VECTOR-VALUED ANALYTIC FUNCTIONS

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In this note we show that a function f defined on a real domain and taking values in the dual of a Fréchet space is analytic if it is weakly analytic. A counterexample shows that this is not generally true for functions taking values in a Fréchet space.

We apply this result to a consideration of partially analytic distributions, and to a theorem of T. Kotake and M. S. Narasimhan on the behavior of elliptic partial differential operators with analytic coefficients.

First, let us fix some notational conventions. We shall denote by $\mathscr{O}$ the space of germs of functions holomorphic in a neighborhood of the point 0 in the complex plane $\mathbf{C}$, and by $\mathscr{P}$ the space of formal power series in z. By $\mathscr{H}(B_r)$ we shall mean the space of holomorphic functions defined on the disc $B_r = \{z \in \mathbf{C}: |z| < r\}$. Then $\mathscr{O}$ is a (generalized) LF-space; $\mathscr{O} = \bigcup_{r>0} \mathscr{H}(B_r)$.

Our first theorem could be proved directly by means of the Baire Category Theorem. However, we prefer to use the theorem of Grothendieck that if $T: E \to F$ is a closed linear map, E is a Fréchet space, and $F = \bigcup F_k$ is an inductive limit of a sequence of Fréchet spaces, then there exists an N such that T maps E into F_N continuously. (See Theorem 3 and the remark at the end of this note.)

THEOREM 1. *Let* E *be a Fréchet space and* E' *its strong dual. If* $\{f_j\}$ *is a sequence of elements of* E' *such that, for each* $u \in E$, *the sequence* $\{|\langle f_j, u\rangle|^{1/j}\}$ *is bounded, then there exists an* $\varepsilon > 0$ *such that* $\{\varepsilon^j f_j\}$ *is strongly bounded.*

Proof. For each $u \in E$, the element $Tu = \sum_{j=0}^{\infty} \langle f_j, u\rangle z^j$ belongs to $\mathscr{O}$. Since the composite map $E \xrightarrow{T} \mathscr{O} \to \mathscr{P}$ is clearly continuous, T is closed. Grothendieck's theorem implies that T maps E continuously into $\mathscr{H}(B_r)$ for some $r > 0$, which implies that $|\langle f_j, u\rangle| \leq C(2/r)^j$ for each $u \in E$, with r independent of u. Our theorem follows, with $\varepsilon = r/2$, by the uniform-boundedness theorem.

THEOREM 2. *Let* Ω *be a region in* $\mathbb{R}^n$, *and suppose* $f: \Omega \to E'$ *is a function such that, for each* u *in the Fréchet space* E, *the map* $x \to \langle f(x), u\rangle$ *is analytic. Then* f *is strongly analytic.*

Proof. If $x_0 \in \Omega$, then for each $u \in E$, $\frac{1}{h}\langle f(x_0 + h\varepsilon_j) - f(x_0), u\rangle$ converges to $(\partial/\partial x_j)\langle f(x), u\rangle|_{x_0}$ as $h \to 0$, where $\{\varepsilon_1, \cdots, \varepsilon_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$. The uniform boundedness theorem implies that there is an $f_j \in E'$ such that

$$\langle f_j, u\rangle = \frac{\partial}{\partial x_j}\langle f(x), u\rangle|_{x_0} \quad \text{for all } u \in E.$$

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Repeating this reasoning, we conclude that for each multi-index α there is an $f_\alpha \in E'$ such that $D^\alpha \langle f(x), u \rangle |_{x_0} = \langle f_\alpha, u \rangle$ for all $u \in E$. Obviously, for each $u \in E$ we have the relation

$$\langle f(x_0 + y), u \rangle = \sum_{\alpha \geq 0} \frac{1}{\alpha!} y^\alpha \langle f_\alpha, u \rangle$$

for all y in some neighborhood of 0. The desired result is now an immediate consequence of Theorem 1, or, to be more precise, of the analogous theorem obtained when $\{f_j\}$ is replaced by $\left\{ \frac{1}{\alpha!} f_\alpha \right\}$, where α is a multi-index.

