THE ESSENTIAL RANGE OF A FUNCTION OF CLASS $H^\infty + C$

Lawrence Zalcman

Let Δ and T denote the open unit disc and the unit circle. Write $C = C(T)$, and denote by $H^\infty = H^\infty(T)$ the subalgebra of $L^\infty(T)$ consisting of the boundary functions (in the sense of radial limits) of bounded analytic functions on Δ. E. M. Klein has asked whether a nonconstant function in $H^\infty + C$ can have a countable essential range. Negative answers have been given by Matthew Lee, who used operator-theoretic techniques, and by Colin Graham [2], who employed methods of Banach algebras. We shall prove a somewhat stronger result, using only simple function theory.

We require some notation from the theory of cluster sets; see [1] for references and an extended treatment. Briefly, for a function h defined on Δ, the *cluster set* at $e^{i\theta}$ (denoted by $C(h, e^{i\theta})$) is the set of all limiting values approached by h as $z \to e^{i\theta}$ unrestrictedly in Δ. Analogously, $C_\rho(h, e^{i\theta})$ is the set of cluster values of h as $z \to e^{i\theta}$ radially. Finally, we define $C_{R\backslash E}(h, e^{i\theta})$, the *boundary radial cluster set* modulo E of h at $e^{i\theta}$, by the condition that $w \in C_{R\backslash E}(h, e^{i\theta})$ provided there exist sequences $\{\theta_n\}$ and $\{w_n\}$ such that $\theta_n \to \theta$, $e^{i\theta_n} \notin E \cup \{e^{i\theta}\}$, $w_n \in C_\rho(h, e^{i\theta_n})$, and $w_n \to w$.

We need the following elementary version of Iversen's theorem (see [1, p. 98]).

LEMMA. *Let* $h \in H^\infty(\Delta)$, *and let* $E \subset T$ *have measure zero. Then* $C_{R\backslash E}(h, e^{i\theta}) \supset \partial C(h, e^{i\theta})$ *for every point of* T.

The proof follows easily from the Poisson integral representation together with the argument of [1, p. 91].

We can now prove our result.

THEOREM. *Let* $h \in H^\infty(T)$, *and suppose* h *is (essentially) discontinuous at* $e^{i\theta}$. *Then the essential range of* h *at* $e^{i\theta}$ *contains a continuum (and is therefore uncountable).*

Proof. Identifying h with its interior function on Δ, we see that $C(h, e^{i\theta})$ is nontrivial. The boundary $\partial C(h, e^{i\theta})$ of this connected set is again nontrivial, since h is bounded. We shall now show that each point w in $\partial C(h, e^{i\theta})$ is in the essential range of h at $e^{i\theta}$.

Indeed, suppose that the point w is an exception. Then we can find an $\varepsilon > 0$ and a neighborhood U in T of $e^{i\theta}$ such that the set

$$E_1 = \{e^{i\theta}: e^{i\theta} \in U, |h(e^{i\theta}) - w| < \varepsilon\}$$

has measure zero. Letting E_2 denote the set of points on T where the radial limit of h fails to exist, and setting $E = E_1 \cup E_2$, we obtain a contradiction to the lemma.

To obtain the result of Lee and Graham, note that if $f = g + h$ ($g \in C$, $h \in H^{\infty}$) and h is (essentially) continuous on T, then either f is constant or f(T) is a continuum. If h is discontinuous at some point of T, the theorem shows that the range of f at that point is uncountable.

Remarks. 1. Since the lemma is actually true in the generality of arbitrary meromorphic functions on Δ, the only real use of boundedness was in concluding that $\partial C(h, e^{i\theta})$ is nontrivial. In the absence of some condition of this sort, the theorem fails. For instance, a well-known construction [1, p. 163] yields a function analytic on Δ whose radial limits exist almost everywhere and take on (essentially) only the values 0 and 1.

2. For more on the class $H^{\infty} + C$, see [4] and [5].

3. For the connection between our lemma and Banach algebras, see [3].

REFERENCES

1. E. F. Collingwood and A. J. Lohwater, *The theory of cluster sets.* Cambridge Univ. Press, Cambridge, 1966.

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3. M. L. Weiss, *Cluster sets of bounded analytic functions from a Banach algebraic viewpoint.* Ann. Acad. Sci. Fenn. Ser. A I No. 367 (1965), 14 pp.

4. L. Zalcman, *Bounded analytic functions on domains of infinite connectivity.* Trans. Amer. Math. Soc. 144 (1969), 241-269.

5. ———, $H^{\infty} + C$. Indiana Univ. Math. J. 20 (1971), 971-974.

University of Maryland
College Park, Maryland 20742