1}$, which is less than $2^{n-2}\varepsilon_{n-2}$. Thus

$$|\beta_{2r-1} + \beta_{2r}| \leq A(2^{i-1}q_{n-i})^{-1}\,\omega(2^{n-2}\varepsilon_{n-2});$$

and since s is no greater than the number of terms in $B_{n,i}$, which is $2^{2i-3}q_{n-i}$, we see that

$$\left|\Sigma\right| \leq A\,2^{i-2}\,\omega(2^{n-2}\varepsilon_{n-2}) < A\,2^{n-2}\,\omega(2^{n-2}\varepsilon_{n-2});$$

the lemma now follows from the first condition in (6).

LEMMA 5. $\mathrm{Lim}_{n\to\infty} M_n = 0$.

Proof. Let us say, to be specific, that M_n is the absolute value of the sum of all the terms in B_n through the sth term in $B_{n,i}$. If i = 1, Lemma 5 reduces to Lemma 3. If i > 1, assume, for the moment, that s is odd. Write

$$M_n = \left|B_{n,1} + B_{n,2} + \cdots + B_{n,i-1} + \Sigma + \tau\right|,$$

where Σ is the sum of the first s - 1 terms in $B_{n,i}$, and where τ is the sth term. The coefficient in τ is $(2^{i-1}q_{n-i})^{-1}$, and therefore $|\tau| \leq K(2^{i-1}q_{n-i})^{-1}$, where K is an upper bound on $|f|$. By Lemma 4, each of $|B_{n,2}|$, $|B_{n,3}|$, $\cdots$, $|B_{n,i-1}|$, and $\left|\Sigma\right|$ is less than $A'(f)n^{-2}$; since $i \leq n-1$, we see that

$$M_n \leq M_n' + A'n^{-1} + D_n^{-1}K,$$

where $D_n = \min\{2q_{n-2},\, 2^2q_{n-3},\, 2^3q_{n-4},\, \cdots,\, 2^{n-1}q_1\}$. Since $q_n \geq n$,

$$D_n \geq \min\{2(n-2),\, 2^2(n-3),\, \cdots,\, 2^n - 1\} = 2n - 4$$

and $M_n = o(1)$. If s is even, the proof goes through more simply, with τ absent.

This completes the proof of the theorem. We obtain an interesting particular case by taking $\omega(x) \equiv x$; there is a simple quadrature for the class of all functions satisfying a Lipschitz condition on [0, 1].

REFERENCES

1. P. J. Davis, *On simple quadratures*. Proc. Amer. Math. Soc. 4 (1953), 127-136.

2. ———, *Interpolation and approximation*. Blaisdell, N.Y., 1963.

3. ———, *Simple quadratures in the complex plane*. Pacific J. Math. 15 (1965), 813-824.

4. ———, *Additional simple quadratures in the complex plane*. Aequationes Math. 3 (1969), 149-155.

5. F. John, *Identitäten zwischen dem Integral einer willkürlichen Funktion und unendlichen Reihen*. Math. Ann. 110 (1935), 718-721.

6. F. John, *A representation of Stieltjes integrals by conditionally convergent series.* Amer. J. Math. 59 (1937), 379-384.

7. H. Rademacher, *Some remarks on F. John's identity.* Amer. J. Math. 58 (1936), 169-176.

National Bureau of Standards
U.S. Department of Commerce
Washington, D.C. 20234