COROLLARY. *If* B *is a Banach space and* $f: \Omega \to B$ *is weakly analytic, then* f *is strongly analytic.*

Proof. This follows from Theorem 2 if we consider B as a closed subspace of B", taking E = B'. We need only remark that the coefficients f_α belong to B. But if $\tilde{\Omega}$ is a small complex neighborhood of Ω to which f has a holomorphic extension $f: \tilde{\Omega} \to B''$, then the Hahn-Banach theorem shows that actually f maps $\tilde{\Omega}$ into B; hence the f_α, being given by Cauchy's integral formula, belong to B.

We note that the corollary is not valid if we replace B by a Fréchet space. Consider the following counterexample. Let $F(x, y) = (x^2 - y^2 + 1)^{-1}$. Denote by I the interval (-1, 1), and define $f: I \to C^\infty(I)$ by $f(x) = F(x, \cdot)$. It is easy to see that for each $\nu \in \mathscr{E}'(I)$, $\langle f(x), \nu \rangle$ is an analytic function of x. However, f does not have a strongly convergent power series expansion about 0.

It is convenient to state an extension of Theorem 2, namely that distributions with values in the strong dual E' of an F-space, which are weakly analytic, are strongly analytic. Recall that a distribution on Ω with values in E' is a continuous linear map $T: C_0^\infty(\Omega) \to E'$.

THEOREM 3. *Let* $T: C_0^\infty(\Omega) \to E'$ *be a distribution with values in the strong dual* E' *of an* F-*space. Suppose that for each* $u \in E$ *the composition* $uT: C_0^\infty(\Omega) \to \mathbb{C}$, *belonging to* $\mathscr{D}'(\Omega)$, *is an analytic function on* Ω. *Then* T *is a strongly analytic function on* Ω *with values in* E'.

Proof. The proof is a simple variation on the previous arguments. For if $x_0 \in \Omega$ and $\mathscr{O}$ is the space of germs of functions analytic in a neighborhood of x_0, then $u \xrightarrow{\tau} uT$ gives a map $\tau: E \to \mathscr{O}$.

Hence $\tau: E \to \mathscr{H}(B_r(x_0))$ for some $r > 0$. If for each $z \in B_r(x_0)$ we define $f(z) \in E'$ by $\langle u, f(z) \rangle = \tau u(z)$, then f is an analytic function in a neighborhood of x_0, its values lie in E', and it agrees with T.

We consider the application of Theorem 3 to partially analytic distributions. Suppose a domain $\Omega = \Omega' \times \Omega''$ is given as a product, and $u \in \mathscr{D}'(\Omega)$. We write $x = (x', x'')$ and say that u is analytic in x' if for each $\phi \in C_0^\infty(\Omega'')$, the product $\langle u, \phi \rangle \in \mathscr{D}'(\Omega')$ is analytic on Ω'. There is another reasonable notion of partial analyticity: we say $u \in \mathscr{D}'(\Omega)$ is strongly analytic in x' provided that each point $x_0 = (x_0', x_0'') \in \Omega$ has a neighborhood $\omega = \omega' \times \omega''$ on which u has an expansion

$$u = \sum_{\alpha \geq 0} C_\alpha (x' - x_0')^\alpha \quad \text{with } C_\alpha \in \mathscr{D}'(\omega''),$$

in the sense that for each $\phi \in C_0^\infty(\omega'')$ and each $x' \in \omega'$, the series

$$\sum_{\alpha \geq u} \langle C_\alpha , \phi \rangle (x' - x_0')^\alpha$$

converges absolutely to $\langle u, \phi \rangle$. Clearly, every strongly analytic distribution in x' is analytic in x'. The next theorem shows that the two notions are indeed equivalent.

THEOREM 4. *If* $u \in \mathscr{D}'(\Omega)$ *is analytic in* x', *then* u *is strongly analytic in* x'.

Proof. By hypothesis $u: C_0^\infty(\Omega') \to E'$, where we take $E = C_0^\infty(\bar{\omega}'')$ with an arbitrary relatively compact $\omega'' \subset \Omega''$. The hypothesis of analyticity implies that for each $\phi \in E$ the composition $\phi u = \langle u, \phi \rangle \in \mathscr{D}'(\Omega')$ is analytic; thus Theorem 4 is an immediate consequence of Theorem 3.

This result stands in sharp contrast to the situation of partially smooth distributions. From it one can deduce without too much difficulty that every distribution, analytic both in x' and in x'', is analytic.

Finally we prove a strengthened form of a result due to T. Kotake and M. S. Narasimhan. First we need a lemma.

LEMMA. *If* Ω *is a region in* $\mathbb{R}^n$ *and* $\{f\}$ *is a sequence of elements of* $\mathscr{D}'(\Omega)$ *such that, for each* $\phi \in C_0^\infty(\Omega)$, *the sequence* $\{|\langle f_j , \phi \rangle|^{1/j}\}$ *is bounded, then for every compact* $K \subset \Omega$ *there is an* r *such that* $|\langle f_j , \phi \rangle| \leq cr^j$ *for each* $\phi \in C_0^\infty(K)$.

Proof. Apply Theorem 1 to $E = C_0^\infty(K)$.

THEOREM 5. *Let* $A = A(x, D)$ *be a strongly elliptic operator of order* $2m$, *with analytic coefficients on* Ω. *Let* $u \in \mathscr{D}'(\Omega)$, *and suppose that for each* $\phi \in C_0^\infty(\Omega)$ *there is a constant* c *such that* $|\langle A^j u, \phi \rangle| \leq c^{j+1} (2m)!$. *Then* u *is analytic.*

Proof. In view of the lemma, our hypothesis implies that, for some ρ independent of ϕ, $|\langle A^j u, \phi \rangle| \leq C\rho^j (2mj)!$, at least for any $\phi \in C_0^\infty(\omega)$ with any preassigned relatively compact $\omega \subset \Omega$. We can now complete the proof as in [3]. Namely, define

$$F(t) = \sum_{j=0}^{\infty} (-1)^{(m+1)j} \frac{1}{(2mj)!} t^{2mj} A^j u .$$

If $\varepsilon = \rho^{-1/2m}$, this series converges on $J = (-\varepsilon, \varepsilon)$ to a continuous function of t with values in $\mathscr{D}'(\omega)$. Hence $F \in \mathscr{D}'(J \times \omega)$. Note that $(\partial^{2m}/\partial t^{2m}) F = (-1)^{m+1} AF$. But $(\partial^{2m}/\partial t^{2m}) + (-1)^m A$ is an elliptic operator with analytic coefficients on $J \times \omega$, so that F is analytic. Hence $u = F(0)$ is analytic.

We remark that the series for $F(t)$ is a natural one to write down. We obtain it by taking the formal power series for $\exp[(-1)^{(m+1)/2m} t A^{1/2m}]u = G(t)$, which clearly satisfies the equation $(\partial^{2m}/\partial t^{2m}) G = (-1)^{m+1} AG$, and throwing away all terms involving nonintegral powers of A. The assumed estimates precisely guarantee convergence of this series.

We also remark that Theorem 5 can be generalized in the following direction. If $A = A(x', D')$ and $B = B(x'', D'')$ are differential operators with analytic coefficients, strongly elliptic of order $2m$ in their respective arguments, and if for each $\phi \in C_0^\infty(\Omega)$, the inequalities

$$|\langle A^j u, \phi \rangle| \leq c^{j+1}(2mj)! \quad \text{and} \quad |\langle B^j u, \phi \rangle| \leq c^{j+1}(2mj)!$$

hold, then the same reasoning as above shows that u is analytic separately in x' and x" and hence, as we remarked previously, is analytic. Indeed it is only necessary to consider ϕ of the form $\phi(x) = \phi_1(x')\phi_2(x'')$, which does not involve a real weakening of hypotheses, as one can see by noticing that Grothendieck's theorem is also valid if E is replaced by an (incompleted!) projective tensor product of Fréchet spaces.

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